Wake Fields between two Parallel Resistive Plates

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Abstract The wake field generated by a point-like particle travelling parallel to two infinite metallic plates with finite resistivity, is calculated.

1 Introduction

Intense electron bunches are known to produce destabilizing wake fields when passing through discontinuous or resistive structures. An example is provided by the CLIC transfer structure[1] which is designed to extract 29 GHz RF power from the passage of a train of bunches containing each about 10^{12} electrons[2]. At present, this structure is a rectangular waveguide with periodic loading 3.2 *mm* away from the beam. Wake fields will be generated both by the periodic structure and by the resistivity of the vertical walls of 4 *mm* aperture. In this paper, we consider the passage of a point like charge between parallel metallic walls with infinite extent and finite resistivity. The impedances per unit length are obtained by solving Maxwell's equation in the f-domain. Thereby we follow closely the mathematical method used by Morton, Neil and Sessler[3] for the case of a resistive pipe. Through a Fourier transform, the wake fields are computed under the form of a series expansion for small and large distance z behind the source particle. The interpolating regime is given by a *real* double integral and is exhibited for both exciting and test particles on axis. Conclusions are drawn for the CLIC transfer structure.

2 Impedance calculation

We consider the geometry of two parallel metallic plates with resistivity κ. As depicted in Fig.1, the two plates are separated by $2a$ along the y-axis. The point-like source particle has charge e and is travelling parallel to the plates along the z-axis with a y-offset value Δ. We directly consider the case of an ultra-relativistic particle $\beta = v/c \simeq 1$. It produces an electromagnetic field $(\mathbf{E}_\kappa, \mathbf{B}_\kappa)$ which can be split in two parts

$$(\mathbf{E}_\kappa, \mathbf{B}_\kappa) = (\mathbf{E}^{(0)}, \mathbf{B}^{(0)}) + (\mathbf{E}, \mathbf{B}) \tag{1}$$

where $(\mathbf{E}^{(0)}, \mathbf{B}^{(0)})$ is the electromagnetic field in the case of perfectly conducting plates, $\kappa = \infty$, and $(\mathbf{E},\mathbf{B})$ is the wake field. It is easy to show that $(\mathbf{E},\mathbf{B})$ obeys the source free Maxwell's equations with a metallic current

$$\mathbf{j} = \kappa \mathbf{E} \tag{2}$$

Its dependance on the source particle comes in only from the boundary conditions which state that $(\mathbf{E}+\mathbf{E}^{(0)})_\parallel$ and $(\mathbf{B}+\mathbf{B}^{(0)})$ are continuous at $y = \pm a$, since the surface current is zero. Since $\mathbf{E}_\parallel^{(0)}$ and $\mathbf{B}_\perp^{(0)}$ are zero at the metallic surfaces, our first task is to calculate $\mathbf{B}_\parallel^{(0)}$ on them. One way to do this is to use the Lorentz invariance of Maxwell's equations in the case of infinite conductivity. First one considers the electrostatic problem where the charge is at rest and then the fields are Lorentz transformed. Defining the Fourier transformation of a field Φ by

$$\Phi(x, y, z - \beta ct) = \int\int dq\, dk\; \Phi(q, y, k) e^{iqx} e^{ik(z-\beta ct)} \tag{3}$$

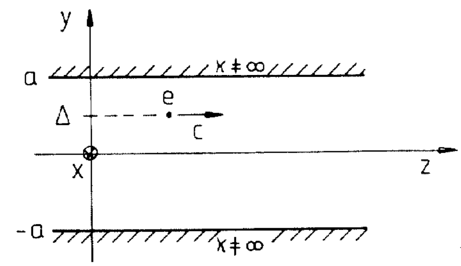

Figure 1: Geometry of the two plates

ones gets on the surfaces, for the $\beta = 0$ static case,

$$\mathbf{E}^{(0)}_{static}(q, y = \pm a, k) = \frac{e}{4\pi^2\epsilon_0} \frac{\sinh(\alpha(\Delta \pm a))}{\sinh(2\alpha a)} \hat{\mathbf{y}} \tag{4}$$

with

$$\alpha = \sqrt{q^2 + k^2} \tag{5}$$

and $\hat{\mathbf{y}}$ the unit vector in the y-direction. For the moving charge, the electric field gives rise to a magnetic field through

$$\mathbf{B}^{(0)} = \frac{\gamma}{c} \boldsymbol{\beta} \times \mathbf{E}^{(0)}_{static} \tag{6}$$

