ASYMPTOTIC ANALYSIS OF ULTRA-RELATIVISTIC CHARGE

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Abstract

A summary of a new approach is given for analysing the dynamic behaviour of distributions of charged particles in an electromagnetic field. Noting the limitations inherent in the Lorentz-Dirac equation for a single point particle, a simple model is proposed for a charged continuum interacting self-consistently with the Maxwell field in vacuo. The model is developed using intrinsic tensor field theory and exploits to the full the symmetry and light-cone structure of Minkowski spacetime. A covariant perturbation scheme is motivated by an exact class of solutions to this system describing the evolution of a charged fluid under the combined effects of both self and external electromagnetic fields. The scheme yields an asymptotic approximation involving inhomogeneous linear equations for the self-consistent Maxwell field, charge current and time-like velocity field of the charged fluid

INTRODUCTION

The intense international activity involved in probing the structure of matter on all scales, with particle beams and radiation, owes much to recent advances in accelerator science and technology. Developments in the production of high power laser radiation also offer new avenues for accelerator design and new diagnostic tools of relevance to medical science, engineering and the communications industry. A common theme in these developments is the interaction between charged particles and the electromagnetic field in domains where relativistic effects cannot be ignored.

It is remarkable that many of the challenges that must be addressed in order to develop and control devices that accelerate charged particles have their origin in the interaction of particles with their *own electromagnetic field*. Despite the fact that the classical laws of electromagnetism were essentially formulated over a century and a half ago the subject of electromagnetic interactions with matter remains incomplete. This incompleteness has had concomitant effects on the development of quantum electrodynamics and renormalisation theory. At root, the difficulties reside in the recognition that the quantum structure of matter at some scale is beyond observation. Furthermore, the classical description of the electron as a point particle leads to singularities in the Maxwell self-fields that inevitably create ambiguities in its interaction with the Maxwell field. The general consensus is that a useful domain of validity of the Lorentz-Dirac equation [1, 2, 3], describing covariantly the radiation reaction on a point electron, can be accommodated by performing a "reduction of order" that effectively replaces the equation by a perturbative second order system for the particle world line. One must then decide whether higher order terms in this expansion should be maintained given the neglect of terms associated with the regularisation scheme. This approach is behind many of the successful applications of approximate radiation reaction dynamics, despite the somewhat delicate and unsatisfactory nature of the arguments that purport to support it. However, an understanding of the interaction of charged particle bunches with their electromagnetic fields is vital for the success of proposed future accelerators.

In this article a summary is given of a new approach for analysing the behaviour of distributions of ultra-relativistic charged particles in a coupled electromagnetic field environment [4]. Rather than mixing particle and field concepts, our model is formulated entirely in the language of smooth tensor fields. Here, the non-linear field equations describing a thermodynamically inert (pressureless) charged fluid are adopted as a dynamical model of charged particle bunches. They are analysed using an ultra-relativistic asymptotic approximation scheme that conserves total energy, momentum and charge at all orders in the expansion parameter.

EQUATIONS OF MOTION FOR A PRESSURELESS CHARGED FLUID

The partial differential system of equations governing the motion of a pressureless charged fluid in MKS form is

$$\begin{aligned}
&\boldsymbol{\nabla}\cdot\boldsymbol{e} = \tfrac{1}{\epsilon_0}\varrho,\ \boldsymbol{\nabla}\times\boldsymbol{b} = \mu_0\varrho\boldsymbol{v} + \tfrac{1}{c^2}\tfrac{\partial\boldsymbol{e}}{\partial t},\\
&\boldsymbol{\nabla}\times\boldsymbol{e} + \tfrac{\partial\boldsymbol{b}}{\partial t} = 0,\ \boldsymbol{\nabla}\cdot\boldsymbol{b} = 0,\\
&\tfrac{\partial\boldsymbol{p}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{p} = q_0\,(\boldsymbol{e} + \boldsymbol{v}\times\boldsymbol{b})\,,\\
&\boldsymbol{p} = \frac{m_0}{\sqrt{1-\boldsymbol{v}\cdot\boldsymbol{v}/c^2}}\boldsymbol{v}
\end{aligned}\tag{1}$$

where $\boldsymbol{e}$ is the electric field, $\boldsymbol{b}$ is the magnetic induction, $\boldsymbol{v}$ is the fluid's velocity field, ϱ is the fluid's charge density, $\boldsymbol{p}$ is the fluid's momentum field, m_0 is the rest mass of the particles comprising the fluid, q_0 is the fundamental charge of the particles comprising the fluid, ϵ_0 is the permittivity of the vacuum and μ_0 is the permeability of the vacuum. Although (1) is the most familiar statement of the fluid's equations of motion, their most succinct and analytically powerful form is expressed without reference to a particular inertial frame i.e. where the inertial time t does not appear explicitly. Let $\mathcal{M}$ be Minkowski spacetime and let g be its flat Lorentzian metric with ∇ the Levi-Civita connection.

