# DYNAMIC APERTURE, A REVIEW OF THEORY AND EXPERIMENT

F. Willeke
DESY, Notkestr. 85, D-2000 Hamburg 52

## Introduction

Charged particles in circular accelerators experience nonlinear forces which limit the amplitudes of betatron oscillations. Beyond the limit, the amplitudes grow until the particles are lost at the beam pipe.

The issue of nonlinear forces has attracted increased attention with the construction of large synchrotrons and storage rings. Sources of nonlinearities include systematic sextupole fields to compensate the chromaticity, octupolar fields might be necessary to damp collective instabilities, and unavoidable imperfections of the guide and focussing fields introduce additional nonlinearities.

In hadron accelerators with superconducting magnets which exhibit large systematic and nonsystematic nonlinear field errors, it becomes one of the most crucial beam dynamics issues. This is especially true for low beam energies. Then the field of the superconducting magnets is additionally distorted by eddy current effects in the superconducting cable (persistent currents) and the beam size is large. Moreover, injection and synchronization procedures require additional aperture. This increases the effective beam size.

The distortion of the motion by nonlinear fields interferes with interaction among the particles, collective effects, interaction with residual gas in the beam pipe, the effect of power supply ripples or radiation effects. The real accelerator is thus quite a complicated object of study. Instead of studying this complex system, it is more useful to start from a reduced model of the machine. One considers single charged particles which move under the influence of conservative linear and nonlinear forces. These forces are perfectly periodic with respect to the cicumference of the accelerator. In the context of this Hamiltonian model one introduces the dynamic aperture as the domain where transverse and longitudinal oscillations around a closed reference trajectory near the center of the beam pipe are stable. This motion is bound to closed surfaces (tori) in phase space. It is called regular motion. Inherent to the nonlinear dynamics is the existence of quasi-stable or chaotic motion for which the tori are destroyed due to the nonlinear forces. Large chaotic domains are likely to occur at large amplitudes. A particle injected in chaotic large amplitude regions of phase space will be subject to amplitude growth. Eventually, sometimes after quite a while (up to $10^8$ turns around the accelerator), the particle will reach the domain where strong nonlinear forces cause a rapid loss of the particle. The border between regular and chaotic motion is therefore defined to be the dynamic aperture.

Our ultimate goal is to optimize the design of the accelerator and its components such that nonlinearities do not limit the performance. Tools are needed to determine the dynamic aperture. Most important for determining the dynamic aperture are numerical simulation procedures (tracking). The equation of motion of the particles is integrated piecewice element by element and the corresponding mapping once around the machine is iterated many times. Once the dynamic aperture as a function of various machine parameters and operating conditions has been found, one needs to understand the mechanism which produces the stability limitations. This enables the designer to optimize machine lattice and corrections schemes.

Since it is in general not possible to include in the simulations all relevant effects which are present in the real machine, the results need further interpretation. The understanding of the interference between various nonlinear effects is greatly improved by analysis and simulation on simple nonlinear systems. However, quantitative predictions for the real machines remain very cumbersome and unreliable. In order to reduce the uncertainty, experimental knowledge of how the results of model calculations translate into the beam behaviour in a real accelerator needs to be gained. This has motivated tests of dynamic aperture calculations by machine experiments.

This article will proceed as follows. In the first section, the important phenomena of nonlinear dynamics as described by various perturbation techniques will be reviewed. The section which follows describes simulation algorithms and interpretation of the results. The report is closed by a review of nonlinear dynamics machine experiments which have been carried out in recent years.

## Description of Nonlinear Phenomena in Accelerators

There is as yet no theory available for an analytic calculation of the dynamic aperture. There are however analytical procedures which enable machine designers to interpret the results of observations and simulations.