In the limit $\gamma \to \infty$, one gets

$$\mathbf{B}^{(0)}(q, y = \pm a, k) = -\frac{e}{4\pi^2\epsilon_0 c} \frac{\sinh(q(\Delta \pm a))}{\sinh(2qa)} \hat{\mathbf{x}}. \tag{7}$$

We can now turn to the calculation of the wake field $(\mathbf{E},\mathbf{B})$. Its Fourier transform obeys the homogeneous Maxwell's equations

$$\nabla.\mathbf{E} = 0 \tag{8}$$
$$\nabla.\mathbf{B} = 0 \tag{9}$$
$$\nabla \times \mathbf{E} = ikc\mathbf{B} \tag{10}$$
$$\nabla \times \mathbf{B} = (\mu_0\kappa - ik/c)\mathbf{E} \tag{11}$$

Of course, one must set $\kappa = 0$ in the vacuum region between the plates ($-a \le y \le a$). Using (10) to evaluate $\mathbf{B}$ in term of $\mathbf{E}$, (9) is fulfilled, and Eqs.(8, 11) become

$$\nabla.\mathbf{E} = 0 \tag{12}$$
$$\left(\frac{\partial^2}{\partial y^2} - K^2\right)\mathbf{E} = 0 \tag{13}$$

with

$$K^2 = q^2 - ikc\mu_0\kappa \tag{14}$$

As already mentionned, the solution $\mathbf{E}$ of these equations must be such that $\mathbf{E}_{//}(y)$ and $(\mathbf{B} + \mathbf{B}^{(0)})_{//}(y)$ are continuous at the boundaries $y = \pm a$. Using (10), the continuity of the normal component of $\mathbf{B}$ follows from that of $\mathbf{E}_{//}$. Note that $\mathbf{B}^{(0)}_{//}$ is the only source of inhomogeneity and thus is similar to a drive term in a homogeneous differential equation. Lengthy but straightforward linear algebra leads to the following solution for the z-component of the electric field between the plates :

$$\mathbf{E}_z(q, y, k) = E_{+,z}\, e^{qy} + E_{-,z}\, e^{-qy} \qquad (-a \le y \le a) \tag{15}$$

with

$$E_{\pm,z} = \frac{ike}{16\pi^2\epsilon_0} q(q^2 - K^2)\{\cosh(q\Delta)[(K\cosh(qa) + q\sinh(qa)) \\ (k^2(K\sinh(qa) + q\cosh(qa)) + q(q^2 - K^2)\cosh(qa))]^{-1} \\ \pm \sinh(q\Delta)[(K\sinh(qa) + q\cosh(qa)) \\ (k^2(K\cosh(qa) + q\sinh(qa)) + q(q^2 - K^2)\sinh(qa))]^{-1}\} \quad (16)$$

K is defined as the solution of (14) with a positive real part.

The force acting on a test particle with charge e' is given by the Laplace formula

$$\mathbf{F} = e'(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (17)$$

For a relativistic particle such that $\mathbf{v} = c\,\hat{\mathbf{z}}$ and using (10) the impedances (i.e. the Fourier transforms of the wake forces) depend only on the longitudinal components $E_{\pm,z}$ as follows :

$$\mathbf{F}(q,y,k)/e' = \begin{pmatrix} q/k \\ -iq/k \\ 1 \end{pmatrix} E_{+,z}e^{qy} + \begin{pmatrix} q/k \\ iq/k \\ 1 \end{pmatrix} E_{-,z}e^{-qy} \quad (18)$$

3 The Longitudinal Wake Potential

In this section, we calculate the longitudinal component of the wake potential $W_{\|}$. We give its asymptotic form for small and large distance $(z-ct)$ of the test charge behind the source, as well as a real integral which provides the interpolating behaviour.

