RENORMALIZATION GROUP REDUCTION OF THE FROBENIUS-PERRON OPERATOR

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Abstract

The Renormalization Group (RG) method is adopted as a tool for a constructive analysis of the properties of the Frobenius-Perron Operator. The renormalization group reduction of a generic symplectic map in the case, where the unperturbed rotation frequency of the map is far from structural resonances driven by the kick perturbation has been performed. It is further shown that if the unperturbed rotation frequency is close to a resonance, the reduced RG map of the Frobenius-Perron operator (or phase-space density propagator) is equivalent to a discrete Fokker-Planck equation for the renormalized distribution function. Some concrete examples have been worked out.

INTRODUCTION

Recursive maps represent a useful and powerful tool to model and to facilitate the understanding of the physical processes taking place in complex nonlinear systems. In particular, they are widely used to study the various transition scenarios from regular to chaotic behaviour in nonlinear dynamical systems, to simulate physical systems exhibiting anomalous diffusion [1], or to analyze the underlying dynamics in time series with $1/f$ noise in their power spectrum [2]. Iterative maps provide a convenient and effective method to investigate single-particle dynamics in accelerators and storage rings [3, 4].

The extremely complicated behaviour of specific trajectories in chaotic systems suggests a probabilistic approach to the dynamics. The Frobenius-Perron operator of a phase-space density (distribution) function provides a tool for studying the dynamics of the iteration of the distribution function itself. The iterative map yields complete information of how the value of an individual phase-space point jumps around during successive iterations, so that one gets a good sense of the point dynamics but no sense of how iteration acts on densities with support on sets in phase space. The latter gap is filled by the Frobenius-Perron operator, which provides a rule to determine how the evolution of densities over repeated iterations is accomplished.

Here we adopt the Renormalization Group (RG) method to analyze the properties of the Frobenius-Perron Operator. The basic idea of the method is to absorb secular or divergent terms of the naive perturbation solution into renormalized integration constants (amplitudes). The stages in the renormalization group reduction of a particular physical system are quite general and well defined, which makes the RG method universal and independent on the concrete details of the underlying dynamics and physical processes involved. [3, 5, 6]

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THE FROBENIUS-PERRON OPERATOR FOR THE HENON MAP

The Henon map is defined by the following expression [3]:

$$\mathbf{z}_{n+1} = \begin{pmatrix} x_{n+1} \\ p_{n+1} \end{pmatrix} = \mathcal{R}_\omega \begin{pmatrix} x_n \\ p_n - \mathcal{S}x_n^2 \end{pmatrix}, \tag{1}$$

where

$$\mathcal{R}_\omega = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}, \tag{2}$$

is the rotation matrix for one period of the map, which is equivalent to one revolution along the accelerator lattice. The frequency ω and the parameter $\mathcal{S}$

$$\omega = 2\pi\nu, \qquad \mathcal{S} = \frac{l\lambda_0(\theta_0)\beta^{3/2}(\theta_0)}{2R^3}, \tag{3}$$

are related to the unperturbed betatron tune ν and to the strength of the sextupole (cubic nonlinearity) perturbation λ_0. Here l is the length of the sextupole, θ_0 is its location on the azimuth of the machine and R is the mean radius.

The Henon map can be alternatively written as

$$\mathbf{Z}_{n+1} = \mathcal{R}_\omega^T \mathbf{z}_{n+1} = \begin{pmatrix} x_n \\ p_n - \mathcal{S}x_n^2 \end{pmatrix}, \tag{4}$$

where $\mathcal{R}_\omega^T$ denotes the transposed of the matrix (2). The Frobenius-Perron operator [3] can be calculated explicitly. We have:

$$f_{n+1}(x,p) = \widehat{\mathbf{U}} f_n(x,p) = f_n\big(X, P + \mathcal{S}X^2\big). \tag{5}$$

Introducing the formal small parameter ϵ and the action-angle variables $x = \sqrt{2J}\cos a$, $p = -\sqrt{2J}\sin a$, we write the Frobenius-Perron operator represented by equation (5) in the form

$$f_{n+1}(a+\omega, J) = f_n\big(x, p + \epsilon\mathcal{S}x^2\big). \tag{6}$$

RENORMALIZATION GROUP TREATMENT OF THE FROBENIUS-PERRON OPERATOR

The generalization of the Frobenius-Perron operator (6) for a generic symplectic map with rotation is straightforward. We have

$$f_{n+1}(a+\omega, J) = f_n(x, p + \epsilon\partial_x V_N), \tag{7}$$

where $V_N(x)$ is a potential and ∂_x denotes partial differentiation with respect to x. Equation (7) can be written as

$$f_{n+1}(a+\omega, J) = e^{\epsilon(\partial_x V_N)\partial_p} f_n(a, J). \tag{8}$$

