LONGITUDINAL COUPLING IMPEDANCE AND TRANSMISSION COEFFICIENT FROM UNIFORM AND HOLLOW RING SOURCES[0]

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Abstract

The longitudinal coupling impedance and the transmission coefficient resulting from a thin ring and from a uniform disk are obtained analytically for a resistive cylindrical beam-pipe of finite wall thickness. The impedances are derived and then compared with the well known corresponding expression for perturbations on a uniform, coasting beam. The transmission coefficients from both sources are found to be exactly the same. Differences do appear in the expressions for the electromagnetic fields within the beam region, and therefore leading to different coupling impedances. By applying the results to parameters relevant for the SIS-18 synchrotron at GSI, it is found that the formula from the ring source underestimates the space-charge impedance at all beam energies and it shows a noticeable deviation from the disk formula for all frequencies. Although their mathematical expressions are different, resistive-wall impedances from the two sources are found to be numerically equal. The space-charge impedances become equal asymptotically only in the so called ultra-relativistic limit.

INTRODUCTION

Reliable impedance calculations are an important issue for the design and optimization of ring accelerators for high currents. Especially in the regime of low and medium beam energies and for low frequencies there still exist relevant differences in the impedances obtained from different approaches (see e.g. [1, 2]). This regime is important for the upgrade of the existing SIS-18 heavy ion synchrotron at GSI and for the design of the new synchrotrons as part of the FAIR project. An important issue is the representation of the source term in Maxwell's equations. At low energies it is important to account for the transverse beam profile. A homogeneously charged disk is used in many studies in order to model a longitudinal perturbation on the beam (e.g. in Refs. [2]). For the calculation of the longitudinal coupling impedance at ultra-relativistic energies the source term is usually reduced to a thin ring charge [3]. This source term is often employed in impedance calculations for arbitrary beam energies (see e.g. [4]).

In some limiting cases, like low or high frequencies, the differences between source terms can be expressed in the form of multiplicative g-factors, cf. [5]. A treatment for arbitrary frequencies, however, was missing. The present work will shed some light on the coupling impedance and transmission coefficient from two different source terms, namely, a thin ring and uniform disk propagating down a resistive cylindrical beam-pipe of finite wall thickness.

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MODEL EQUATIONS

The general wave equations for the external (free) charge and current densities ρ and $\vec{j}$, respectively, satisfied by the magnetic induction $\vec{B}$ and electric field $\vec{E}$ in a conducting medium of conductivity S, permittivity ϵ_0 and permeability μ_0 are obtained from Faradays and Amperes laws and by the continuity equation. We assume a source term of total charge Q and transverse charge distribution $\sigma(r,\theta)$ that is moving in a cylindrical pipe of radius b with constant longitudinal velocity $\vec{v} = \beta c\hat{z}$ along the z axis.

For a uniformly charged disk, the surface charge density distribution in the transverse direction is $\sigma_d = Q/\pi a^2$ [2, 6]. A thin ring of radius a, on the other hand, moving parallel to the z-axis is represented by $\sigma_r = Q\delta(a-r)/2\pi a$ [3]. Accordingly, the Fourier time-transformed charge densities for both distributions are

$$\rho_d(r,z,\omega) = \frac{Q}{\pi a^2 \beta c}\, e^{ik_z z} \tag{1}$$

$$\rho_r(r,z,\omega) = \frac{Q\delta(a-r)}{2\pi a\beta c}\, e^{ik_z z}\ , \tag{2}$$

where $\omega = k_z v$ has been used and k_z is the wave number in the direction of beam propagation. Due to the axial symmetry of the particle beams under consideration, only transverse-magnetic (TM) cylindrical waveguide modes couple to the propagating beams such that $B_z = 0$. All other field components are obtained from $E_z(r,z,\omega)$ via Maxwells equations, where $E_\theta(r,z,\omega)$ and $B_r(r,z,\omega)$ vanish identically because of axial symmetry and periodicity demands $E_z(r,z,\omega) = E_z(r,\omega)\ e^{ik_z z}$. Fourier transforming in time and in transverse space coordinates, and making use of the density and current of eqs. (1) and (2), respectively, we get differential equations for the field components in the region $0 \le r \le a$, in the beam-pipe region $a \le r \le b$, in the pipe $b \le r \le h$, and outside $h \le r < \infty$.