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The system (1) may be recast in the coordinate-free form

$$\begin{aligned} &dF = 0, \quad d\star F = -\rho \star \widetilde{V}, \\ &\nabla_V \widetilde{V} = i_V F, \quad g(V,V) = -1 \end{aligned} \tag{2}$$

where V is the fluid's 4-velocity field, F is an electromagnetic field 2-form, ρ is proportional to the proper charge density[1] and is called the *reduced* proper charge density, d is the exterior derivative, $\star$ is the Hodge map associated with g, i_V is the interior operator and the 1-form $\widetilde{V}$ is the metric dual of V. For more information on the precise relationship between (2) and (1) see [4].

APPROXIMATION SCHEME

In general situations, exact solutions to the non-linear partial differential system (2) are extremely difficult to find. A self-consistent method for obtaining approximate solutions is to let V, F and ρ depend on a running parameter ε and to solve the hierarchy of equations generated by (2) for the 1-parameter families of fields V^ε, F^ε and ρ^ε order by order in ε. Whether or not the hierarchy of equations so obtained is easier to solve than the original system depends on the ε dependences of the fields. Although there are many possibilities, a viable choice is strongly suggested by the behaviour of a class of highly symmetric *exact* solutions relevant to accelerator physics. These solutions describe "walls of charge" of infinite extent accelerating due to their self-field ("space-charge") and an applied constant electric field. Let (t, x, y, z) be an inertial coordinate system where t is the laboratory time and (x, y, z) are Cartesian coordinates. The spacetime metric tensor g has the form[2]

$$g = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz. \tag{3}$$

The "wall of charge" solutions have the following properties in the (dt, dx, dy, dz) frame : the magnetic field vanishes, the electric field has only a z component, the 4-velocity field has only t and z components and all field components depend only on t and z. The z component of the reduced electric field has the form

$$\mathcal{E}(t,z) = \zeta\big(\hat{\sigma}(t,z)\big) \tag{4}$$

where $\sigma = \hat{\sigma}(t,z)$ is a solution to the implicit equation

$$\sigma = z - \frac{1}{\zeta(\sigma)}\left(\sqrt{1 + [\zeta(\sigma)]^2 t^2} - 1\right) \tag{5}$$

and the lines of constant σ are the integral curves of V. Splitting the initial ($t = 0$) electric field into a constant external part ζ_{-1}/ε and a self part $\zeta_0(z)$:

$$\mathcal{E}^\varepsilon(0,z) = \frac{1}{\varepsilon}\zeta_{-1} + \zeta_0(z) \tag{6}$$

leads to

$$\begin{aligned} &\hat{\sigma}^\varepsilon(t,z) = z - t + \tfrac{\varepsilon}{\zeta_{-1}} - \tfrac{1+2t\zeta_0(z-t)}{2t\zeta_{-1}^2}\varepsilon^2 + O(\varepsilon^3), \\ &\mathcal{E}^\varepsilon(t,z) = \tfrac{1}{\varepsilon}\zeta_{-1} + \zeta_0(z-t) + \tfrac{\zeta_0'(z-t)}{\zeta_{-1}}\varepsilon + O(\varepsilon^2), \end{aligned} \tag{7}$$

for $t > 0$ and $\zeta_{-1} > 0$ and where $\zeta_0'(\sigma) = \frac{d\zeta_0}{d\sigma}(\sigma)$. Using (7), it can be shown that the 4-velocity field V^ε and reduced proper charge density ρ^ε take the form

$$\begin{aligned} V^\varepsilon \;&= \Big[\tfrac{1}{\varepsilon}\zeta_{-1} + \zeta_0(z-t)\Big]t\big(\partial_t + \partial_z\big) \\ &\quad + \Big[\tfrac{1+2t^2\zeta_0'(z-t)}{2t\zeta_{-1}}\partial_t + \tfrac{\zeta_0'(z-t)t}{\zeta_{-1}}\partial_z\Big]\varepsilon + O(\varepsilon^2), \\ \rho^\varepsilon &= \varepsilon\tfrac{\zeta_0'(z-t)}{\zeta_{-1}t} + O(\varepsilon^2). \end{aligned} \tag{8}$$