Since the potential V_N does not depend on the momentum variable p, we have $\widehat{\mathbf{L}}_V = (\partial_x V_N)\partial_p - (\partial_p V_N)\partial_x = (\partial_x V_N)\partial_p$, where $\widehat{\mathbf{L}}_V$ is the Liouvillean operator associated with V_N. Therefore, equation (8) becomes

$$f_{n+1}(a+\omega, J) = e^{\epsilon \widehat{\mathbf{L}}_V} f_n(a, J). \tag{9}$$

We assume that the potential V_N, written in action-angle variables can be split as follows $V_N(a, J) = V_0(J) + V(a, J)$. Respectively, the Liouvillean operator can be written as $\widehat{\mathbf{L}}_V = \widehat{\mathbf{L}}_0 + \widehat{\mathbf{L}}$, where $\widehat{\mathbf{L}}_0 = -\omega_V(J)\partial_a$, and

$$\widehat{\mathbf{L}} = (\partial_a V)\partial_J - (\partial_J V)\partial_a, \qquad \omega_V(J) = \frac{\partial V_0}{\partial J}. \tag{10}$$

First of all, we consider the case, where the rotation frequency ω is away from nonlinear resonances driven by the potential V. Following the standard procedure of the RG method [3, 5], we seek a solution to equation (9) by naive perturbation expansion

$$f_n(a, J) = \sum_{k=0}^{\infty} \epsilon^k f_n^{(k)}(a, J), \tag{11}$$

where the unknown functions $f_n^{(k)}(a, J)$ have to be determined order by order. The zero-order perturbation equation $f_{n+1}^{(0)}(a+\omega, J) = f_n^{(0)}(a, J)$ has the obvious solution $f_n^{(0)}(a, J) = e^{-n\omega\partial_a} F(a, J) = F(a - n\omega, J)$. To this end $F(a, J)$ is an arbitrary function of its arguments, and will be the subject of the renormalization group reduction in the sequel.

The first-order perturbation equation can be written as follows $f_{n+1}^{(1)}(a+\omega, J) - f_n^{(1)}(a, J) = \widehat{\mathbf{L}}_V F(a - n\omega, J)$. Standard but cumbersome algebra yields its the solution in the form

$$f_n^{(1)}(a, J) = \left(n\widehat{\mathbf{L}}_0 + \widehat{\mathcal{L}}_\omega\right) F(a - n\omega, J), \tag{12}$$

where $\widehat{\mathcal{L}}_\omega = (\partial_a V_\omega)\partial_J - (\partial_J V_\omega)\partial_a$. Furthermore, the potential $V_\omega(a, J)$ is defined according to the expression

$$V_\omega(a, J) = \sum_{m \neq 0} \frac{V_m(J) e^{im(a - \omega/2)}}{2i \sin(m\omega/2)}. \tag{13}$$

The second order equation is

$$f_{n+1}^{(2)}(a+\omega) - f_n^{(2)}(a) = \widehat{\mathbf{L}}_V f_n^{(1)}(a) + \frac{\widehat{\mathbf{L}}_V^2}{2} F(a - n\omega). \tag{14}$$

Since we are interested in the secular solution of equation (14), we retain on its right-hand-side only terms that would yield a secular contribution. Omitting the details of the calculation and the non secular terms, we can write the second-order solution as

$$f_n^{(2)}(a, J) = \left[\frac{n^2}{2}\widehat{\mathbf{L}}_0^2 + n\widehat{\mathcal{L}}_\omega \widehat{\mathbf{L}}_0 + n\Omega(\omega, J)\partial_a\right] F(a_n, J), \tag{15}$$

where $a_n = a - n\omega$ and

$$\Omega(\omega, J) = \sum_{m=1}^{\infty} m \cot\left(\frac{m\omega}{2}\right) \partial_J (V_m \partial_J V_m). \tag{16}$$

To remove secular terms (proportional to n and n^2) in the first-order and the second-order solution (15), we define a renormalization group transformation $F(a, J) \to \widetilde{F}(a, J; n)$ by collecting all terms proportional to $F(a_n, J)$. Following Reference [3, 6], we define a discrete version of the RG equation by considering the difference

$$\widetilde{F}(a_n, J; n+1) - \widetilde{F}(a_n, J; n) = \left\{\epsilon\widehat{\mathbf{L}}_0 + \epsilon^2\left[\left(n + \frac{1}{2}\right)\widehat{\mathbf{L}}_0^2 + \Omega\partial_a\right]\right\} F(a_n, J). \tag{17}$$