The concept of nonlinear resonances has proved to be a powerful tool to interpret and to control nonlinear effects in accelerators. A nonlinear resonance occurs if the nonlinear force has a component which oscillates with the betatron frequency. This is the case if the tunes $Q_x, Q_y, Q_s$ are rational numbers. The motion in the vicinity of nonlinear resonances is potentially unstable. The design and the performance of accelerators is usually optimized by avoiding or canceling strong resonant components of the nonlinear forces (driving terms).

Another important concept is nonlinear detuning. In any nonlinear system, the tune depends on the oscillation amplitude. This phenomenon of detuning is closely related to nonlinear instability. On one hand, detuning tends to stabilize nonlinear resonances. On the other hand, strong detuning induces instability by causing close stabilized resonances to overlap in amplitude space. This was first discovered by Chirikov [1] as a mechanism and criterion of chaotic motion.

The description of these phenomena accompagnies with the attempt to comprehend the nonlinear motion by means of perturbation theory. There is a large variety of techniques which has been introduced into our field. Many of them share a common concept; one attempts to express the motion in a transformed coordinate system where the solutions are harmonic oscillations (rotations in phase space) with amplitude dependent frequencies. In the following, this concept is demonstrated with classical perturbation theory [2]. Explicit formulae will be given for one degree of freedom only.

Classical Perturbation Theory. Consider how the constants $J, \Phi$ of the linear betatron motion $x(s) = \sqrt{2J\beta(s)}\cos(\Psi(s) + \Phi)$ ($\beta(s)$ is the envelope function, $\Psi(s)$ is the betatron phase advance, both defined by the linear focussing system; $s$ is the pathlength around the closed orbit, the "time") are affected by nonlinear forces derived from a potential $H$, so that $dJ/ds = -\partial H/\partial\Phi$, $d\Phi/ds = -\partial H/\partial J$. $H$ (the Hamiltonian) may be given in terms of polynominals of the coordinates (multipole expansion)

$$H = \sum_n a_n x^n = \sum_{nm} \left( \begin{array}{c} n \\ \frac{n-m}{2} \end{array} \right) \left(\frac{\beta}{2}\right)^{n/2} a_n(s) J^{n/2} e^{im(\Psi(s)+\Phi)}$$

($m \in \{-n, -n-2...n\}$). Exploiting the fact that $a_n(s)$ and $\beta(s)$ and $\Psi(s) - Q \cdot 2\pi s/C$ ($Q$ is the linear tune; $C$ is the machine circumference) are periodic in $s$, $H$ is expanded in a Fourier series (index $q$) in $2\pi s/C$

$$H = \sum_{nm} h_{nmq} J^{n/2} e^{i(m\Phi+(mQ-q)2\pi s/C)}.$$

The canonical transformation one is aiming for is $H, J, \Phi \rightarrow K, I, \varphi$, where $K$ depends only on the new amplitude (action) variable $I$. In this case the action varies according to $I' = \partial K/\partial\varphi = 0$; thus $I$ is a constant and the phase variable varies according to $\varphi' = \partial K/\partial I = f(I)$ which results in the amplitude dependent tune $\tilde{Q} = Q + \oint \frac{ds}{C}\varphi' = Q + f(I)$. Thus the solutions are harmonic oscillations with an amplitude dependent tune. Such transformations may be generated by a function $S(I, \Phi, s)$ mixed in old and new variables from which the transformation equations are derived as $J = \partial S/\partial\Phi$ and $\varphi = \partial S/\partial I$ according to the theory of canonical transformations. $S$ relates the original and the transformed potential or Hamiltonian by the Hamilton-Jacobi equation

$$\partial S/\partial s = K - H$$

We choose for $S$ an expansion

$$S = I\Phi + \sum_{nmq} \sigma_{nmq} I^{n/2} e^{i(m\Phi+(mQ-q)2\pi s/C)}.$$

An approximate solution can then be constructed by setting the coefficients $\sigma_{nmq}$ of the generating function equal to

