SIMPLE METHOD OF THE DYNAMIC APERTURE ESTIMATION

Yu.S. Fedotov
Institute for High Energy Physics, Serpukhov, USSR

ABSTRACT. The method of numeric definition of the stable motion boundaries in a nonlinear magnetic field with the use of the matrix methods developed for periodical linear systems is described. Examples with sextupolar nonlinearities are considered.

With nonlinear field perturbations present in accelerators and storage rings, the phase plane of betatron motion close to the frequencies satisying the relation $Q=p/q$, where p,q are integers, is broken into domains possessing different qualities of motion. Inside the domains in closed separatrix the motion is stable whereas beyond them is unstable though it may be limited. The position of separatrix is determined by that of the fixedpoints, both stable and unstable, satisfying the equation [1]

$$\frac{dJ}{dn} = -\frac{\partial \mathcal{H}(\Psi,J)}{\partial\Psi} = 0, \quad \frac{d\Psi}{dn} = \frac{\partial\mathcal{H}(\Psi,J)}{\partial J} = 0 \tag{1}$$

Here $\mathcal{H}(\Psi,J)$ is the Hamiltonian of motion close to the resonance $q\cdot Q=p$, Ψ,J are the standart canonically conjugate action-angle variables, n is the turn number.

Even with a simple distribution of the nonlinearities across the machine azimuth, the analytical techniques used for determining the boundaries of stable motion are very complicated especially if some nonlinearities take place simultaneously. Therefore numeric simulation [2] has recently come into use for these purposes.

The matrix methods developed for establishing the conditions of stable motion in periodic linear systems [3] can also be applicable for nonlinear systems. For betatron frequencies close to the rational ratio p/q the trajectory of a particle obeying the initial conditions $\vec{r}_1=(r_1,r_1')$ is nearly closed after q turns. Computing the trajectory with any integration method applied [4,5,6] calculate the tranfer matrix $N_q(2\times2)$ corresponding to the relevant "period" of q turns such that $\vec{r}_2=N_q\cdot\vec{r}_1$.

The elements of the N_q matrix are the functions of the initial conditions, i.e. of the point $\vec{r}_1$. For a fixed point (r_f,r_f') we have a relation $\vec{r}_f=N_q\cdot\vec{r}_f$ or

$$\begin{aligned} n_{11}\cdot r_f + n_{12}\cdot r_f' &= r_f \\ n_{21}\cdot r_f + n_{22}\cdot r_f' &= r_f' \end{aligned} \tag{2}$$

which is a system of equations for finding the eigenvectors of the matrix N_q having the eigenvalues $\lambda_1=\lambda_2=1$. The relation $\mathrm{Tr}N_q=n_{11}+n_{22}=2$ is therefore fulfilled for the matrix of fixed points, which is the general condition for the boundary of stable motion. The slope of the eigenvectors $\vec{x}$ of the matrix N_q in (2) is

$$\frac{x'}{x} = \frac{r_f'}{r_f} \tag{3}$$

With the initial conditions varied by a small value $\Delta\vec{r}_1$, the final point also becomes somewhat shifted by $\Delta\vec{r}_2$. Since $\vec{r}_2=f(r_1,r_1')$ then

$$\begin{aligned} \Delta r_2 &= \frac{\partial r_2}{\partial r_1}\Delta r_1 + \frac{\partial r_2}{\partial r_1'}\Delta r_1' = m_{11}\cdot\Delta r_1 + m_{12}\cdot\Delta r_1' \\ \Delta r_2' &= \frac{\partial r_2'}{\partial r_1}\Delta r_1 + \frac{\partial r_2'}{\partial r_1'}\Delta r_1' = m_{21}\cdot\Delta r_1 + m_{22}\cdot\Delta r_1' \end{aligned} \tag{4}$$

where $M_q=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22}\end{pmatrix}$ is the matrix of small deviations from the trajectory possessing the initial conditions $\vec{r}_1$ such that $\Delta\vec{r}_2=M_q\cdot\Delta\vec{r}_1$.