Eliminating $F(a_n, J)$ in terms of $\widetilde{F}(a_n, J; n)$, we finally obtain

$$\widetilde{F}(n+1) - \widetilde{F}(n) = \left[\epsilon\widehat{\mathbf{L}}_0 + \epsilon^2\left(\frac{\widehat{\mathbf{L}}_0^2}{2} + \Omega\partial_a\right)\right]\widetilde{F}(n). \tag{18}$$

Equation (18) is the RG equation. It describes the evolution of the distribution function on slow time scales in addition to the fast oscillations with a fundamental frequency ω.

To first order in the perturbation parameter ϵ the renormalized solution to equation (9) can be written as $f_n(a, J) = \left(1 + \epsilon\widehat{\mathcal{L}}_\omega\right)\widetilde{F}(a_n, J; n)$, where the renormalized "amplitude" $\widetilde{F}(a_n, J; n)$ satisfies the RG equation (18). In the continuous limit equation (18) acquires the form

$$\partial_n \widetilde{F}(a_n, J; n) = \left[\epsilon\widehat{\mathbf{L}}_0 + \epsilon^2\left(\frac{\widehat{\mathbf{L}}_0^2}{2} + \Omega\partial_a\right)\right]\widetilde{F}(a_n, J; n). \tag{19}$$

Provided $\widehat{\mathbf{L}}_0 \neq 0$ (in the case, where the potential V_N is not antisymmetric) the latter is a Fokker-Planck equation with the Fokker-Planck operator acting on the angle variable only.

RESONANCE STRUCTURE OF A SYMPLECTIC MAP

The solution (12) to the first-order perturbation equation was obtained under the assumption that the unperturbed betatron tune ν is sufficiently far from any structural nonlinear resonance of the form $m_0\nu = 1$, where m_0 is an integer. In the present paragraph, we assume that $\omega = \omega_0 + \epsilon\delta_1 + \epsilon^2\delta_2 + \ldots$, where $m_0\omega_0 = 2\pi$. Moreover, for the sake of simplicity, we assume that there are no higher angle-dependent harmonics in the Fourier spectrum of $V(a, J)$ that would drive higher-order resonances of the form $pm_0\nu = p$, where p is an integer. However, results can be generalized easily to take into account this case as well.

Proceeding as in the previous paragraph, we can calculate the naive perturbation solution to second order in ϵ. Repeating the steps that brought us along to equation (18) in the previous paragraph, we obtain the RG equation in the resonant case

$$\widetilde{F}(a_n, J; n+1) - \widetilde{F}(a_n, J; n) = \left\{\epsilon\left(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}_R\right) + \epsilon^2\left[(\Omega' - \delta_2)\partial_a + \frac{\delta_1}{2}\left[\widehat{\mathbf{L}}_R, \partial_a\right] + \frac{1}{2}\left(\widehat{\mathbf{L}}_1 + \widehat{\mathbf{L}}_R\right)^2\right]\right\}\widetilde{F}(a_n, J; n), \tag{20}$$

where now $a_n = a - n\omega_0$. Here $\left[\widehat{\mathbf{L}}_R, \partial_a\right] = \widehat{\mathbf{L}}_R\partial_a - \partial_a\widehat{\mathbf{L}}_R$ is the commutator of the operators $\widehat{\mathbf{L}}_R$ and ∂_a, and

$$\Omega'(\omega_0, J) = \sum_{m\neq m_0}^{\infty} m\cot\left(\frac{m\omega_0}{2}\right)\partial_J(V_m\partial_J V_m). \quad (21)$$

is the new nonlinear tune shift. In addition, $\widehat{\mathbf{L}}_1 = -\delta_1\partial_a + \widehat{\mathbf{L}}_0$, and $\widehat{\mathbf{L}}_R = (\partial_a V_R)\partial_J - (\partial_J V_R)\partial_a$ is the resonant Liouvillean operator, where

$$V_R(a, J) = \sum_{m=\pm m_0} V_m(J)e^{ima} = 2V_{m0}(J)\cos m_0 a, \quad (22)$$

is the resonant potential.