$$\sigma_{nmq} = -h_{nmq}/i(m\tilde{Q}(I) - q)$$

which, when inserted in the Hamilton-Jacobi equation, cancel the terms $h_{nmq}$ so that $k_{nmq} = 0$ in first order according to the general concept. Terms in the old Hamiltonian which depend only on the action $J$ ($m = 0, q = 0$) cannot be included in the transformation. They are the lowest order terms to be absorbed in the new Hamiltonian. The tune $\tilde{Q}(I)$ in the denominator of $\sigma_{nmq}$ contains amplitude corrections (detuning) which result from expanding the terms $\sum_n h_{n00} J^{n/2}$ in terms of the new action variable $I$. Consider trajectories with various amplitudes. Whenever the amplitude dependent tune $\tilde{Q}(I)$ is close to a resonance $m\tilde{Q} - q$ of order ($m \leq n$), the coefficients $\sigma_{nmq}$ would become arbitrary large. This is prevented by absorbing the corresponding terms, the resonance driving terms (secular terms), in the new Hamiltonian $K$ as well. Expansion of the old action value in terms of the new action value introduces additional higher order terms in $K$. These may be removed by a second transformation which creates terms of third and higher order. A sequence of canonical transformations is formed this way. At every step, new high order detuning and secular terms are created which may drive nonlinear resonances up to any high order (even for quadratic nonlinearities). The procedure is known to converge poorly and it is in general not possible to prove that convergence occurs at all. One of the reasons for the difficulty in the mathematical prove of the existence of closed tori is that the amplitude dependent tune $\tilde{Q}$ is modified throughout the procedure as new detuning terms appear at each step. This requires readjustment of the initial amplitudes in order to fix the tunes in order to prevent secularities developing in terms previously considered nonsecular.

If strong low order secular terms driving a single resonance $m\tilde{Q} - q = 0$ occur early on in this procedure, they quite likely dominate the dynamics and may cause a drastic reduction of the stability limit. In this case one proceeds as follows. The explicit "time" dependence of the truncated Hamiltonian can the absorbed into a new phase variable $\phi = \varphi + (Q - q/m) \cdot 2\pi s/C$ and the corresponding Hamiltonian $G$ becomes independent of time at the expense of an additional term $\Delta \cdot I$, $\Delta$ being the distance of the linear tune from the resonance $\Delta = (Q - q/m)$. Hence, the motion is integrable in this approximation.

$$G = \Delta I + \sum_n k_{n00} I^{n/2} + \sum_n |k_{nmq}| I^{n/2} \cos(m\phi + \Phi_{nmq})$$

If the detuning is small compared to the driving force, unstable motion occurs for amplitudes beyond the separatrix, the contour $G(I, \phi) = G(I_0, \phi_0)$ contains the fixed points $I_0, \phi_0$ defined by $\partial G/\partial I = \partial G/\partial\phi = 0$. Inside the separatrix the motion is bound to closed tori.

If the detuning is strong compared to the driving force, the particles will always get out of phase with the driving force and the branches of the unstable orbits will close to a stabilized resonance island.

Since the tune depends on the amplitude, the occurrence of secular terms in the transformation depends on the actual amplitude. Consequently, a dense web of resonant island chains covers the nonlinear phase space. The variation of the amplitude along these islands, the island width, depends on the square root of the ratio of driving forces and detuning. The distance between islands is inversly proportional to the detuning. Thus with increasing detuning the island chains approach each other faster than they shrink and they eventually overlap. As mentioned earlier, overlapping island chains are considered as a mechanism for chaotic behaviour. The onset of massive chaos in accelerator phase space marks the dynamic aperture limit. The detuning is thus a central parameter connected to single particle stability.

The build up of strong low order resonant harmonics and detuning terms which considerably reduce the dynamic aperture, can easily be avoided by appropriate lattice design and correction schemes. This is good practice in designing and operating accelerators. For large hadron colliders the relevant question is what is the stability limit for a well designed and optimized machine. From this consideration arises the desire to carry on the perturbation series to high order to extract and analyze high order resonance driving terms and detuning.

