HIGH ORDER ABERRATION CORRECTION

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Abstract

It is known that modern accelerators fall under nonlinear aberrations influence. The most of these aberrations have harmful character, and their effect must be maximally decreased. There are a set of approaches and codes to solving this problem. In this paper, we consider an approach for solving this problem using the matrix formalism for Lie algebraic tools. This formalism allows reducing the starting problem to linear algebraic equations for aberration coefficients, which are elements of corresponding matrices. There are discussed results evaluated using suggested approach and nonlinear programming tools. Some examples of corresponding results are given.

INTRODUCTION

The problem of nonlinear aberrations correction is one of the most important problems in beam physics. It is necessary to note that all types of aberrations can be separated into two ones. The first type consists of undesirable (or harmful) aberrations, and the second type accumulates useful aberrations, which characteristics can be used as control parameters and/or functions. In the last case an investigator exploits these aberrations (mainly nonlinear) for his special goals, whereas the first type lead to deterioration of particle beam quality. Here it is necessary to divide built-in control field (for example, octupole components in the field of a quadrupole lense and control field of special input multipole lenses). It is know that the searching of appropriate correcting fields (accordingly correcting elements — correctors) is a complex enough problem from the computational point of view. The choice of optimal configuration of locations and forces of correctors should supply flexibility and effectiveness of the computational process. In this paper we discuss an approach, which allows to find correcting control fields and has the above mentioned properties.

Let numerate some important problems which we mention in this paper: full or partly correction of chromatic and geometric aberrations up to some order of nonlinearity, aberrations induced by space charge forces, optimization of fringe field influence for nonlinear dynamics. In the base of the suggested method there lie matrix formalism for Lie algebraic tools [1, 2].

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THE MATHEMATICAL BACKGROUNDS FOR ABERRATIONS DESCRIPTION

Here we give necessary mathematical definition and concepts of the matrix formalism for Lie algebraic tools.

Beam Propagator

The motion equation of beam particles can be written in the following vector form:

$$\frac{d\mathbf{X}}{ds} = \mathbf{F}(\mathbf{X}, \mathbf{U}; s), \tag{1}$$

where $\mathbf{X}$, $\mathbf{U}$ are phase vector and control functions (or parameters) vector correspondingly, s — the independent parameters measured along the reference orbit. In accordance with Lie algebraic approach [3] the solution of eq. (1) can be written in the following form:

$$\mathbf{X}(\mathbf{U}; s) = \mathcal{M}(\mathbf{U}; s|s_0) \circ \mathbf{X}_0, \quad \mathbf{X}_0 = \mathbf{X}(s_0). \tag{2}$$

Here $\mathcal{M}(s|s_0)$ is a Lie map, generated by eq. (1), or a *beam propagator*.

Usually the perturbation character of control and self consistent fields allows us to write eq. (1) in the form of multivariate Taylor matrix series

$$\frac{d\mathbf{X}}{ds} = \sum_{k=0}^{\infty} \mathbb{P}^{1k}(s)\mathbf{X}^{[k]}, \tag{3}$$

where $\mathbf{X}^{[k]}$ is a Kronecker power of k-th order. Solutions of the initial problem for eq. (3) can be written in the following form

$$\mathbf{X}(s) = \sum_{k=0}^{\infty} \mathbb{R}^{1k}(\mathbf{U}; s|s_0)\mathbf{X}_0^{[k]}, \tag{4}$$

where $\mathbb{R}^{11}$ is a matriciant for linearized motion equation $d\mathbf{X}/ds = \mathbb{P}^{11}\mathbf{X}$ [4], and $\mathbb{R}^{1k}(s|s_0)$ are aberration matrices of k-th order (usual two-dimensional matrices), which accumulate corresponding aberrations of k-th order. For some behavioral assumption of matrices $\mathbb{P}^{11}$ one can evaluate these matrices $\mathbb{R}^{1k}$ in a symbolic mode (using computer algebra codes, for example, Maple, Mathematica and so on). This practically automatic allows to evaluate aberration matrices $\mathbb{R}^{1k}$ symbolically too. These matrices have dimensions equal to $n \times \binom{n+k-1}{k}$. Of course all necessary evaluations numerically.