EXAMPLES

Let us now consider a few examples. In the non resonant case of the Henon map $\left(\widehat{\mathbf{L}}_0 \equiv 0\right)$ equation (19) can be written in the form $\partial_n\widetilde{F}(a_n, J; n) = \Omega_H(a, J)\partial_a\widetilde{F}(a_n, J; n)$, where

$$\Omega_H(\omega, J) = \frac{\mathcal{S}^2 J}{8}\left[3\cot\left(\frac{\omega}{2}\right) + \cot\left(\frac{3\omega}{2}\right)\right]. \quad (23)$$

The above equation for $\widetilde{F}(a_n, J; n)$ written in alternative form $\partial_n\widetilde{F}(a, J; n) = [-\omega + \Omega_H(a, J)]\partial_a\widetilde{F}(a, J; n)$, describes regular motion with a frequency $\omega - \Omega_H$, and effective Hamiltonian

$$H_{eff}(J) = \omega J - \frac{\mathcal{S}^2}{16}\left[3\cot\left(\frac{\omega}{2}\right) + \cot\left(\frac{3\omega}{2}\right)\right]J^2. \quad (24)$$

The next example we would like to consider is the case of a cubic map with a potential of the form $V_N(x) = \mathcal{C}x^4/4$. We also have

$$V_0(J) = \frac{3\mathcal{C}J^2}{8}, \quad V(a, J) = \frac{\mathcal{C}J^2}{2}\cos 2a + \frac{\mathcal{C}J^2}{8}\cos 4a. \quad (25)$$

Furthermore, the nonlinear tune shift can be expressed according to the expressions $\omega_V(J) = 3\mathcal{C}J/4$, $\Omega_C(\omega, J) = 3\mathcal{C}^2J^2(8\cot\omega + \cot 2\omega)/32$. Thus, equation (19) can be written as

$$\partial_n\widetilde{F}(a, J; n) = -\widetilde{\omega}\partial_a\widetilde{F}(a, J; n) + \frac{\omega_V^2}{2}\partial_{aa}^2\widetilde{F}(a, J; n), \quad (26)$$

where

$$\widetilde{\omega}(\omega, J) = \omega + \omega_V - \frac{3\mathcal{C}^2J^2}{32}(8\cot\omega + \cot 2\omega). \quad (27)$$

Equation (26) can be readily solved, yielding the result

$$\widetilde{F}(a, J; n) = \sum_k \widetilde{F}_k(J; 0)e^{ik\left(a - n\widetilde{\omega}\right)}e^{-k^2\omega_V^2 n/2}. \quad (28)$$

The latter indicates that the renormalized distribution function $\widetilde{F}(a, J; n)$ rapidly relaxes towards the invariant density $\widetilde{F}_0(J)$.

To study the resonance case for the Henon map, we assume that the unperturbed betatron tune is close to a third order resonance $3\nu_0 = 1$. In this particular case, we have $\widehat{\mathbf{L}}_1 = -\delta_1\partial_a$, and $\Omega'(\omega_0, J) = \sqrt{3}\mathcal{S}^2J/8$. In the continuous limit equation (20) can be written as

$$\frac{\partial\widetilde{F}}{\partial n} = -(\omega - \Omega')\partial_a\widetilde{F} + \widehat{\mathbf{L}}_R\widetilde{F} + \frac{1}{2}\left(\widehat{\mathbf{L}}_1^2 + 2\widehat{\mathbf{L}}_1\widehat{\mathbf{L}}_R + \widehat{\mathbf{L}}_R^2\right)\widetilde{F}, \quad (29)$$

where $\widehat{\mathbf{L}}_R = -\mathcal{S}\sqrt{2J}[2J\sin(3a)\partial_J + \cos(3a)\partial_a]/4$, and the renormalized distribution function $\widetilde{F} = \widetilde{F}(a, J; n)$ is a function of the phase-space variables and the "time" n.

Equation (29) is a Fokker-Planck equation describing the slow evolution of the phase-space density in the case where the rotation frequency of the Henon map is close to a third order resonance.

CONCLUDING REMARKS

We have applied the Renormalization Group (RG) method to study the stochastic properties of the Frobenius-Perron operator for a variety of symplectic maps. After a brief introduction and derivation of the Frobenius-Perron operator for a generic symplectic map with rotation, the case, where the unperturbed rotation frequency of the map is far from structural resonances driven by the kick perturbation has been analyzed in detail. It has been shown that up to second order in the strength of the perturbation kick, the renormalized propagator for maps with nonlinear stabilization $\left(\widehat{\mathbf{L}}_0 \neq 0\right)$ describes random wandering of the angle variable. Further, the resonance structure of a symplectic map has been investigated. It has been shown that in the case, where the unperturbed rotation frequency is close to a resonance, the reduced RG map of the Frobenius-Perron operator (or reduced phase-space density propagator) is equivalent to a discrete Fokker-Planck equation for the renormalized distribution function.

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