Perturbation Theory Based on Lie Transformations. The classical procedure, turns out to be quite ineffective for calculating higher order detuning and resonance driving forces. The main reason is that the classical procedure is nonrecursive and therefore involves carrying out multiple integrals of increasing complexity over the nonlinear fields around the accelerator lattice. The method using Lie transformations as developed by Horiand Deprit[3] (for an application to accelerators see [4]) has proved to be more effective. The basic concept however is the same. The system is to be transformed such that the Hamiltonian contains only secular and detuning terms.

The objective is to transform the system $J, \Phi$ with the Hamiltonian $H(\Phi, J, s) = H_0 + \sum_n \frac{\varepsilon^n}{n!} H_n$ ($\varepsilon$ is a formal parameter which characterizes the strength of the nonlinear force) into a system $I, \varphi$ with a Hamiltonian $K(\varphi, I, s) = K_0 + \sum_n \frac{\varepsilon^n}{n!} K_n$ which depends only on the action variable, or more generally contains only secular terms. This transformation is to be constructed by

Poisson bracket operations or Lie generators $\hat{L} = \frac{\partial L}{\partial \Phi}\frac{\partial}{\partial J} - \frac{\partial L}{\partial J}\frac{\partial}{\partial \Phi} = [\,, L]$, which act upon the phase space variables. The operators $\hat{L}$ may be expanded in the form $\hat{L} = \sum_n \frac{\varepsilon^n}{n!}\hat{L}_n$. They generate an infinitesimal canonical transformation of a phase space variable. The corresponding finite transformation $(J, \Phi) \rightarrow^{\hat{M}} \rightarrow (I, \varphi)$ is to be expanded in powers of $\varepsilon$ as well so that $\hat{M} = \sum_n \frac{\varepsilon^n}{n!}\hat{M}_n$, where $\hat{M} = \sum_{m=0}^{n-1} \hat{M}_m \hat{L}_{n-m}$. The following expression (to be compared with the Hamilton-Jacobi equation) relates $K$ to $H$.

$$K = \hat{M}^{-1}H + \hat{M}^{-1}\int_0^{\varepsilon} d\varepsilon' \hat{M}\partial L(\varepsilon')/\partial s.$$

Inserting the power expansion for $H, K, M, L$ into this equation, one obtains a recursive set of differential equations

$$(\partial/\partial s + [\,, H_0])L_n = n(K_n - H_n) - \sum_{m=1}^{n-1}(\hat{L}_{n-m}K_m + m\hat{M}_{n-m}^{-1}H_m)$$

In lowest order one obtains ($H_0 = Q \cdot J$)

$$\begin{aligned}(\partial/\partial s - Q\cdot\partial/\partial\Phi)L_1 &= (K_1 - H_1)\\ (\partial/\partial s - Q\cdot\partial/\partial\Phi)L_2 &= (K_2 - H_2) \quad -\hat{L}_1(K_1 + H_1)\end{aligned}$$

(Q is the linear tune). This allows us to calculate recursively the Lie generating functions $L_n$ order by order. Quite in analogy to the classical procedure, the functions $(\partial/\partial s + Q\cdot\partial/\partial\Phi)L_n$ may be chosen to cancel all terms on the rhs of the equations excluding $K_n$. Then $K_n$ vanishes except for secular terms which cannot be included in the transformation. The equation of motion are integrable in the new system as long as only one secular term has to be taken into account.

This recursive algorithm is very well suited for analytic calculation by computer to any high order. Moreover, the transformation $\hat{M}^{-1}$ is given explicitly in the new variables $I, \varphi$ so that one can solve immediately for $J$ and $\Phi$ in order to reconstruct the tori in the original system. This procedure is used in celestial mechanics. There are tools available for accelerator applications (see ref [4]).

