Three Dimensional Computer Design of Multi-Cell Accelerating Cavities

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Abstract: Multi-cell linac structures such as side-coupled linacs or the recently proposed slotted linac cavities for TeV colliders often do not have cylindrical symmetry. Since the numerical analysis of very long waveguides becomes too expensive in terms of CPU time, it is an appropriate approximation to consider the structure as infinitely long. In this case, it suffices to analyse a single cell provided that one allows for periodic boundary conditions. Such capabilities have been added to the frequency solver in the new release of the MAFIA codes. Several examples show the usefulness of this new feature in MAFIA.

I. Introduction

In order to achieve high accelerating voltages in linear accelerators, it is necessary to use many cavities in a row. As of today, it is impossible to exactly analyze such systems consisting of (very) many identical cells if the unit cell does not have a simple geometry. But except for the cavities near the ends of the structure the fields are the same as if there were infinitely many subcells. It is therefore very useful to have a computer code that allows to analyze arbitrary periodic waveguide structures. Such a code has been added to the MAFIA programs [1]. The finite difference method used in these codes makes almost no assumptions about the problem geometry and allows the analysis of very general problems. In particular, the unit cells may be filled with arbitrary three-dimensional material distributions. It will be shown how, for example, the loss parameter of a quasiperiodic structure can be calculated.

II. Theory

By Floquet's theorem, the fields supported by periodic structures are periodic themselves up to an exponential factor. To be more specific, let us consider a structure periodic in z and having unit cells of length L. In the following a time dependence of $e^{j\omega t}$ is understood and suppressed. Then the electric field will have the form

$$\vec{E}(x,y,z) = \vec{E}_P(x,y,z)e^{+jkz} \tag{1}$$

where $\vec{E}_P$ is some periodic vector, i.e.,

$$\vec{E}_P(x,y,z+L) = \vec{E}_P(x,y,z) \tag{2}$$

and k is the wave number. The upper (lower) sign in equation (1) corresponds to a wave traveling in positive (negative) z-direction.

Because of eqn.(1), everything is known about the field in a periodic structure once the field in a unit cell is known. The problem is, therefore, to solve for the electromagnetic fields in a unit cell subject to the boundary condition

$$\vec{E}(x,y,z_{max}) = \vec{E}(x,y,z_{min})e^{\pm j\psi} \tag{3}$$

where by definition $\psi := kL$, and z_{min} and $z_{max} = z_{min} + L$ are the boundaries of the unit cell chosen.

It is obvious that one has to use a general numerical method if one aims at studying general periodic structures. Especially finite difference techniques have proven very useful in this respect. In the FIT (Finite Integration Technique) method used in the MAFIA program codes, the three-dimensional volume in question is covered by a grid [2, 3]. Thus, the volume is divided up into a set of elementary cells which are right parallelepipeds if a rectangular grid is used. As shown in Fig. 1, the electric field components are allocated at the mid-points of the sides of the rectangular cells and the magnetic field components at the center of each face. This defines a dual grid with the origin of each dual cell lying at the center of an original cell. The magnetic field components are allocated on the dual grid in the same way as the electric field components on the original grid. By this allocation, the transition from one cell to the next only involves continuous components. This enables one to fill each cell with an arbitrary material without having to worry about continuity conditions.

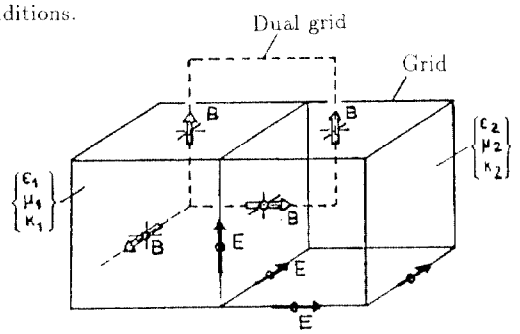

Figure 1: Geometry and allocation of the field components for the FIT method

On the grid, Maxwell's equations in their integral form are discretized by a simple linear approximation to the line and surface integrals. For RF resonator problems, in which the solutions of the homogeneous Maxwell equations are searched for, the algorithm leads to a matrix eigenvalue problem of the form

$$\tilde{A}\vec{e} = (\frac{\omega}{c})^2\vec{e}. \tag{4}$$

We consider only lossless materials, and then $\tilde{A}$ is a real symmetric matrix. ω denotes the angular frequency and c the velocity of light in vacuum. The vector $\vec{e}$ contains the square roots of the electric energy densities associated with each of the electric field components allocated on the grid. The reason for choosing the energy density as unknown instead of the electric field itself is that, in the first case, the matrix $\tilde{A}$ is symmetric whereas this would not be true in the latter case.