The trace of the matrix M_q determines the stability of small deviations close to the fixed point. The eigenvectors of the matrix M_q $\vec{V}=(V,V')$ are found from the relation

$$M_q\cdot\vec{V} = \lambda\cdot\vec{V}, \tag{5}$$

where the eigenvalues are

$$\lambda_{1,2} = \frac{\mathrm{Tr}M_q}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}M_q}{2}\right)^2 - 1}\ . \tag{6}$$

Since $\lambda_1\cdot\lambda_2=1$, the slopes of the eigenvectors are

$$\frac{V_1'}{V_1} = \frac{\lambda_1 - m_{11}}{m_{12}}, \quad \frac{V_2'}{V_2} = \frac{1/\lambda_1 - m_{11}}{m_{12}}\ . \tag{7}$$

The eigenvalues $\lambda_{1,2}$ are real if $\mathrm{Tr}M_q>2$. In this case the fixed point is unstable, i.e. saddle-shaped. It is the intersection of two separatrises whose slopes are equal to those of the eigenvectors (7). If $\mathrm{Tr}M_q<2$, the fixed point is stable, i.e. of the centre type.

Figure 1 shows the triangle of the separatrix of stable motion and the position of the boundaries $\mathrm{Tr}M_3=2$ and $\mathrm{Tr}N_3=2$ close to the third-order resonance $3Q=110$. This triangle has been calculated for the following conditions: the resonant harmonic is excited by two sextupole lenses of strength

$$K_1 = K_2 = \frac{1}{2B_oR_o}\cdot\frac{d^2B}{dr^2}\cdot l = 0.087\ ^1/_{m^2}$$

placed symmetrically in the accelerator ring at the points where β=144m and having the same sign of the sextupole field nonlinearity, Q =36.685.

The section of the half ring between the lenses is given by the matrix

$$N = \begin{pmatrix} \cos\pi Q & \beta\sin\pi Q \\ -^1/_\beta\sin\pi Q & \cos\pi Q \end{pmatrix} \tag{8}$$

For the sextupole lens in a thin-lens approximation, the matrix of "large" deviations is

$$N_s = \begin{pmatrix} 1 & 0 \\ -K\cdot r & 1 \end{pmatrix} \tag{9}$$

where r is the deviation of a particle from the lens axis, while the matrix of "small" deviations is

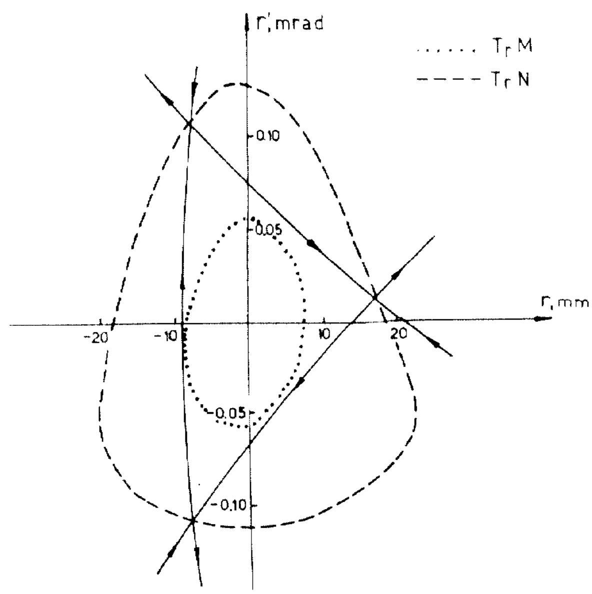

Fig.1. Stability triangle and boundaries $TrM_3=2$ and $TrN_3=2$ of the third-order resonance.