Normal Forms. This perturbation theory can be applied to a description of nonlinear dynamics in terms of maps as well as to a Hamiltonian description. In simulation procedures, nonlinear maps which describe the propagation of particles once around the accelerator are constructed on each turn. It seems to be natural to investigate the maps to analyse the tracking data and to extract from the map the relevant parameters like tune shifts which govern the dynamics. Since the complete map is in general very complex and contains very high powers of the coordinates, one usually deals with truncated maps.

A powerful way of analysing maps is the normal form procedure [5,6] which is again based on the principle that the mapping is to be expressed in a new coordinate system where it is reduced to a rotation with an amplitude dependent tune. A very elegant technique uses Lie operators[7]. Consider a mapping of a phase space vector around the accelerator $\vec{x}(s + C) = \hat{M}\vec{x}(s)$. If this form is inserted into the equation of motion $d\vec{x}/ds = S\vec{\nabla}H = [\vec{x}, H] = \hat{H}\vec{x}$ ($S$ is the symplectic form, $H$ is the Hamiltonian) we obtain a differential equation for the map $d\hat{M}/ds = -\hat{M}\hat{H}$ with the formal solution $\hat{M} = \exp(T\int_s^{s+C}\hat{H}) = \exp(\hat{L})$ ($T$ stands for time ordered) which is called a Lie operator. Any map of this form can be factorized into a rotation and a nonlinear part [8] $\hat{M} = \hat{R}\cdot\exp(\hat{F})$. A coordinate transformation generated $\vec{y} = \exp(\hat{K})\vec{x}$ by a Lie operator $\hat{K}$ is to be found so that the original map is reduced to an amplitude dependent rotation $\Omega(J)$

$$\Omega = e^{-\hat{K}}\hat{R}e^{\hat{F}}e^{\hat{K}}$$

Following the basic rules for operating with Lie operators (see for example ref[8]) one obtains

$$\Omega = \hat{R}\cdot e^{[-(1-\hat{R}^{-1})K + F]} + higher\ order\ terms$$

The goal is accomplished to lowest order if $K$ is chosen such that $-(1 - \hat{R}^{-1})K$ cancels the nonsecular terms in $F$. Having expanded $K$ and $F$ in eigenfunctions $h = \sqrt{2J}\exp(i\Phi)$ of the rotation operator $\hat{R}$, with $\hat{R}\cdot h = \exp(i\pi Q)\cdot h$, ($Q$ is the linear tune) then $K = \sum_{nm} k_{nm}h^n h^{*m}$, $F = \sum_{nm} f_{nm}h^n h^{*m}$, and we see that we have to require

$$k_{nm} = f_{nm}/(1 - \exp(-i\pi(n - m)Q))$$

for nonsecular terms $(m - n)Q \neq integer$, and $k_{nm} = 0$ for secular terms. The remaining terms in $F$ are tuneshift terms and resonance driving terms.

This concept has been developed into a recursive algorithm which allows calculation of the elements of the generator $\hat{K}$ order by order. It has been applied to calculate tune shift and nonlinear distortions of tori ("smear") for the Superconducting Super Collider[7]. Another application for accelerators using a modified formalism is the layout of tuneshift correction schemes for the LEP Hadron Collider [10].

Perturbation Theory and Dynamic Aperture. So far we may summarize by saying that perturbation theory provides qualitative understanding of nonlinear dynamics and supplies us with tools to calculate the relevant parameters for the design of an accelerator. Nothing has been said so far about the dynamic aperture. Heuristic models have been proposed which give an estimate of the dynamic aperture which is associated with the breakdown of perturbation theory. For example, the dynamic aperture has been defined as the limit $I_0$ with $(\partial J/\partial I)_{I=I_0} = 0$. The transformation between the old action variable $J$ and the new action variable $I$ is accomplished by a second order canonical transformation which interpolates across resonance islands [11]. In another approach, the Hamilton-Jacobi equation is solved directly by an iterative procedure based on Newton's method. Nonconvergence of the procedure is interpreted as the dynamic aperture limit [12]. Other procedures are based on successive linearisations of the nonlinear equation of motion and subsequent investigation of linear stability [13]. There are examples for all these methods which seem to indicate that they provide a reasonable estimate for the dynamic aperture. However, there is as yet no method considered reliable enough to replace numerical tracking.