N o t e 1. The convergence of the series (4) can be proved only for some interval for s. In general this series is divergent. But for enough small interval $[s_0, s_1]$ there is a convenient estimation for rate of convergence.

Grouped Aberration Matrix Elements

As stated above aberration matrices $\mathbb{R}^{1k}$, $k \geq 2$ accumulate all coefficients, which combine nonlinear effects of k-th order, generated by control elements. Their location in $\mathbb{R}^{1k}$ can be determined by a researcher using permutation matrices. For example, the well known spherical aberrations are determined by coefficients by monomials of third order: x'^3, x'^2y', $x'y'^2$, y'^3. But these monomials are distributed among other monomials of third order. From computational point of view it is convenient to place correct monomials in front of the rest ones. Similar redistribution can be realized for chromatic aberrations of second order and other types of aberrations. This technology is very useful for computational process and will be used for our goals.

Particular Propagators Concatenation

According to the Lie algebraic tools the whole beam propagator, describing some investigated machine, can be evaluated using one of known procedures (see, for example, [5]). Using the matrix representation for beam propagator we have to use the corresponding matrix analogue. Let us write the necessary equalities up to N-th order of nonlinear terms for some value of independent variable s

$$\begin{gathered}\mathbb{R}^{1k}(s_m|s_0) = \sum_{l=1}^{k} \mathbb{R}^{1l}(s_m|s_{m-1})\mathbb{R}^{lm}(s_{m-1}|s_0), \\ \mathbb{R}^{ll} = \left(\mathbb{R}^{11}\right)^{[l]}, \quad \forall k = \overline{1, N}.\end{gathered} \tag{5}$$

where $[s_{m-1}, s_m]$ is a current interval and s_0 — an initial value of s. It is necessary to note that equalities (5) can be evaluated for $k \in [1, N]$ in a symbolic form and then can be used for practical simulation.

Symplectic Correction of the Beam Propagator

For practical computing we have to truncate the series (4). This procedure leads to sacrifice of computational accuracy. Here we mean, first of all, qualitative character of errors. In the paper [9] it was demonstrated how one should correct the matrices $\mathbb{R}^{1k}$ to retain the symplecticity property of the matrix presentation of beam propagator up to desired order of nonlinearity. In the next our manipulations we suggest that the corresponding symplectification procedure is supported.

ABERRATION CORRECTION APPROACHES

The problem of the aberration correction concept can be formulated in two forms. The first of them is based on nonlinear programming approach. According this approach for each particular problem one should formulate an objective function, which describes some quality criterion. In general similar criteria can be written as

$$\mathrm{F}\left[\mathbf{U}\right](s_{\mathrm{fin}}|s_0) = \int_{s_0}^{s_{\mathrm{fin}}} \int_{\mathfrak{M}(\tau)} \mathrm{G}\left(\mathbf{X}(\tau), \mathbf{U}(\tau), \tau\right) d\mathbf{X} d\tau,$$

where G as an appropriate function describing an investigated physical phenomenon, $\mathfrak{M}(\tau)$ is a current phase manifold occupied beam particles. In particular, a researcher can taking into account an information on envelopes along the beam line axis and an information on some terminal characteristics of the beam. The unremovable harmful effects lead to aberrations precipitation. The beam evolution equations (1), (3) allows us to separate contributions of necessary and unwanted effects:

$$\frac{d\mathbf{X}}{ds} = \sum_{k=0}^{\infty} \left(\mathbb{P}^{1k}_{\mathrm{ext}}(s) + \mathbb{P}^{1k}_{\mathrm{corr}}(s)\right) \mathbf{X}^{[k]},$$

where $\mathbb{P}^{1k}_{\mathrm{ext}}$ and $\mathbb{P}^{1k}_{\mathrm{corr}}$ are necessary matrices. The above described matrix representation (see (4)) allows to write (see, for example, [2]).

$$\mathbb{R}^{1k}\left(s_{\mathrm{fin}}|s_0\right) = \mathbb{R}^{1k}_{\mathrm{ext}}\left(s_{\mathrm{fin}}|s_0\right) + \mathbb{R}^{1k}_{\mathrm{corr}}\left(\mathbf{\Gamma}; s_{\mathrm{fin}}|s_0\right), \tag{6}$$

where $\mathbf{\Gamma}$ is a vector of parameters γ_k, $k = \overline{1, N_{\mathrm{corr}}}$, describing corresponding characteristics of correcting multipoles.

