PARTICLE TRACKING IN A SEXTUPOLE FIELD USING THE EULER METHOD APPROXIMATION

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Abstract

The purpose of this paper is to evaluate any differences in the single particle tracking through a magnetic lattice when sextupoles are treated either like sliced or single-kick elements. Only on-energy transverse motion is considered. Convergence and symplecticity of the method of sliced sextupoles are discussed. Dynamic apertures and transverse phase spaces applied to the Elettra synchrotron lattice are compared for the two cases.

INTRODUCTION

The purpose of this paper is to investigate any differences in a single particle tracking performed through a nonlinear magnetic lattice when sextupoles are described like single-kick elements or multiple kicks (sliced sextupoles). Simply speaking, can a sextupole be replaced by many slices? And if yes, is the method converging and symplectic? A first brief theoretical address is followed by calculations applied to the motion for nonlinear fields; finally, plots of dynamic aperture and transverse phase space obtained with Mad code [1] are shown. Comments on the simplecticity and comparison of the two techniques conclude the paper.

DISCRETIZATION

Euler method [2] allows approximating a physical continous dynamical system with one discrete. Solutions of the latter approximate solutions of the first in a rigorous way defined by the uniform convergence [3]. Let us consider a discrete dynamical system, linear in the coordinates and described by:

$$\dot{X} = AX \tag{1}$$

$$X_{m+1} = (I + \frac{At}{m})X_m$$

where $A \in M_{n\times n}$, I is the identity matrix, $t \in \Re$ is the variable, $m \in N^+$ the number of step used for the discretization and X(t) the vector of coordinates through which we describe the system, with $X(t=0) = X_0$. From (1) and by definition of the exponential function, it is possible to write in matrix formalism:

$$\lim_{m\to+\infty}(I + \frac{At}{m})^m X_0 = e^{At}X_0 \tag{2}$$

For each X_0, given $m_{ij}(t)$ the generic element of the matrix e^{At} and $x_{0,j}$ the j-th component of the X_0 vector:

$$m_{ij}x_{0,j} = e^{a_{ij}t}\cdot x_{0,j}(t) = x_j(t) \tag{3}$$

$x_j(t)$ is the (j-th component of) solution of (1) at a certain t. The l.h.s. of equation (3) illustrates the integral flux of a continous dynamical system for a given initial condition, while the r.h.s. is the representation of the solution of the analogous discrete system, linear in X_0, and described by (1). The equivalence can be graphically thought like a linearization of the continous solution over a large number *m* of steps, so that it is approximated at each interval h=Δt/m with its differential. A solution of (1) for the k-th step is:

The error computed in the discretization is a function

$$X(0)=X_0;$$

$$X_1 \cong X_0 + (\frac{dX}{dt})_0\delta = X_0 + A\frac{\Delta}{m}X_0;$$

$$X_2 \cong X_1 + (\frac{dX}{dt})_1\delta = (X_0 + A\frac{\Delta}{m}X_0) + A\frac{\Delta}{m}X_1 = X_0(I + A\frac{\Delta}{m}) + (A\frac{\Delta}{m})^2X_0;$$

$$X_3 \cong X_0\{A\frac{\Delta}{m} + (A\frac{\Delta}{m})^2 + (A\frac{\Delta}{m})^3\};$$

....

$$X_k \cong X_0\{A\frac{\Delta}{m} + (A\frac{\Delta}{m})^2 + (A\frac{\Delta}{m})^3 + ...+ (A\frac{\Delta}{m})^k\}$$

rapidly growing with the step of the approximation; consequently, the required power of calculus is great. It can be shown [3] that the *error of local truncation* Δ_l, relative to the single step approximation, satisfies:

$$\Delta_l = \|\exp(Ah) - (I + Ah)\| \leq \frac{h^2}{2}e^{ah} \tag{4}$$

As for the *error of cumulative truncation* Δ_c – the total error computed in the approximation after k-steps – we refer to the general result of the theorem of convergence [3]. A generic field *F*(*X*) with $X \in \Re^n$ is considered:

$$\begin{aligned}&\dot{X} = F(X),\ X(0) = X_0\\ &X(t_1) \cong X_1 = X_0 + hF(X_0)\\ &X_{k+1} = X_k + hF(X_k)\end{aligned} \tag{5}$$