In order to treat a periodic structure, the volume of a unit cell is discretized and the eigenvalue equation (4) is set up as usual. In this process, it has to be taken care that integrals over field components allocated at the periodic boundaries of the grid are calculated properly. Any part of an integral that would have to be performed outside the grid is substituted by an integral calculated at the other end of the grid. Once the matrix eigenvalue equation has been set up, the periodicity of the problem is used to eliminate all field components allocated at $z = z_{max}$ by using the boundary condition (3). Thus, if the grid consists of N_z planes perpendicular to the direction of periodicity, the fields of only $N_z - 1$ planes are solved for. This procedure yields a new

matrix equation of the form

$$\tilde{B}\vec{e} = (\frac{\omega}{c})^2\vec{e} \tag{5}$$

where $\tilde{B}$ is a complex Hermitian matrix. The eigenvalues $(\omega/c)^2$ will still be real, but the eigenvectors $\vec{e}$ now are complex in general.

Instead of solving eqn.(5) directly, it was chosen to transform it into the equivalent real problem

$$\begin{pmatrix} \Re\{\tilde{B}\} & -\Im\{\tilde{B}\} \\ \Im\{\tilde{B}\} & \Re\{\tilde{B}\} \end{pmatrix} \begin{pmatrix} \Re\{\vec{e}\} \\ \Im\{\vec{e}\} \end{pmatrix} = (\frac{\omega}{c})^2 \begin{pmatrix} \Re\{\vec{e}\} \\ \Im\{\vec{e}\} \end{pmatrix} \tag{6}$$

Note that the matrix on the left hand side of this equation is real and symmetric, thus giving real eigenvalues and real eigenvectors. The eigenvalues are degenerate, however, because the complex conjugate of eqn.(5) leads to the same real problem as the original equation. In view of the Floquet condition (1), this means that the solution of the real problem will yield both the forward and the backward traveling waves at the same time.

From a logical point of view, it would be advantageous if one could choose a frequency (independent variable) and calculate the wave numbers of the waves at the frequency chosen. But k is hidden in the system matrix, and the above approach would mean that a nonlinear eigenvalue problem has to be solved. For the preparation of dispersion curves it is therefore necessary to first choose some k and then calculate the corresponding frequencies as the eigenvalues of a linear eigenvalue problem.

In the design of linacs one is particularly interested in the loss parameter κ which describes how well a given structure couples energy from an RF resonator mode to a passing particle bunch. For a structure consisting of N identical cells of length L each, the total complex accelerating voltage seen by a particle moving at speed $v = \beta c \approx c$ along the trajectory $r = r_p, \varphi = \varphi_p, z$ is given by

$$\underline{V}_N = \int_0^{N\cdot L} \underline{E}_z(r_p, \varphi_p, z) e^{j(\frac{\omega z}{\beta c} + \Psi)} dz \tag{7}$$

where Ψ is the phase of the RF oscillation at the time when the particle enters the structure. The total field energy stored in the cavities is

$$\mathcal{E}_N = \int_0^{N\cdot L} \iint_{A(z)} |\frac{\varepsilon}{4}\underline{\vec{E}}(r, \varphi, z)|^2 dA\, dz \tag{8}$$

with $A(z)$ being the z-dependent (periodic) cross section of the structure. The loss parameter κ_N is now defined by

$$\kappa_N = \frac{\underline{V}_N \underline{V}_N^*}{4\mathcal{E}_N} \quad . \tag{9}$$

The quasiperiodic structure supports waves that have a phase advance per unit cell of

$$\psi_{N,n} = n \cdot \frac{\pi}{N}, \; n = 1,2,\ldots,2N \quad . \tag{10}$$

By treating such a structure as infinitely long but using only phase advances obeying eqn.(10), a straightforward calculation shows that the loss parameter is equal to

$$\kappa_N = \kappa_1 \cdot \frac{1}{N} \cdot (\frac{\sin\frac{N\alpha}{2}}{\sin\frac{\alpha}{2}})^2 \tag{11}$$

where

$$\alpha = \frac{\omega L}{\beta c} - \psi_{N,n} \tag{12}$$

and κ_1 is the loss parameter associated with a single unit cell of the *infinite* periodic structure. It can be calculated from the knowledge of the field inside a unit cell by setting $N = 1$ in (7), (8) and (9).