Nonlinear Programming Approach

As we mentioned above the most general approach for solving similar problems is based on nonlinear programming methods (see. e. g. [6]). Unfortunately, in the case of high number of variety parameters nonlinear programming methods make difficult to investigate and leads to efficient increasing of computational time. Therefore in this case as a first step it would be more effective to use some variant of Monte–Carlo methods. And only on the second step we can put into numerical simulation some deterministic methods. In particular, for some beam physics problems we used the well known flexible tolerance method [7]. This approach has the adequate effectiveness. But these methods can be realized only numerically. Some approaches for beam physics optimization problems were described in [10].

An Algebraic Approach

Therefore it is necessary to suggest some another method: more flexible and more effective. In the previous works authors (see, e.g. [8]) demonstrated the advantages of matrix representation for some special cases. Let us describe the main features of this approach. On the first step one writes out the equalities similar to (4) using the additive presentation for aberrations matrices (6). On the second step the matrices $\mathbb{R}_{\mathrm{corr}}$ should be presented in the following form

$$\mathbb{R}^{1k}_{\mathrm{corr}} = \sum_{j=1}^{N_{\mathrm{corr}}} \gamma_j \mathbb{R}^{1k}_{j-\mathrm{corr}},$$

where $\mathbb{R}^{1k}_{j-\mathrm{corr}}$ are aberration matrices for correcting elements. Let us explain this procedure in the presence of nonlinear effects of second and third order. For this one should evaluate two matrices $\mathbb{R}^{12}_{\mathrm{corr}} = \mathbb{R}^{12}_{\mathrm{corr}}(s_{\mathrm{fin}}|s_0)$ and $\mathbb{R}^{13}_{\mathrm{corr}} = \mathbb{R}^{13}_{\mathrm{corr}}(s_{\mathrm{fin}}|s_0)$.

$$\mathbb{R}^{12}_{\mathrm{corr}} = \int_{s_0}^{s_{\mathrm{fin}}} \mathbb{R}^{11}(s_{\mathrm{fin}}|\tau)\mathbb{P}^{12}(\tau)\mathbb{R}^{22}(\tau|s_0)d\tau.$$

$$\mathbb{R}^{13}_{\mathrm{corr}} = \int_{s_0}^{s_{\mathrm{fin}}} \mathbb{R}^{11}(s_{\mathrm{fin}}|\tau)\mathbb{P}^{13}(\tau)\mathbb{R}^{33}(\tau|s_0)d\tau + \int_{s_0}^{s_{\mathrm{fin}}}\int_{s_0}^{\tau} \mathbb{R}^{11}(s_{\mathrm{fin}}|\tau)\mathbb{P}^{13}(\tau)\mathbb{R}^{22}(\tau|\tau')\times \times \mathbb{P}^{23}(\tau')\mathbb{R}^{33}(\tau'|s_0)d\tau' d\tau.$$

For the next step we should use the following equalities [2]:

$$\mathbb{P}^{23} = \mathbb{P}^{12}\otimes\mathbb{E} + \mathbb{E}\otimes\mathbb{P}^{12},\ \ \mathbb{R}^{kk} = \left(\mathbb{R}^{11}\right)^{[k]}.$$

Let α, β are forces of second and third order nonlinearities correspondingly. Then we can write

$$\mathbb{P}^{12} = \alpha\widehat{\mathbb{P}}^{12},\ \ \mathbb{P}^{13} = \beta\widehat{\mathbb{P}}^{13},\ \ \mathbb{P}^{23} = \alpha\widehat{\mathbb{P}}^{23}.$$