The longitudinal wake potential per unit length is given by[4]

$$W_{\|}(x,y,\varsigma) = -\frac{1}{ee'}\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dk\; F_z(q,y,k)e^{iqx}e^{ik\varsigma} \quad (19)$$

The calculation of this double integral from the expression of $F_z(q,y,k)$ as given by (18), is described elsewhere[5]. As expected, ahead of the relativistic source particle one obtains a vanishing wake $W_{\|} = 0$. For $(z-ct) \geq 0$, the results are as follows.

The case $(z-ct) = 0$

One finds the following expression for the longitudinal wake

$$W_{\|}(x,y,z-ct=0) = \frac{\pi}{16\epsilon_0 a^2}\frac{1+\cosh(2u)\cos(2v_+)}{(\cosh(2u)+\cos(2v_+))^2} \quad (20)$$

where we have introduced the dimensionless variables

$$u = \frac{\pi x}{4a} \quad (21)$$

$$v_\pm = \frac{\pi(\Delta \pm y)}{4a} \quad (22)$$

In particular, the loss factor per meter for a particle on axis is

$$k_z = W_{\|}(x=0,y=0,z-ct=0) = \frac{\pi}{32\epsilon_0 a^2} \quad (23)$$

The case $(z-ct) < 0$

The long range behaviour Introducing the characteristic length

$$\lambda = (\mu_0 c\kappa)^{-1} \quad (24)$$

of the order of $10^{-10}\,m$ for metals, the asymptotic expression of $W_{\|}$ is reached when $|z-ct| \gg (a^2\lambda)^{1/3}$. The x,y dependance can be expressed as power series[5]. For $x=0$ one gets

$$W_{\|}(0,v_\pm,z-ct) = -\frac{c}{4\pi a}\sqrt{\frac{Z_0}{\pi\kappa}}\,|z-ct|^{-3/2}(1+v_+^2-\frac{1}{3}v_-^2) + O(v_\pm^4) \quad (25)$$

Thus, the long range on-axis longitudinal wake field is

$$W_{\|}(z-ct) = -\frac{c}{4\pi a}\sqrt{\frac{Z_0}{\pi\kappa}}\,|z-ct|^{-3/2} \quad (26)$$

It decays like $|z-ct|^{-3/2}$, as in the case of a pipe[3].

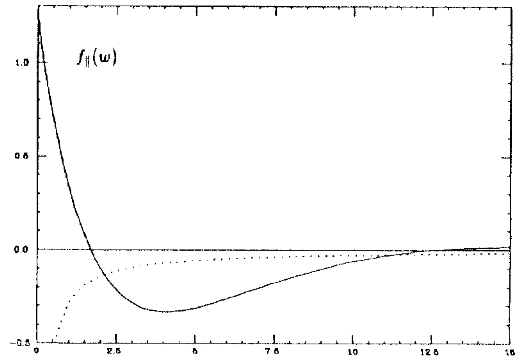

Figure 2: The longitudinal wake function $f_{\|}(w)$.

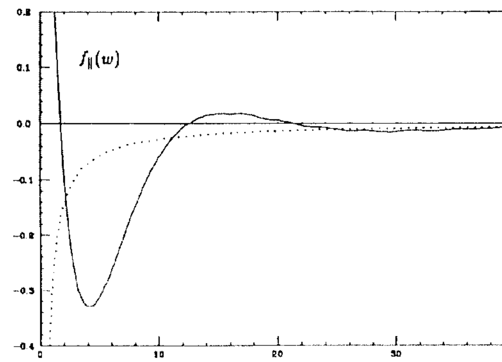

Figure 3: The longitudinal wake function $f_{\|}(w)$. The large w asymptotic regime is in dotted line