If *F(x)* is Lipschitzian of constant L and limited in module by a constant M,

$$|F(x) - F(y)| \leq L|x - y| \text{ and } |F(x)| \leq M \tag{6}$$

then Δ_c will depend on the step size h, M and L:

$$|\Delta_c^{(k)}| \equiv |\Delta_c(t = kh)| \leq Mh[e^{Lt} - 1] \tag{7}$$

This result is completely general, thus it can be applied to any nonlinear field satisfying (5) and (6).

SEXTUPOLE

Two dimensional dynamics of a particle passing through a sextupole may be described, in analogy with (1), in the following form:

$$\dot{X} = F(X)$$

$$F(X) = \begin{pmatrix} \alpha t(x^2 - y^2) \\ -2\alpha t(xy) \end{pmatrix} \quad (8)$$

where $\alpha = e\beta m_s/2m_e$ and m_s the sextupole strength. For a bore radius R_s, conditions in (6) are satisfied by $L=\alpha l_s R_s/\beta c$ and $M=2R_sL$. Thus, the error of cumulative truncation in (7) extended over a large number of turns N becomes:

$$|\Delta_c| \sim m_s (Nl_s)^2 R_s^2 [e^{\alpha(\frac{N\times l_s}{\beta c})^2 R_s} - 1] \quad (9)$$

As a practical case, let us consider a sextupole defined by $l_s = 0.1$ m, $m_s = 30$ m^{-2}, $R_s = 25$ mm ($\beta = 1$). We obtain $|\Delta_c\ (10^4 \text{ turns})| \leq 10^{-3}$ m, i.e. comparable with a realistic particle transverse excursion performing betatron oscillations.

TRACKING

Single particle tracking has been performed using the Mad code applied to the bare lattice of the Elettra storage ring [4]. Main parameters used in the simulations are listed in Table 1.

Table 1: Parameters of the single particle tracking in the Elettra lattice

Energy (GeV)	0.9
$\varepsilon_x = \varepsilon_y$ (m rad)	3.5×10^{-8}
σ_x (μm) / σ_y (μm) at observation point	560 / 190
chromaticities ξ_x / ξ_y	0.0 / 0.0
$\Delta p/p$ (%)	0.0

The lattice includes two families of chromatic sextupoles, called SF and SD, and one family of harmonic sextupoles, called S1 (see, Table 2). In the case of sliced sextupoles, each one has been divided into ten pieces of identical strength.

Table 2. Sextupoles in the Elettra lattice

	# per cell / total number	magnetic length (m)	strength (m^{-2})
SF	1 / 12	0.230	35.166
SD	1 / 12	0.230	32.744
S1	1 / 12	0.115	19.130

In Figure 1 is the transverse dynamic aperture (in $\sigma_{x,y}$ units) resulting from tracking into single-kick (left) and sliced (right) sextupoles over 10^4 turns. It shows a stronger limitation in the horizontal plane with respect to the vertical. For the sliced lattice, the horizontal dynamic aperture is somewhat shifted so to be reduced in the right region but increased in the left one; it is also slightly larger in the vertical plane (right side).

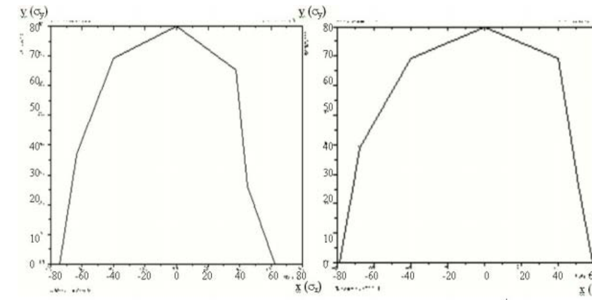

Figure 1: Dynamic aperture in sigma unit (σ_x = 560 μm; σ_y = 190 μm) after 10^4 turns, for single-kick (left) and sliced (right).