## Numerical Methods

The so called kick codes are the most used tracking codes. They are conceptionally simple describing the motion by successions of phase space rotations followed respectively by a nonlinear kick which depends on the particle position. Kick codes are symplectic so that they describe a solution of Hamilton's equation of motion. Due to the large number of turns needed to make a reliable prediction of stability, kick codes turn out to be very expensive tools for investigating the dynamic aperture of future large hadron colliders.

Maps which carry only the essential information about of the motion could be more effective. The mapping generated by a kick codes includes extremely high powers of the coordinates to be tracked. The calculation can only be speeded up by truncating the map at a given maximum order which is appropriate to the problem investigated. It is thus evident that describing the motion by a map implies serious approximations. On the other hand, the kick procedure is approximate too, since the action of the nonlinear forces is concentrated in a minimum number of thin lens kicks, an approximation which the generation of a truncated map does not rely on. On the contrary, there is much less restriction on the amount of information which can be taken into account in building the map. The effort to iterate the map remains the same.

A truncation procedure is provided by using Lie operators to describe the motion through single elements. There are well established rules to concatenate and factorize these propagators which allow the map to be represented as a product of operators $\Pi_k \exp(\hat{L}_k)$ [8] where the Lie generating function $L_k$ is composed of homogenious polynominals in the phase space variables of order $k$. The product can be truncated at any order. It always will provide a symplectic mapping. The procedure is implemented in the computer code MARYLIE [9].

Mapping by Lie operators $\exp(\hat{L}) = 1 + \hat{L} + \frac{1}{2}\hat{L}^2 + ....$ implies an infinite series of iterations which is evaluated numerically in MARYLIE by a Newton's method. Considerable progress has been made recently [14] by expressing the action of a Lie operator by a number of kicks. This is accomplished by writing the Lie generating functions which are usually represented as $L_k = \sum_{n+m=k} a_{nm} x^n p_x^m$ in the form

$$L_k = \sum_{ki} A_{ki}(x \cdot \cos\Phi_i + p_x \cdot \sin\Phi_i)^k$$

A Lie operator of the form $\exp([\ ,f(ax + bp_x)])$ produces just a generalized kick $\Delta x = -\partial f/\partial p_x$, $\Delta p_x = \partial f/\partial x$. The $\Phi_i$ are angles to be chosen appropriately to assure that one can solve for the coefficients $A_{ki}$. The so called kick factorization may considerably speed up the evaluation of Lie generated maps, at least for one or two degrees of freedom.

Another way to produce truncated maps is by using differential algebra [15]. The essence of this technique is that it propagates partial derivatives $\partial^n \vec{f}/\partial \vec{x}^n$ up to any order $n$ of a function $\vec{f}(\vec{x})$ through a series of mathematical operations on $f$ by applying differentiation rules such as $(g \cdot f)' = g'f + f'g$. Applied to tracking algorithms, $f$ is to be identified with a phase space vector $\vec{x}$ which is propagated around the machine. It has the initial form $\vec{f}(\vec{x}_0) = \vec{x}_0$ with $\partial\vec{f}/\partial\vec{x}_0$ equal to the unit matrix, and $\partial^n \vec{f}/\partial \vec{x}^n = 0$ for $n \geq 1$. After one turn one obtains $\partial^n \vec{x}/\partial \vec{x}_0^n$, the coefficients of a Taylor expansion representation of the one turn map. Unfortunately, this map is in general not symplectic and symplectification cannot be performed unambiguously. Nonetheless, this new tool provides a powerful truncating mechanism. The resulting map can be easily evaluated.

From the many methods to determine the dynamic aperture, we will select and discuss three examples.