If a particle moves with the same velocity as the phase of the electromagnetic wave traveling down the periodic structure, the angle α vanishes, and eqn.(11) simplifies to

$$\kappa_N = N\kappa_1 \text{ for synchronism between wave and particle .} \tag{13}$$

One can also see from (11) that κ_N becomes arbitralily small if N gets very large and α is not zero at the same time. It is therefore important to design a multi-cell linac structure in such a way that the phase velocity of the accelerating mode matches the particle speed very well. For ultrarelativistic particles like electrons or positrons this means that the accelerating mode must have the phase velocity c.

III. Examples

(1) *S-band linac structure:*
The first example deals with a SLAC linac structure consisting of 84 cells. Although the cells are not identical, the taper is so small that one can take the center cell and consider it as part of an infinite periodic structure. In a first approximation, this will give the dispersion curves of the real structure. Fig. 2 shows the discretized version of this center cell where only 729 mesh points were used. The dispersion curves for the lowest monopole and dipole modes are shown in Fig. 3.

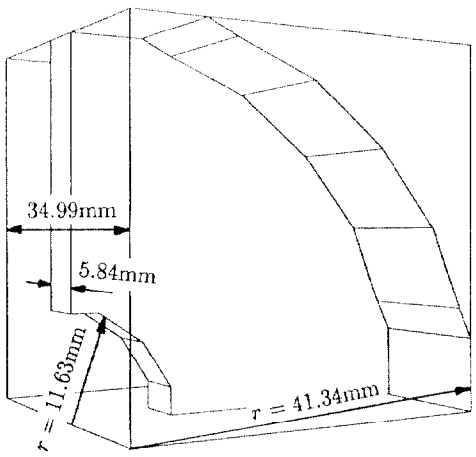

Figure 2: Discretization of one quadrant of an S-band cavity.

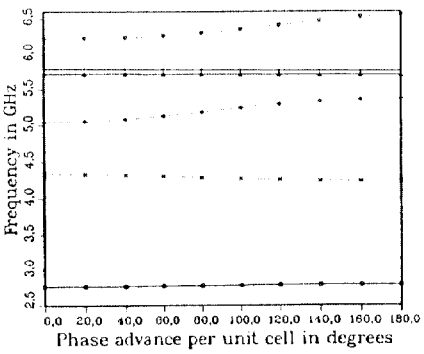

Figure 3: Dispersion curves for the lowest monopole (solid lines) and dipole modes (dotted lines) of the S-band cavity.

These results have been compared to calculations with many more points and are accurate to about 1%. On a SUN SPARC workstation, the CPU time needed to compute the first few modes was less than 3 minutes. This shows that it is possible to produce dispersion curves quite fast if there are not too many mesh points needed to model the geometry in question.

(2) *Backward wave structure:*
Now consider a case where there are coupling slits in the diaphragms separating the different cavities (Fig. 4). These slits lead to an inductive coupling of the fields in neighboring cells which has the interesting effect that the structure now supports backward waves (i.e., waves whose group and phase velocities point in opposite directions) of low order. The dispersion diagram in Fig. 5 shows that the lowest modes and in particular the accelerating mode are backward waves.

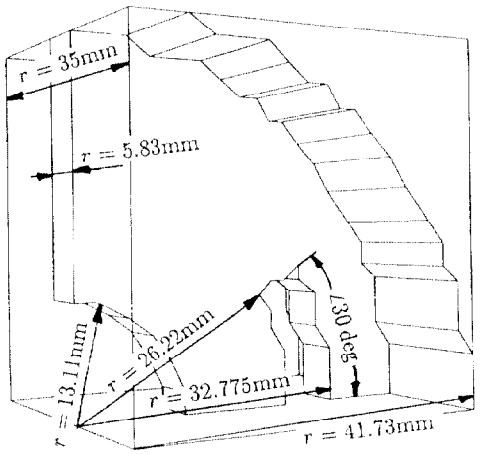

Figure 4: Discretization of one quadrant of a unit cell of a backward wave structure.