$$M_s = \begin{bmatrix} 1 & 0 \\ -2K\cdot r & 1 \end{bmatrix} \tag{10}$$

The position of the boundaries is found easily with any integration technique applied. A fixed point on the boundary $TrN_q = 2$ can be found using relation (2). The iteration process

$$\vec{r}_{1,k+1} = \vec{r}_{1,k} + \Delta\vec{r}_1 , \tag{11}$$

where

$$\Delta r_1 = \frac{(m_{22}-1)\left[(n_{11}-1)r_{1,k}+n_{12}r'_{1,k}\right]}{TrM_q-2} - \frac{m_{12}\left[n_{21}r_{1,k}+(n_{22}-1)r'_{1,k}\right]}{TrM_q-2}$$

$$\Delta r'_1 = \frac{(m_{11}-1)\left[n_{21}r_{1,k}+(n_{22}-1)r'_{1,k}\right]}{TrM_q-2} - \frac{m_{21}\left[(n_{11}-1)r_{1,k}+n_{12}r'_{1,k}\right]}{TrM_q-2}$$

is faster. Here the elements of the matrixes N_q and M_q are calculated for the trajectory originating at point $\vec{r}_{1,k}$. With nonlinearities absent, when $M_q=N_q$, the coordinate transformation (11) transforms any point of $\vec{r}_1$ into the only one fixed point of the phase plane, i.e. into the coordinate origin.

Figure 2 shows the separatrises for the case with two octupole lenses of strength

$$K = \frac{1}{6B_oR_o}\cdot\frac{d^3B}{dr^3}\cdot l = 3.0\ {}^1/_{m^3}$$

introducing amplitude - dependent betatron frequency shift, added to the conditions of the above example. In this case, the motion became limited and, in contrast with fig.1, two more, "distant", boundaries emerged, $TrN_3=2$ and $TrM_3=2$. Hence, three stable points are added up to three unstable fixed ones. The stable and unstable fixed points are on different boundaries of $TrN_3=2$. This is an example of "strong" instability.

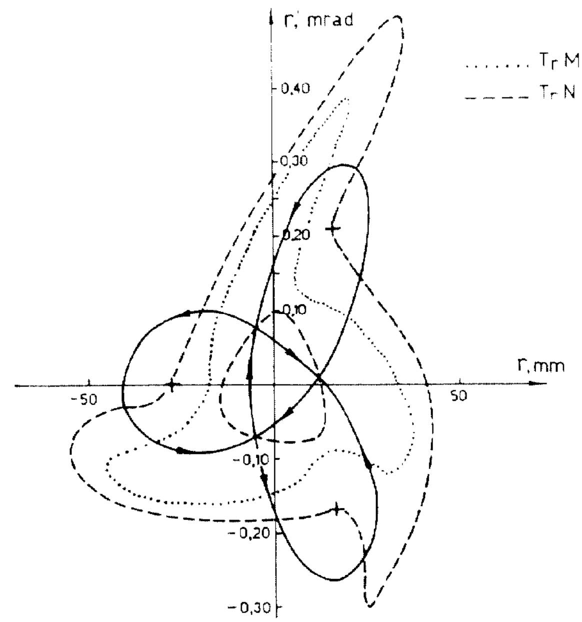

Fig.2. Stability triangle and boundaries $TrM_3=2$ and $TrN_3=2$ of the third-order resonance with the octupole field nonlinearity added.

Figure 3 shows the separatrises and boundaries $TrN_5=2$, $TrM_5=2$ for $Q=36.805$ and $K=0.047\ m^{-2}$. As is seen, the stable and unstable fixed points are on the same boundary, $TrN_5=2$. This is "weak" instability. It should be noted that sextupole field nonlinearity can excite such instability close to any line of the $Q=p/q$ type.