For the TM-modes with azimuthal symmetry, the electromagnetic field components $B_\theta(r,z,\omega)$ and $E_r(r,z,\omega)$ are needed for matching the solutions at the different interfaces involved in the problem under consideration and are obtained from $E_z(r,z,\omega)$ via the Maxwell equations as follows,

$$E_r(r,z,\omega) = -i\,\frac{\gamma^2}{k_z}\,\frac{\partial E_z(r,z,\omega)}{\partial r}\ , \tag{3}$$

$$B_\theta(r,z,\omega) = \left(\frac{\beta}{c} + i\frac{\mu_0\beta c S}{\omega}\right)\ E_r(r,z,\omega)\ ,$$

where γ stands for γ_0 in the non–conducting regions, and for $\underline{\gamma}$ from $\underline{\gamma}^{-2} = \gamma_0^{-2}\left(1 - 2i\gamma_0^2/k_z^2\delta_s^2\right)$, where

$\delta_s = \sqrt{2/\mu_0 S\omega}$ is the skin depth at frequency ω and $\gamma_0^{-2} = 1 - \beta^2$.

In the following we will solve the wave equations for E_z for both types of beams and then find the corresponding expressions for the coupling impedance for a resistive beam–pipe of arbitrary wall thickness.

THIN RING SOURCE

For a point charge Q moving down a beam-pipe with an offset a in the $\theta_0 = 0$ direction with a constant longitudinal velocity $\vec{v} = \beta c\hat{z}$, and in decomposing the corresponding charge and current densities in terms of multipole moments, the lowest monopole moment gives,

$$\rho_r(\vec{r},t) = \frac{Q}{2\pi a}\,\delta(a-r)\,\delta(z-\beta ct)\ , \qquad (4)$$
$$\vec{j}_r(\vec{r},t) = \frac{Q}{2\pi a}\,\delta(a-r)\,\delta(z-\beta ct)\ \beta c\,\hat{z}\ .$$

This monopole source has an axially symmetric transverse charge distribution and it represents an infinitesimally thin ring with radius a. Time Fourier–transformed charge and current densities in equations (4) and (5) are already given in equation (2). For a thin ring beam moving inside a metallic cylindrical pipe of radius b and finite wall thickness d extending from $r = b$ to $r = b + d$ with vacuum outside $r > b + d$, the longitudinal electric field component $E_z^{(r)}$ within each region of interest is calculated.

The overall regular general solution for the electric field $E_z^{(r)}$ in each region is written in terms of ther Bessel functions I_0 and K_0 of arguments $\sigma_0 = k_z/\gamma_0$ and $\underline{\sigma} = k_z/\underline{\gamma}$. The system of equations is solved with the discontinuity at $r = a$,

$$\frac{\partial E_z^{r\geq a}}{\partial r} - \frac{\partial E_z^{r\leq a}}{\partial r} = i\frac{k_z}{\epsilon_0\gamma_0^2\beta c}\,\frac{Q}{2\pi a}\ . \qquad (5)$$

and the continuity of E_z at $r = a$, the continuity of E_z and B_θ at $r = b$, and $r = h$. where $\eta = \omega\epsilon_0\gamma_0/i\underline{\gamma}\,(S - i\omega\epsilon_0)$.

We now calculate the corresponding longitudinal coupling impedance as a volume integral over the transverse distribution of the beam as follows,

$$Z_{\parallel,r}(\omega) = -\frac{1}{Q^2}\int_{V_{beam}} d^3x'\ \vec{E}^{(r)}(r',z,\omega)\cdot\vec{j}_r^{\,*}(r',z,\omega)$$
$$= i\frac{nZ_0}{\gamma_0^2\beta}I_0^2(\sigma_0 a)\left[\frac{K_0(\sigma_0 a)}{I_0(\sigma_0 a)} + \frac{K_1(\sigma_0 b)+G\,K_0(\sigma_0 b)}{I_1(\sigma_0 b)-G\,I_0(\sigma_0 b)}\right], \qquad (6)$$

where G is a constant to be found in [2].

In the ultra-relativistic limit such that $\sigma_0 a \ll 1$ and $\sigma_0 b \ll 1$, we have $I_0(\sigma_0 a) \to 1$, $I_0(\sigma_0 a) \to 1$, $K_0(x) \to -\left(\ln\frac{x}{2} + \gamma_e\right) I_0(x)$, where $\gamma_e = 0.5772$ is the Euler constant. Accordingly, the4 space charge impedance takes on the following form,

$$Z_{\parallel,r}^{(\mathrm{sc})}(\omega,\gamma\to\infty) \ = \ i\,\frac{nZ_0}{2\gamma_0^2\beta}\left[2\ln\frac{b}{a}\right] \qquad (7)$$

and the resistive wall impedance is

$$Z_{\parallel,r}^{(\mathrm{rw})}(\omega) = Z_{\parallel,h}(\omega) - Z_{\parallel,h}^{(\mathrm{sc})}(\omega)$$
$$= i\frac{nZ_0}{\gamma_0^2\beta}I_0^2(\sigma_0 a)\left[\frac{K_0(\sigma_0 b)}{I_0(\sigma_0 b)} + \frac{K_1(\sigma_0 b)+G\,K_0(\sigma_0 b)}{I_1(\sigma_0 b)-G\,I_0(\sigma_0 b)}\right]. \qquad (8)$$