The short range behaviour For the small $(z-ct)$ asymptotic regime, a series expansion for $W_{\|}(z-ct)$ can be derived:

$$W_{\|}(z-ct) = \frac{cZ_0}{2\pi a^2}\sum_{k=0}^{+\infty} C_k\left(-\frac{|z-ct|^{3/2}}{a\sqrt{\lambda}}\right)^k \quad (27)$$

where

$$C_k = \frac{(-2)^{3k}}{\sqrt{\pi}}\frac{\Gamma((3k-1)/2)}{\Gamma(3k+1)}\int_0^\infty dt\, t^{k+1}\frac{\cosh^{k-1}(t)}{\sinh^{k+1}(t)} \quad (28)$$

As expected, immediately behind the exciting charge $(z-ct \to 0^-)$, the wake potential is positive and, from

$$C_0 = \frac{\pi^2}{8} \quad (29)$$

and (20), one gets the well-known relation

$$W_{\|}(z-ct \to 0^-) = 2k_z \quad (30)$$

The interpolating regime Although it is valid over the whole $(z-ct)$-range, the series in (27) is slowly converging and could not be used to see the onset of the asymptotic long range behaviour of $W_{\|}(z-ct)$ given by (26). The interpolating regime between the short and long range asymptotic behaviours is exhibited by writing the longitudinal wake potential $W_{\|}(z-ct)$, for the on axis particle case $x = y = \Delta = 0$, as a real two-dimensional integral. Introducing the dimensionless variable

$$w = \frac{|z-ct|^{3/2}}{a\sqrt{\lambda}} \quad (31)$$

one gets

$$W_{\|}(z-ct) = \frac{cZ_0}{2\pi a^2} f_{\|}(w) \quad (32)$$

where the dimensionless function $f_{\parallel}(w)$ is such that $f_{\parallel}(0) = \pi^2/8$, and is given by

$$f_{\parallel}(w) = \frac{2}{3\pi}\int_0^\infty dt\,\frac{t\coth t}{\cosh^2 t}\int_{-1}^\infty du\,\frac{u\sin\theta+\cos\theta}{u^2+1} \qquad (33)$$

with

$$\theta = \left(\frac{1}{\sqrt{2}}(u+1)t\,\coth(t)\,w)\right)^{2/3} \qquad (34)$$

The two-dimensional integration can be numerically performed. The function $f_{\parallel}(w)$ is shown in Figs.2,3. For $w < 15$, it is calculated from (27), with 64 terms in the series, and for $15 < w < 40$, from the integral in (33). One sees that the the asymptotic regime occurs for $w \geq 30$, after one oscillation through the horizontal axis in the region of positive $W_{\parallel}$.

4 The Transverse Wake Potential

The transverse wake potential per unit length is defined[4] by

$$\mathbf{W}_\perp(x,y,\varsigma) = \frac{1}{ee'}\int_{-\infty}^{+\infty}dq\int_{-\infty}^{+\infty}dk\,\mathbf{F}_\perp(q,y,k)e^{iqx}e^{ik\varsigma} \qquad (35)$$

It is related to the longitudinal wake through Panofsky-Wenzel theorem

$$\partial_{(z-ct)}\mathbf{W}_\perp(x,y,z-ct) = -\boldsymbol{\nabla}_\perp W_{\parallel}(x,y,z-ct) \qquad (36)$$

We use this relation to derive the long range behaviour as well as the integral form of the transverse potential $\mathbf{W}_\perp$ from the corresponding expressions, given by (25), (32) and (33), of the longitudinal potential $W_{\parallel}$.

