ON THE LONGITUDINAL COUPLING IMPEDANCE OF A TOROIDAL BEAM TUBE*

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Abstract

In this paper, the longitudinal coupling impedance of a smooth toroidal beam tube is derived. By treating the torus as a slow-wave structure, the well-known method of describing the impedance in terms of cavity resonances can be used. A simple analytical expression for the coupling impedance of a toroidal beam tube with square cross section valid in the low-frequency limit is obtained. The results from the present study are compared with previously published solutions and qualitative differences are pointed out.

Introduction

The longitudinal coupling impedance of a toroidal beam tube is a topic of long standing. A general formal treatment of the fields induced by a beam in a toroidal chamber was published by van Bladel, however without apparent impact on the subsequent studies.[1] The impedance review paper by Faltens and Laslett[2] at the *1975 ISABELLE Summer Study* summarized Neil's doctoral dissertation[3] on this topic and pointed to this potentially important source of coupling impedance.

Any electromagnetic wave in a straight beam tube propagates with a phase velocity faster than light. In contrast, a curved beam tube acts like a slow-wave structure allowing synchronism between particles and wave resulting in coupling impedance resonances above cutoff. Whereas Faltens & Laslett focused on these resonances, Zotter addressed the low frequency end and showed that a curved beam tube has essentially the same space charge term as obtained from the straight-tube analysis.[4] The renewed interest in this topic was demonstrated at the *1987 Workshop on Impedance Beyond Cutoff*,[5,6] by the papers of Warnock & Morton[5] and Ng.[6]

Recently, Ruggiero conjectured the possibility of a high Z/n near cutoff resulting from the curved beam tubes in RHIC dipoles.[7] This was discussed at the *1988 Workshop on the RHIC Performance*,[8,9] however papers by Ng & Warnock discount the possibility of a significant impedance below the resonance region.[10]

The analysis of a toroidal beam tube demands non-routine mathematical skills. A torus with circular cross section requires a toroidal coordinate system and leads to non-separable differential equations.[4] A rectangular cross section allows exact analytical solutions in terms of Bessel functions. Extracting a low-frequency approximation involves however the use of asymptotic expansions and cancellation of large terms rendering the results suspect.

Most of the mathematical difficulties are avoided in the present paper by using the well-known method in which the coupling impedance at all frequencies is expressed in terms of its resonances.[11,12] The low-frequency approximation is then determined with sufficient accuracy by the lowest resonance alone. To achieve simple results, the analysis is limited to filamentary beams of extreme relativistic particles, however radiation effects are neglected. Numerical results are obtained using Bessel-function routines in the SLATEC program library.

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Torus with Circular Cross Section

A qualitative estimate of the coupling impedance Z/n in a torus with circular cross section (beam tube radius b, bending radius R) can be obtained by assuming that the beam induced fields are essentially TM_{01}-like.[13] Wave propagation is possible above cutoff

$$n_{co} = j_{01}\frac{R}{b}$$

and synchronism, i.e. $v = c$ occurs at the mode number n_s. In first approximation, the phase velocity is not changed by the curvature[14] but a change similar to curved rectangular waveguides[6,15] must be expected, implying

$$n_s \sim n_{co}\sqrt{R/b}$$

Assuming a lossless structure, the coupling impedance is then given by

$$\frac{Z}{n} = j\left[1-\left(\frac{n}{n_s}\right)^2\right]^{-1}\left(\frac{Z}{nQ}\right)_{syn}$$

with the usual definition (in natural units $c = \mu_o = 1$)

$$\left(\frac{Z}{nQ}\right)_{syn} = \frac{2\pi R}{n_s}\left\{\frac{E_0^* E_0}{\omega_s W}\right\}$$

where W represents the stored energy per unit length and the circular frequency at synchronism

$$\omega_s = n_s/R$$

It follows that

$$\left(\frac{Z}{nQ}\right)_{syn} = \frac{4\pi R^2}{n_s^2}\,\frac{E_0^* E_0}{\int H^* H dA} \sim \frac{4}{j_{01}^2 J_1^2(j_{01})}\,\frac{b^2}{R^2}$$

The curvature effect contributes an inductive impedance of the order of b^2/R^2 which is essentially constant up to frequencies well above cutoff.