For a perfectly conducting wall, $\eta \to 0$ and $G \to 1$, the resistive-wall impedance of eq. (8) vanishes identically. In the thick wall limit such that $d \to \infty$ we get $G \to 1$ and, hence, eq. (8) becomes,

$$Z_{\parallel,r}^{(\mathrm{rw})}(\omega) = \qquad (9)$$
$$= i\,\frac{nZ_0}{\gamma_0^2\beta\sigma_0 b}\,\frac{I_0^2(\sigma_0 a)}{I_0^2(\sigma_0 b)}\,\frac{\eta}{1+\eta I_1(\sigma_0 b)/I_0(\sigma_0 b)}\ .$$

For a thin ring source in a pipe the transmission coefficient as the ratio of transmitted to incident fields at $r = b$ is found to be exactly the same expression already reported for a disk source [2].

UNIFORM DISK

The uniform disk source has been treated already in our previous paper [2] and here we review the results of the calculations of the coupling impedance and the transmission coefficient of a cylindrical pipe of finite wall thickness. The corresponding longitudinal coupling impedance is [2],

$$Z_{\parallel,d}(\omega) =$$
$$i\frac{nZ_0}{2\beta\gamma^2}\frac{4\gamma_0^2}{k_z^2a^2}\left[1-2I_1(\sigma_0 a)\left(K_1(\sigma_0 a)-\frac{1}{H}I_1(\sigma_0 a)\right)\right], \qquad (10)$$

and the resistive wall part alone amounts to

$$Z_{\parallel,d}^{(\mathrm{rw})}(\omega) = i\,\frac{nZ_0}{2\beta\gamma_0^2}\,\frac{8\gamma_0^2}{k_z^2a^2}\,I_1^2(\sigma_0 a)\left[\frac{K_0(\sigma_0 b)}{I_0(\sigma_0 b)}+\frac{1}{H}\right], \qquad (11)$$

For $\sigma_0 a \ll 1$ and $\sigma_0 b \ll 1$, the space-charge impedance of a uniform beam takes on the following form,

$$Z_{\parallel,d}^{(\mathrm{sc})}(\omega) =\approx i\,\frac{nZ_0}{2\beta\gamma_0^2}\ \left[2\ln\frac{b}{a}\right]\ . \qquad (12)$$

Note that, according to the results in eqs. (7) and (12), both source terms produce asymptotically the same space-charge impedance in the ultra-relativistic limit. For a thick wall the resistive wall part takes on the following form,

$$Z_{\parallel,d}^{(\mathrm{rw})}(\omega) =$$
$$i\,\frac{nZ_0}{\beta\gamma_0^2\sigma_0 b}\,\frac{4I_1^2(\sigma_0 a)}{\sigma_0^2a^2I_0^2(\sigma_0 b)}\,\frac{\eta}{1+\eta\,I_1(\sigma_0 b)/I_0(\sigma_0 b)}\ . \qquad (13)$$

If we compare eq. (13) with (10), we see that the resistive-wall impedances for a thick wall have similar structures. Differences appear only in the factor $I_0^2(\sigma_0 a)$ in eq. (10) which is replaced by the factor $4I_1^2(\sigma_0 a)/\sigma_0^2a^2$ in eq. (13). Again, in the limit $\sigma_0 a \ll 1$ and $\sigma_0 b \ll 1$, both resistive-wall expressions become identical. For a uniform disk the transmission coefficient $\tau_d(d)$ is found to be exactly equal to the thin ring transmission coefficient τ_r. According to Gauss' integral law, as expected, integral quantities like resistive-wall impedance and transmission factor outside the sources of charge depend only on the total charge and not on the details of the charge distribution within the beam.

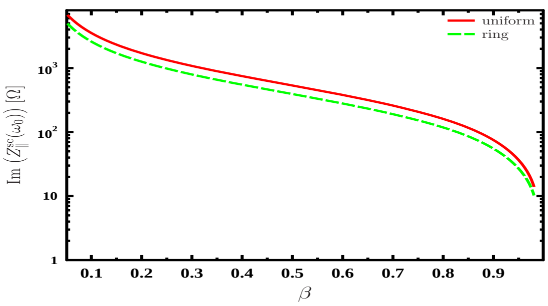

Figure 1: Space charge impedance as a function of beam velocity.