The straightforward approach for determining the dynamic aperture is long term tracking. Particles are tracked for many turns. The initial amplitude is varied until the betatron oscillation are stable for a large number ($\simeq 10^6 - 10^8$) of turns. This procedure is very expensive (simulation of $10^6$ turns in HERA using the kick code RACETRACK [16] takes $\simeq 10^5 s$ for one initial condition on an IBM3090). It is therefore not well suited for studying the dynamic aperture as a function of many parameters like tunes, chromaticities, different statistical seeds of magnet errors, tests of different correction schemes,etc. More effective methods are very desirable.

The early detection of chaotic motion can speed up considerably the process of determing the dynamic aperture. The exponential divergence of two, initially very close trajectories is a criterion for chaos [17]. A measure for this divergence is the average growth rate of the separation in phase space, the so called Lyapunov exponent defined as $\sigma = \lim_{d_0 \to 0} \lim_{N \to \infty} \frac{1}{N} \ln \frac{d_N}{d_0}$. Here $d$ is the separation of two vectors in phase space $d_N = ||\Delta\vec{x}(N)||$ and $N$ measures the time in turns around the accelerator. This parameter can be related to the eigenvalues $\lambda_k^N$ of the $N$-turn Jacobian $J_N = \partial\vec{x}(N)/\partial\vec{x}(0)$, obtained by linear expansion of the fields about the trajectory under consideration. Choosing $\Delta\vec{x}_0$ along the eigenvectors $\vec{e}_k$ of $J_N$ results in

$$\sigma = Max\{\ lim_{N\to\infty}\ \sigma_N\} = Max\{\ lim_{N\to\infty}\ \frac{1}{N}\ln\sqrt{\lambda_k^N \lambda_k^{N*}}\}$$

Thus the Lyapunov exponent measures essentially the density of unstable fixed points associated with nonlinear resonances along the particle trajectory. For linear motion, $\sigma$ is zero. For regular motion where one expects linear divergence of $d/d_0$, $\sigma_N$ evolves as $\ln(a \cdot N)/N$, approaching zero quickly. For chaotic trajectories, $\sigma$ is different from zero. Such methods have been used to determine the dynamic aperture in the HERA proton ring [18]. Stability of the motion can be predicted for $\simeq 10^6 - 10^7$ turns after $10^4 - 10^5$ simulated turns. This saves a factor of $10 - 100$ in computing time.

In the same spirit, stability is predicted for large times by analysing the motion for shorter times by a method following Nekhoroshev's theorem. Nekhoroshev's theorem tells that perturbation theory is able to predict stability for a finite time which increases exponentially with the size of the phase space domain considered. On the basis of this theorem, the following principle has been established [20]. Consider the deviation of the action $\delta J(t)$ from an approximate smooth torus observed over a time $T$. If the maximum variation $\Delta J$ within a domain $J_1 \leq \Omega_1$ is $\delta J$ over a period $T$, an orbit which is confined for the same period $T$ in a smaller domain $J \leq \Omega_2 \leq \Omega_1$, will remain within $\Omega_1$ for a time of at least $Min(J_1 - J_2)/\delta J \cdot T$. The approximate torus necessary for this analysis is found by fitting numerical phase space data from tracking to a generating function. Normal form analysis of a one turn map might be another way of providing approximate tori. The method has been applied to a model lattice with sextupolar nonlinearities. Stability over up to $10^8$ turns has been predicted by this method [20].

## Experiments

The physics of particle beams in accelerators comprises many effects which have not been taken into account in the Hamiltonian model. The meaning of the dynamic aperture for a real accelerator is therefore not obvious. Experimental tests of calculations are necessary.