Torus with Square Cross Section

A beam centered in a straight waveguide with square cross section, as shown in Fig. 1 (horizontal width w, vertical height $h = w$) induces TM_{11}-like fields. Since the phase velocity is greater than light, no synchronism is possible. The field components (z, r, θ) of the two lowest waves in a torus with square cross section, here referred to as H_{11} and E_{11} mode, are listed in Table I and compared with the TM_{11} mode of a straight tube. Note that the H_{11} and E_{11} modes revert to the so-called Longitudinal-Section modes in the straight waveguide which in contrast to the degenerate TM_{11}/TE_{11} modes are orthogonal and thus stable in the presence of perturbations such as losses or curvatures.[16]

In Table I the common factor $e^{-jn\theta}e^{j\Omega t}$ is suppressed. The sinus-like combination of Bessel functions is defined by

$$S(\kappa r) = \frac{\pi}{2}\left\{J_n(\kappa r)\,Y_n'(\kappa R_i) - J_n'(\kappa R_i)\,Y_n(\kappa r)\right\}$$

and the cosinus-like combination

$$C(\kappa r) = 2^{-2/3} n^{2/3}\left\{Y_n(\kappa r)\,J_n(\kappa R_i) - Y_n(\kappa R_i)\,J_n(\kappa r)\right\}$$

Table I: Wave propagation in beam tube with square cross section*.

	TM_{11}	LSE_{11}	H_{11}	E_{11}
$E_z = -\sin\zeta z$	$\cos\zeta x$	0	0	$(v^2-\epsilon^2)\,C(\kappa r)$
$E_r = -\cos\zeta z$	$\sin\zeta x$	$\sqrt{1+2\epsilon^2}\sin\zeta x$	$v\frac{R}{r}S(\kappa r)$	$\epsilon\sqrt{v^2-\epsilon^2}\,C'(\kappa r)$
$E_\Theta = j\cos\zeta z$	$2\epsilon\cos\zeta x$	$\epsilon\sqrt{1+2\epsilon^2}\cos\zeta x$	$v\sqrt{v^2-\epsilon^2}\,S'(\kappa r)$	$\epsilon\frac{R}{r}C(\kappa r)$
$H_z = \cos\zeta z$	$\sqrt{1+2\epsilon^2}\sin\zeta x$	$(1+\epsilon^2)\sin\zeta x$	$(v^2-\epsilon^2)\,S(\kappa r)$	0
$H_r = -\sin\zeta z$	$\sqrt{1+2\epsilon^2}\cos\zeta x$	$\epsilon^2\cos\zeta x$	$\epsilon\sqrt{v^2-\epsilon^2}\,S'(\kappa r)$	$v\frac{R}{r}C(\kappa r)$
$H_\Theta = j\sin\zeta z$	0	$\epsilon\sin\zeta x$	$\epsilon\frac{R}{r}S(\kappa r)$	$v\sqrt{v^2-\epsilon^2}\,C'(\kappa r)$

* The common factor $e^{-jn\Theta}\, e^{j\Omega t}$ suppressed; $\zeta = \pi/h$; $\epsilon = \zeta R/n$.

where the prime denotes the derivative with respect to the argument. The frequency Ω is given by $\Omega^2 = \kappa^2 + \zeta^2$ with the vertical wave number fixed by $\zeta = \pi/h$ and the horizontal wave number κ by the boundary condition $S'(\kappa^H R_0) = 0$ and $C(\kappa^E R_0) = 0$. The phase velocity at the center of the torus ($z = 0, r = R$) then follows from

$$v^2 = \left(\frac{\kappa R}{n}\right)^2 + \left(\frac{\zeta R}{n}\right)^2$$

Inspection of Table I suggests that the low-frequency coupling impedance is dominated by the H_{11} mode. An approximate value for the mode number n_s at which $v = 1$ is given by[6]

$$n_s^H \approx 2.057\left(\frac{R}{w}\right)^{3/2}$$

In RHIC $R = 243.24$ m and $w = 7.29$ cm resulting in the approximate $n_s^H \approx 0.4\times10^6$ which differs substantially from the exact value $n_s^H = 1.564\times10^6$ obtained with the SLATEC routines. For comparison, the TM_{11} cutoff mode number is $n_{co}^{TM} = \sqrt{2}\pi R/w = 1.48\times10^4$.

The synchronous coupling impedance follows to be

$$\left(\frac{Z}{nQ}\right)_{syn} \approx \frac{8\pi R}{n_s^2 h}\frac{R\,S'^2(\kappa R)}{\int S^2(\kappa r)\,dr}$$

The functions $S(\kappa r)$ and $S'(\kappa r)$ are shown in Fig. 2 for the RHIC geometry from which

$$\frac{R\,S'^2}{\int_{R_i}^{R_0} S^2 dr} \approx 2.43$$

and the low-frequency coupling impedance

$$\frac{Z}{n} \approx j0.93\left(\frac{h}{R}\right)^2\left\{1+\left(\frac{n}{n_s}\right)^2\right\}$$

in good agreement with approximate results for the torus with circular cross section.