Using these equalities one can write

$$\mathbb{R}^{12}_{\mathrm{corr}} = \sum_{i=1}^{N^1_{\mathrm{corr}}} \alpha_i \int_{s_{i-1}}^{s_i} \mathbb{R}^{11}(s_{\mathrm{fin}}|\tau)\widehat{\mathbb{P}}^{12}(\tau)\mathbb{R}^{22}(\tau|s_0)d\tau = \sum_{i=1}^{N^1_{\mathrm{corr}}} \alpha_i\widetilde{\mathbb{R}}^{12}_i(s_{\mathrm{fin}}|s_0).$$

$$\mathbb{R}^{13}_{\mathrm{corr}} = \sum_{i=1}^{N^2_{\mathrm{corr}}} \beta_i \int_{s_{i-1}}^{s_i} \mathbb{R}^{11}(s_{\mathrm{fin}}|\tau)\widehat{\mathbb{P}}^{13}(\tau)\mathbb{R}^{33}(\tau|s_0)d\tau + \sum_{i=1}^{N^1_{\mathrm{corr}}}\sum_{k=1}^{N^2_{\mathrm{corr}}} \alpha_i\beta_k \int_{s_{i-1}}^{s_i}\int_{s_{k-1}}^{s_k} \mathbb{R}^{11}(s_{\mathrm{fin}}|\tau)\widehat{\mathbb{P}}^{13}(\tau)\mathbb{R}^{22}(\tau|\tau')\times \times \widehat{\mathbb{P}}^{23}(\tau')\mathbb{R}^{33}(\tau'|s_0)d\tau' d\tau = \sum_{i=1}^{N^2_{\mathrm{corr}}} \beta_i\widetilde{\mathbb{R}}^{13}_i(s_{\mathrm{fin}}|s_0) + \sum_{i=1}^{N^1_{\mathrm{corr}}}\sum_{k=1}^{N^2_{\mathrm{corr}}} \alpha_i\beta_k\widehat{\mathbb{R}}^{13}_{ik}(s_{\mathrm{fin}}|s_0).$$

Here N^1_{corr} and N^2_{corr} are numbers of correcting elements and $\widetilde{\mathbb{R}}^{12}_i$, $\widetilde{\mathbb{R}}^{13}_k$, $\widehat{\mathbb{R}}^{13}_{ik}$ are partial matrices chargeable effects of second and third orders under the assumption that corresponding forces of correcting elements are equal to *unit* on partial intervals and to *zero* for all other intervals. These matrices can be evaluated in advance as in symbolic or numerical form.

The above described approach permits to formulate the problem searching correcting elements forces in the following way. On the first step we evaluate matrices $\mathbb{R}^{12}$, $\mathbb{R}^{13}$, $\mathbb{R}^{23}$ which describe the control elements with harmful aberrations and $\widetilde{\mathbb{R}}^{12}_i$, $\widetilde{\mathbb{R}}^{13}_k$, $\widehat{\mathbb{R}}^{13}_{ik}$, which describe the correcting elements using the above declared algorithm. As a result we form special matrices and vectors, which generate the following system of linear algebraic equations

$$\mathbb{A}\mathbf{\Gamma} = \mathbf{B}, \tag{7}$$

where the matrix $\mathbb{A}$ consists of the elements of matrices $\widetilde{\mathbb{R}}^{12}_i$, $\widetilde{\mathbb{R}}^{13}_k$, $\widehat{\mathbb{R}}^{13}_{ik}$.

For example, see an example of third order spherical aberration correction in [2]. This approach allows us to reduce the problem of correcting elements to the problem of the linear algebra equation (7). The dimension of the vector $\mathbf{\Gamma} = (\gamma_1, \ldots, \gamma_{N_{\mathrm{corr}}})$ depends on the type of harmful aberrations and our possibilities to use corresponding correcting elements.

CONCLUSION

As we mentioned previously these procedures can be evaluated in symbolic or in numerical mode. In previous papers similar matrices were evaluated for some special cases of geometrical, chromatic aberrations up to third order. More over the equations (7) are solved also symbolically. All this information can be kept in a special database and can be used for numerical simulation *on-demand*. For long or circular systems as a previous step we suggest to use nonlinear programming methods, as we pointed out above. These numerical methods allows to find the starting values of required parameters as initial ones for the next steps, which are realized using the described method.

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