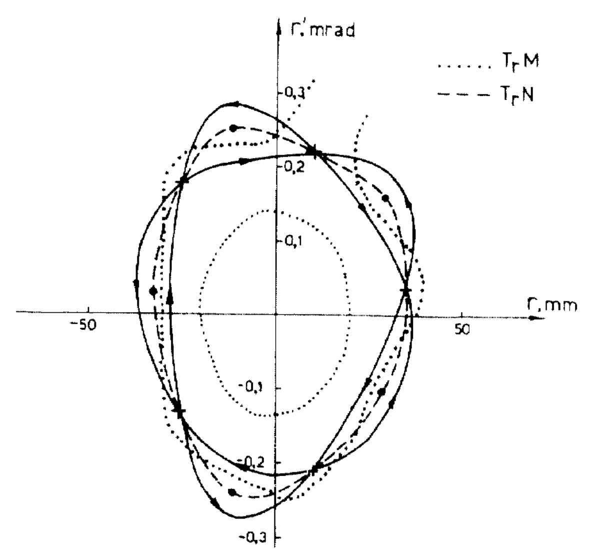

Fig.3. Phase trajectories and boundaries $TrM_5=2$ and $TrN_5=2$ of the fifth-order resonance.

Figure 4 shows the phase trajectories and boundaries $TrR_3=2$, $TrM_3=2$ for $Q=36.652$, $K_1=-K_2=0.145m^{-2}$. This is the manifestation of the sixth-order resonance on the harmonics of the sextupole nonlinearity $6Q=220$ in the second approximation.

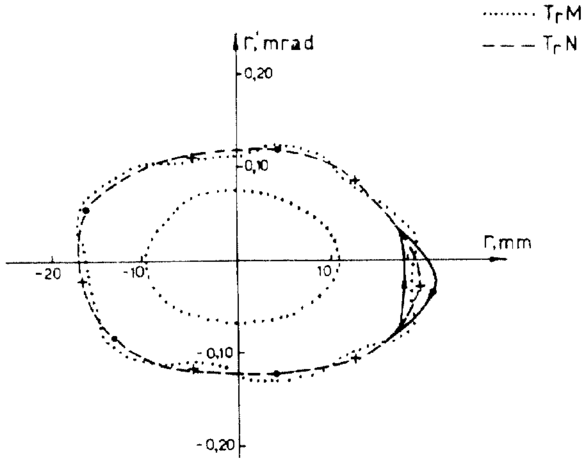

Fig.4. Phase trajectories and boundaries $TrM_3=2$ and $TrN_3=2$ of the sixth-order resonance. Shown is one area of closed motion around the stable fixed point.

This technique of determining the boundaries of stable motion is simple and demonstrable. The calculation technique applying the thin-lens approximation is accurate, simplectic, though somewhat slow. But since no calculation during tens of thousands turns is not required, in this case much less calculation time is spent as compared with ordinary tracking. On determining the boundaries of stable motion for different frequencies Q , one can estimate the effect of frequency ripples due to, for example, synchrotron oscillations with nonzero chromaticity.

This technique also allows one to detect the deviation from the Hamiltonian character of motion when in some areas of the phase plane the boundaries of stability of some resonances overlap. Figure 5 shows an example of such a motion for $Q=36.595$, $K_1=K_2=0.07\ m^{-2}$.

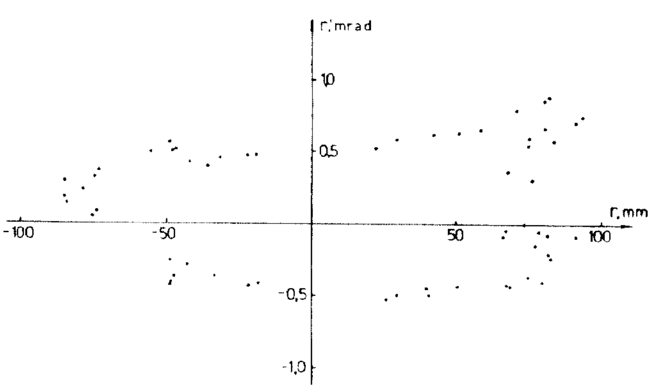

Fig.5. Example of quasidiffusive motion close to the fifth-order resonance.

For the conditions specified and with r>74 mm the areas of unstable motion of the fifth- and seventh-order resonances overlap. The manifestation of the diffusive nature of motion is seen.

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