In the long range asymptotic regime $|z-ct| \gg (a^2\lambda)^{1/3}$, one gets for $x=0$,

$$W_x(0,y,z-ct) = 0 \qquad (37)$$

and, to first order in y and Δ,

$$W_y(0,y,z-ct) = \frac{c}{24a^3}\sqrt{\frac{\pi Z_0}{\kappa}}\,|z-ct|^{-1/2}\,(y+2\,\Delta) \qquad (38)$$

By symmetry, the transverse wake fields vanish when both the source and test particles are on axis. In this case, the interesting quantity is the gradient of the transverse potential. One finds

$$\partial_y W_y(z-ct) = -\partial_x W_x(z-ct) = \frac{c}{24a^3}\sqrt{\frac{\pi Z_0}{\kappa}}\,|z-ct|^{-1/2} \qquad (39)$$

for $x=y=\Delta=0$.

These gradients can also be obtained, over the whole $(z-ct)$-range, under an integral form. The calculation proceeds in the same way than for $W_{\parallel}(z-ct)$, and leads to

$$\partial_y W_y(z-ct) = -\partial_x W_x(z-ct) = \frac{cZ_0^{2/3}}{2\pi a^3(a\kappa)^{1/3}}\,f_\perp(w) \qquad (40)$$

where the dimensionless $f_\perp(w)$ is given by

$$f_\perp(w) = -\frac{2}{\pi}\int_0^\infty dt\,\frac{t^{7/3}\coth^{1/3}t}{\cosh^2 t}\int_0^\infty du\,\frac{u^3\cos\theta-\sin(\theta+\pi/4)}{u^6-\sqrt{2}\,u^3+1} \qquad (41)$$

with $\theta = u^2(t\,\coth(t)\,w)^{2/3}$. $f_\perp(w)$ is drawn in Fig.4. One can see that the asymptotic regime is reached for $w \geq 18$.

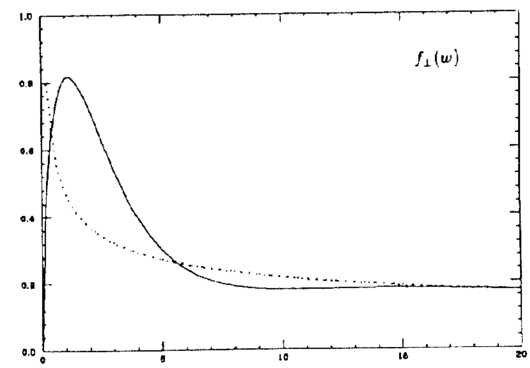

Figure 4: The transverse wake function $f_\perp(w)$. The large w asymptotic regime is in dotted line

5 Conclusion

The wake fields are calculated for a highly relativistic point-like charge between two metallic (finite conductivity κ) plates. As well known, the longitudinal wake experienced by the charge itself, (20), has half the strength of the wake behind the charge. Away from the exciting charge there is an intermediate region where the wakes could be given only numerically (Figs.2,3,4). Even further behind, at distances larger than $4.10^{-3}\,a^{2/3}\,m$, where a is half the distance between the plates (in meter), simple asymptotic expressions could be derived (26, 38). In this region, the longitudinal wake is proportional to $a^{-1}\,\kappa^{-1/2}\,|z-ct|^{-3/2}$ and the transverse wake proportional to $a^{-3}\,\kappa^{-1/2}\,|z-ct|^{-1/2}$ ($|z-ct|$ is the distance behind the exciting charge). This is the same asymptotic behaviour of the wakes as for a charge travelling in a metallic pipe.

In case of Gaussian bunches with r.m.s. length σ_z, one can use the electric field given in (15,16) but with e replaced by $e.\exp(-k^2\sigma_z^2/2)$. Using the same approximations as in Section 3, the resulting wake fields are identical to the one derived by Piwinski[6].

In order to illustrate the effect of resistive wall wakes, we take the example of the CLIC transfer structure. The aperture between copper plates is $2a = 4\,mm$ and a bunch of 10^{12} electrons as an r.m.s. length $\sigma_z = 1\,mm$. Then the integrated wake forces, called *loss factors*, are $k_z = 0.27\,V/pC\,m$ and $k_y = 490.\,V/pC\,m^2$. The power loss of one bunch is $35\,MW$.

References

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