Resonant behaviour has been studied in many accelerators. The performance has been improved considerably by the compensation of resonance driving terms. The compensation of third order resonances in the CERN-PS [21] is a typical example. In almost every accelerator operating in fixed target mode, slow resonant extraction has been controlled with high precision by exciting nonlinear resonances on purpose and by adjusting the tunes to be close to a nonlinear resonance (see for example [22]). More recently, nonlinear phase space trajectories of electron and proton beams in the presence of strong sextupole fields have been observed directly by analysing digitized signals from two beam position monitors. A small beam has been kicked transversly and the beam position has been recorded subsequently for about 1000 successive turns. After taking into account filamentation effects in the multiparticle beam, the measurements confirm well the predictions [23,24]. One can summarize that resonant behaviour and the stability limit near resonances in a real machine can be reliably predicted qualitatively and quantitatively by analytical and numerical calculations.

Further experiments have been performed to measure the short term dynamic aperture in the case of strong nonlinearities but for operating far from low order resonances. The dynamic aperture in presence of strong sextupole fields has been measured in the TEVATRON as a part of the E778 machine experiment [25]. The beam emittance was increased by many small transverse kicks while the beam profile was monitored using wire scanners. The beam width, after increasing for a while, reached some final value which was interpreted as the short term dynamic aperture. The results obtained for different values of the strength of the sextupoles was compared with a 500-turn dynamic aperture from tracking. The experimental values are always smaller than the predicted values. The discrepancy is 40% for low excitation of sextupoles and 20% for strong sextupoles. In experiments performed at the CERN-SPS [26] dynamic aperture has been measured under similar conditions. A small proton beam is kicked by a single kick. Beam loss is taken as an indication that the edge of the beam reached the dynamic aperture.The dynamic aperture in the corresponding simulation was defined as the border between regular and chaotic trajectories determined by detection of a nonvanishing Lyapunov exponent.Measured and simulated dynamic apertures ($\simeq 17mm$) agree within a few millimetres. These experiments seem to indicate that if the actual field errors in an accelerator are known, dynamic aperture calculations do indeed predict with resonable accuracy the amplitude limit for short term stability, for time scales of the order of $10^5$ turns or a few seconds in real time.

More experiments have been carried out to explore the validity of dynamic predictions for a larger time scale. In the CERN SPS experiment [26] diffusion induced by strong sextupolar nonlinearities has been investigated. The beam size is controlled by scraping the beam. When the scrapers are released, the beam life time improves for a while until the beam is blown up by diffusion processes. This diffusion is clearly related to the strength of the sextupoles. In the corresponding long term tracking calculation (up to $10^6$ turns) no signs of diffusion have been detected. To suppress the diffusion, amplitudes have to be reduced to at least half the short term dynamic aperture.

There are models which describe how diffusion is induced by an interference of tune modulation and nonlinearities. Tune modulation causes sidebands around stable resonance islands in phase space. For low modulation frequency (induced for example by power supply ripples) but large modulation depth, these sidebands overlap and this then satisfies the Chirikov criterion for chaos [27]. This mechanism can be investigated analytically in the (integrable) single resonance model. Small amplitude oscillation around the closed orbit associated with a stabilized resonance exposed to external tune modulation corresponds to the motion of a driven pendulum. The analysis of this model reveals a phase transition between regular and chaotic in the parameter space of modulation depth and frequency. The validity of this model has been explored in the Fermilab E778 experiment [28]. The beam is kicked as a whole, populating a resonance island generated by strong sextupoles and has a working point near the resonance $5Q_x = 97$. Whereas the coherent signal from the part of the beam which is outside the island decays quickly due to detuning induced filamentation, the beam which is trapped in the island provides a persistent signal since all amplitudes inside the island are on average the same. Slow decay of this persistent signal is due to diffusion processes by which particles leak out off the island. This has been stimulated in the E778 experiment by tune modulation. Enhanced decay rates have been observed for modulation parameters which correspond to the chaotic phase in the pendulum model.

## Conclusion

There has been considerable progress in understanding the impact of nonlinear phenomena in accelerators. New analytic and numerical methods have been developed and to be used to determine, to analyze and to improve the dynamic aperture in large hadron colliders.


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