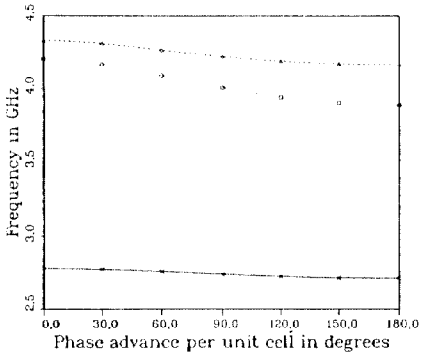

Figure 5: Dispersion diagram for the backward wave structure. The fundamental mode (solid line) and the first dipole modes are shown.

In the above calculations, the mesh grid consisted of 4446 points. Again on a SUN workstation, the total time needed for matrix setup and solution (including all I/O) was typically 40 minutes for the lowest 7 modes of a given type and a given phase advance per cell. This means that the preparation of a diagram like Fig. 7 involving three mode types and 7 different phase advances takes about 14 hours. Although one can hardly speak of a CAD tool under these circumstances, it is still possible to optimize single parameters as the group or phase velocity of a given mode in a reasonable time.

(3) *Wakefield transformer:*
The last example deals with a wakefield transformer which has been proposed as a means of achieving very high voltage gradients [4]. Fig. 6 shows a section of the transformer. One should

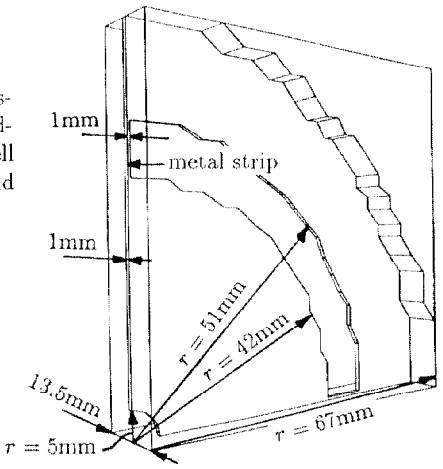

Figure 6: Discretized quadrant of a unit cell of the wakefield transformer.

think that the small strips joining the inner disk to the outer one play almost no role. But the calculations show that this is far from true. For instance, without strips the fundamental mode is a forward wave with the passband limits 1.706 GHz ($\psi = 0^0$) and 3.861 GHz ($\psi = 180^0$). But with the small strips, the passband limits are 1.706 GHz ($\psi = 0^0$) and 0.856 GHz ($\psi = 180^0$), showing that now this mode has turned into a backward wave.

The transformer works with the second quasi monopole mode at about 4 GHz (which is almost not affected by the strips and stays a forward wave in their presence). The loss parameter calculations for particles that move with $\beta = 1$ give values of $\kappa \approx 0.930 V/pC$ for 3 cells and $\kappa \approx 3.456 V/pC$ for 12 cells. The ratio of the two values is 3.72 which says that the mode chosen is almost but not quite synchronous with the particles (ideally it should be $12/3 = 4$).

IV. Conclusion

A new code has been presented which allows the numerical analysis of arbitrary periodic structures. With the help of three illustrative examples it was demonstrated that the program can be a valuable tool for calculating dispersion curves and loss parameters even if one deals not with infinite periodic structures but with systems consisting of a number of identical subcells.

References

[1] see, for instance, S.G.Wipf et al., "Status and Future of the 3D MAFIA Group of Codes", DESY M-88-15, December 1988.

[2] K.S.Yee, "Numerical Solution of Initial Boundary Value Problems Involving Maxwell's Equations in Isotropic Media", IEEE Trans. AP-14, No. 3, May 1966, pp. 302-307.

[3] T.Weiland, "On the Unique Numerical Solution of Maxwellian Eigenvalue Problems in Three Dimensions", Part. Acc. 17 (1985), pp. 227-242.

[4] G.-A. Voss, T. Weiland, "Particle Acceleration by Wake Fields", DESY M-82-10, April 1982.