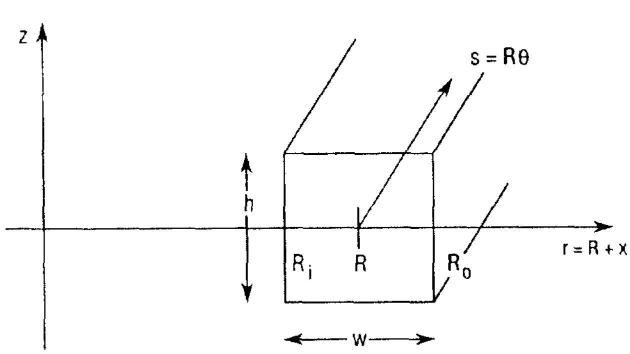

Fig. 1: Toroidal beam tube geometry.

Asymptotic Expansions

Although not required, it is convenient to replace the Bessel functions by their principal asymptotic forms for large n in terms of Airy functions leading to

$$S(\kappa r) \approx \mathrm{Ai}(\rho)\,\mathrm{Bi}'(\rho_i) - \mathrm{Ai}'(\rho_i)\,\mathrm{Bi}(\rho)$$

and

$$S'(\kappa r) \approx -\frac{2^{1/3}}{n^{1/3}}\left\{\mathrm{Ai}'(\rho)\,\mathrm{Bi}'(\rho_i) - \mathrm{Ai}'(\rho_i)\,\mathrm{Bi}'(\rho)\right\}$$

with $\rho = -\frac{2^{1/3}}{n^{1/3}}(\kappa r - n)$.

Furthermore, taking into account the relation

$$\kappa = \frac{n}{R}\sqrt{1-\gamma^{-2}-\epsilon^2}$$

where $\gamma^2 = \left(1-v^2\right)^{-1}$ one can approximate

$$\rho \approx -2^{1/3}n^{2/3}\left(\frac{x}{R} - \frac{1}{2\gamma^2} - \frac{1}{2}\epsilon^2\right)$$

with $x = r - R$ and ϵ as defined above.

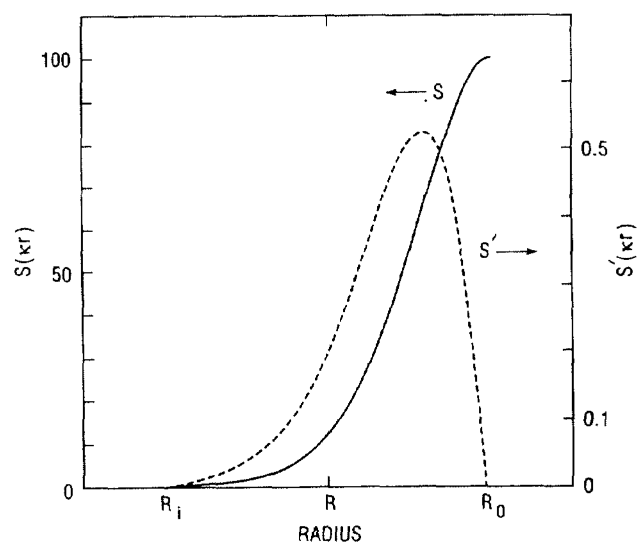

Fig. 2: Functions $S(\kappa r)$ and $S'(\kappa r)$.

Discussion

It is instructive to compare the results in this paper with previous publications. Zotter[4] gives an expression for the space charge term of a beam (radius a) in a toroidal beam tube (radius b) in terms of associated Legendre functions from which one obtains the low-frequency approximation

$$\frac{Z}{n} = -j\frac{1}{v\gamma^2}\left\{\ln\frac{b}{a} + \frac{1}{8}\frac{n^2}{\gamma^2}\frac{b^2}{R^2}\ln\frac{1}{2}\frac{R^2}{a^2}\right\}$$

As expected, the space charge term vanishes for extreme relativistic energies, $v = 1$; however in this approximation, no curvature term is obtained.

Ng and Warnock[10] have derived a curvature term which survives at $v = 1$. Their expression has an apparent similarity to the present results but differs qualitatively due to its capacitive character at low frequencies and due to the $(n/n_{co})^2$ frequency dependence which is stronger than the present $(n/n_s)^2$ dependence.

Either result allows the conclusion that the curvature effect will represent a negligible contribution at frequencies up to the vicinity of cutoff. As to future studies, it would be desirable to obtain a more exact expression for the phase velocity of the TM_{01}-mode in a torus with circular cross section.

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