Figure 2 and Figure 3 represent the horizontal pseudo-canonical phase space after 10^3 turns; initial conditions for tracking have been set at large amplitude in proximity of the dynamic aperture limits, where some differences appear between the two tracking technique and a stronger nonlinear motion is expected to be (see, Figure 1). As for Figure 2, the sliced lattice leads to a stronger distortion of the ellipse and finally to an unstable motion (notice the point at maximum divergence), while in the single-kick case the motion is still stable. The same result is also obtained after 10^4 turns (not shown here). In Figure 3 the single-kick case leads to unstable motion and particle loss, while in the sliced case, the vertical dynamic aperture being larger (see Fig.1), the phase space ellipse is only distorted and the particle motion remains bounded.

SIMPLECTICITY

Mad code uses the Lie-algebric tracking method up to the 3rd order in the particle's coordinates [5]. It is known that truncating the exponential series of the generating functions for the beam transport at a finite order does not produce a symplectic map. Therefore, Mad tracks the linear terms of the 4 dimensional beam phase space using the linear transfer matrix; as for the nonlinear terms, it applies a generating function defined such that the resulting canonical transformation agrees to the desired order with the mapping; the generating function is truncated at the given order of the Lie transformation. In this way, simplecticity is guaranteed for the single-kick approximation up to the 3rd order. Since each individual sextupole slice is described by a single-kick matrix, their product generates a transfer matrix still simplectic up to the 3rd order. Nevertheless, the resulting effect from the nonlinear field on the transverse phase space is different. If the error of truncation cumulated in the Euler method is not too large, the slice option is expected to be more realistic as it approximates to an integral.

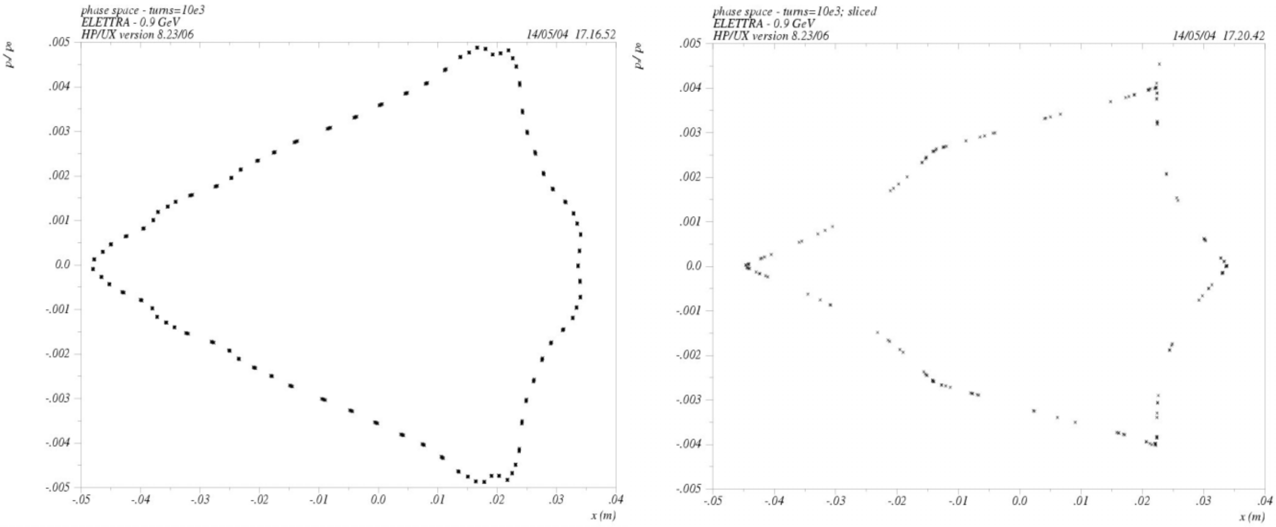

Figure 2: Horizontal phase space after 10^3 turns with initial conditions $x_o = 60\sigma_x$, $y_o = 0$, zero divergences; single-kick (left) and sliced (right) sextupoles.