NUMERICAL EXAMPLES

To visualize possible differences between impedances from the different source terms, the theory above will be applied to the heavy ion synchroton SIS18 at GSI Darmstadt with a ring circumference of L=216 m. It has a stainless steel wall at radius b =10 cm of thickness $d \approx 0.3$ mm with a conductivity of $S = 1.1 \times 10^6/\Omega$m. In Fig. 1, we plot the space charge impedances vs. beam velocity β at revolution frequency $\omega = \omega_0$ and beam radius $a = 5$ cm. The figure show that the impedances from the thin ring source are always lower than those from the uniform disk. At the beam reference velocity $\beta = 0.155$ of SIS18, we see an impedance difference of about 600 Ω which corresponds to 25 % of the space-charge impedance at this energy.

In Fig. 2, we plot the space charge impedances per harmonic number $Z_{\parallel}(\omega)/n$ vs. mode frequency $f = \omega/2\pi$, where $\omega = n\omega_0$, $\omega_0 = \beta c/R$, and n is the harmonic number. Here we fixed the beam velocity at $\beta = 0.155$ with a beam size $a = 5$ cm. For all frequencies, the figure shows an obvious difference between the space charge impedances resulting from the two source terms. Approximately up to 50 MHz the difference is about 600 Ω and is nearly constant. It becomes smaller by increasing the frequency until it becomes zero at about 300 MHz. For the higher frequencies above 300 MHz, the thin ring impedance becomes larger than the space charge impedance from the uniform disk. Variation of the SIS18 space-charge impedance with beam size at revolution frequency is shown in Fig 3. We see that the space-charge impedance from the uniform disk is greater than the thin ring source impedance.

Interestingly, the variation of the impedance difference with beam size is constant for fixed beam energy. At the beam reference velocity $\beta = 0.155$, the impedance difference is again about 25 % of the disk impedance at the same beam energy.

CONCLUSIONS

The longitudinal coupling impedances and transmission coefficients from a thin ring source term and from a uniformly charged disk propagating down a resistive cylindrical beam-pipe of finite wall thickness are investigated and limiting cases of space-charge and resistive-wall impedances from the two source terms have also been obtained and discussed. Transmission coefficients from both source terms are exactly the same. Differences appear in the electromagnetic fields within the sources leading to different coupling impedances. For the SIS18 synchrotron of GSI Darmstadt it is found that the space-charge impedance from the thin ring source deviates appreciably from the disk formula for a wide range of beam energies and mode frequencies. Although their mathematical expressions are different, resistive-wall impedances from the two source terms are found to be numerically equal. The space-charge impedances become equal in the ultra-relativistic limit.

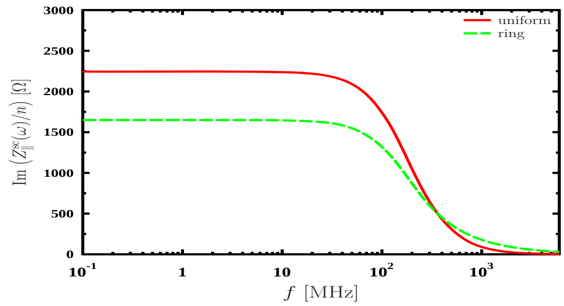

Figure 2: Space charge impedance as a function of frequency.

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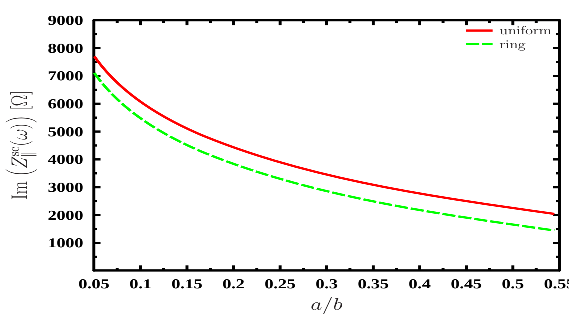

Figure 3: Space-charge impedance as a function of beam radius.

REFERENCES

[1] F. Zimmermann, K. Oide, Phys. Rev. ST Accel. Beams 7, 044201 (2004).

[2] A. M. Al-Khateeb *et al.*, Phys. Rev. E **63**, 026503 (2001), Phys. Rev. E **71**, 026501 (2005), NIM A **587**, 236 (2008), NIM in Phys. Res. A (2008), *in print*.

[3] R.L. Gluckstern, CERN Yellow Report, CERN 2000-011 (2000)

[4] N. Wang, Q. Qin, Phys. Rev. ST Accel. Beams 10, 111003 (2007)

[5] J. G. Wang, H.Suk, D.X. Wang and M. Reiser, Phys. Rev. Lett. **72**, 2029 (1994).

[6] B. W. Zotter and S. A. Kheifets, Impedances and Wakes in High -Energy Particle Accelerators, Ch. 6, World Scientific (1998).