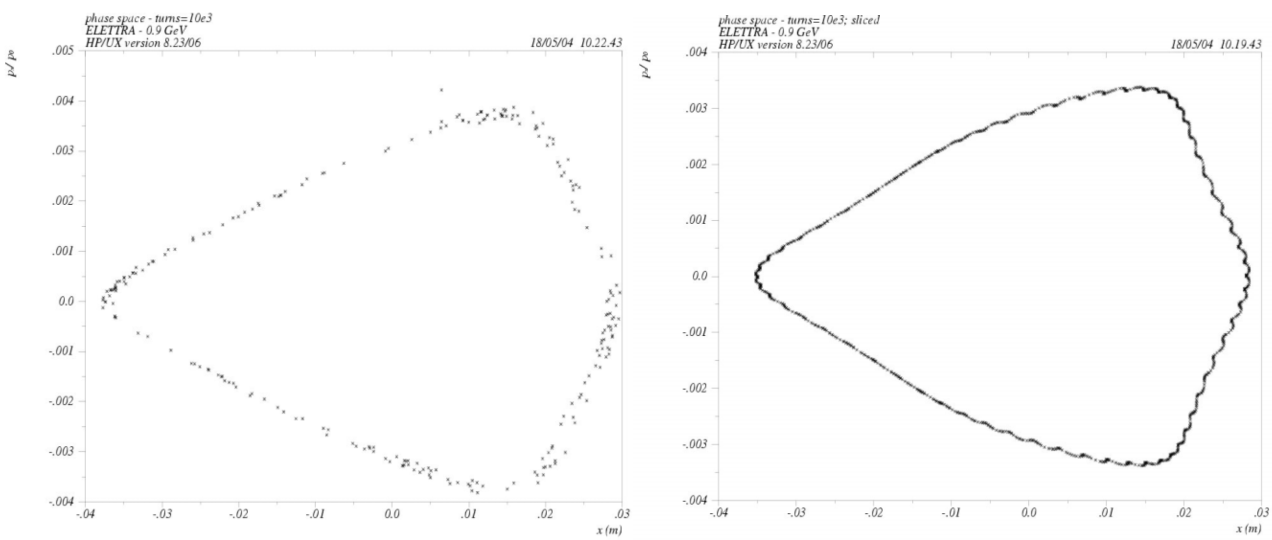

Figure 3: Horizontal phase space after 10^3 turns with initial conditions $x_o = 50\sigma_x$, $y_o = 20\sigma_y$, zero divergences; single-kick (left) and sliced (right) sextupoles.

CONCLUSIONS

It has been verified that a sextupole field satisfies the theorem of convergence of Euler. The error of truncation cumulated when applying the Euler method has been analytically evaluated for a realistic example. If the error is made sufficiently small, the Euler method is expected to be more precise with respect to the simple single-kick element description because it approximates an integral. Single particle tracking has been performed with Mad and applied to the Elettra storage ring lattice. Dynamic aperture and phase space have been compared for the two cases of single-kick and sliced sextupoles. Differences in the motion stability become important just after 100 turns (passing through 36 sextupoles per turn) at large betatron amplitudes, where nonlinearities are stronger. In such cases, the Euler method predicts a different range for the stability of the particle motion.

REFERENCES

[1] H. Grote and F.C. Iselin, *The MAD Program, Version 8.19, User's reference Manual*, CERN/SL/90-13 (AP)

[2] Any text of advanced mathematical analysis.

[3] http://copernico.dm.unipi.it/~milani/dinsisno/node 25.html or any text of advanced mathematical analysis.

[4] *Elettra Conceptual Design*, Sincrotrone Trieste (1989)

[5] F.C. Iselin, *The MAD Program, Version 8.19, Physical Methods Manual*, CERN/SL/92 (AP)