Note that the coefficient of $1/\varepsilon$ in V^ε is light-like; therefore, such field configurations are called *ultra-relativistic*. More generally, it can be shown that the ε expansions of F^ε, V^ε and ρ^ε take the form

$$\begin{aligned} &F^\varepsilon = \sum_{n=-1}^{\infty} \varepsilon^n F_n, \;\; V^\varepsilon = \sum_{n=-1}^{\infty} \varepsilon^n V_n, \\ &\rho^\varepsilon = \sum_{n=1}^{\infty} \varepsilon^n \rho_n, \end{aligned} \tag{9}$$

where F_{-1} is an external field (a solution to the source-free Maxwell equations). Therefore, (9) is adopted as an ansätz for the ε dependence of general solutions that include the effects of guiding and accelerating electromagnetic fields. Indeed, inserting (9) in (2) leads to a *partially decoupled* (in comparison with (2)) infinite set of differential equations whose myriad solutions include (7) and (8).

The infinite system is solved order by order in ε according to the following iterated pattern:

solve Maxwell's equations
↓
solve for the motion of the continuum (10)
↓
enforce charge conservation

The first step in the solution procedure is to note that the external electromagnetic field F_{-1} is a solution to the source-free Maxwell equations:

$$dF_{-1} = 0, \quad d\star F_{-1} = 0. \tag{11}$$

The second step involves solving

$$\nabla_{V_{-1}}\widetilde{V}_{-1} = i_{V_{-1}}F_{-1}, \quad g(V_{-1}, V_{-1}) = 0 \tag{12}$$

for the leading order 4-velocity field V_{-1}. Note that, to leading order, the motion of the charged continuum is governed by the external field F_{-1}. The third step is to enforce charge conservation to leading order by solving

$$d\star\left(\rho_1 \widetilde{V}_{-1}\right) = 0 \tag{13}$$

[1]The proper charge density is $\frac{\epsilon_0 m_0 c^2}{q_0}\rho$.
[2]Units are chosen in which the speed of light $c = 1$.

for ρ_1 and ensures that the following step is consistent. The fourth step is to solve the Maxwell equations for F_0 with the 4-current $\rho_1 V_{-1}$ as a source:

$$dF_0 = 0, \quad d \star F_0 = -\star \rho_1 \widetilde{V}_{-1} \tag{14}$$

and the fifth step is to solve

$$\begin{aligned} &\nabla_{V_{-1}} \widetilde{V}_0 + \nabla_{V_0} \widetilde{V}_{-1} = i_{V_{-1}} F_0 + i_{V_0} F_{-1}, \\ &g(V_{-1}, V_0) = 0 \end{aligned} \tag{15}$$

for V_0. The sixth step is to enforce charge conservation to next-to-leading order by solving

$$d \star \left(\rho_2 \widetilde{V}_{-1}\right) + d \star \left(\rho_1 \widetilde{V}_0\right) = 0 \tag{16}$$

for ρ_2, leading to the seventh step which is to solve the Maxwell equations

$$dF_1 = 0, \; d \star F_1 = -\star \rho_2 \widetilde{V}_{-1} - \star \rho_1 \widetilde{V}_0 \tag{17}$$

for F_1. This procedure is continued to any order in ε and leads to expressions for F^ε, V^ε and ρ^ε to any desired level of accuracy.

The key point to note is that the only equation non-linear in its unknown (the vector field V_{-1}) is (12). However, (12) is straightforward to analyse because it can be reduced to a quasi-linear second order ordinary differential equation for the integral curves of V_{-1}. It can be shown that the equations for V_n with $n \geq 0$, such as (15) for V_0, written out in a frame adapted to V_{-1} lead to to three inhomogeneous linear differential equations determining three of the components of V_n. The remaining component is the solution to a linear algebraic equation involving fields of lower order in ε.

EXAMPLE

A simple application of the approximation scheme is to consider a high-energy charged beam propagating in free space where the external field F_{-1} is zero. Following the solution method introduced in the previous section, it may be shown that the following 1-parameter fields are approximate solutions to (2):

$$\begin{aligned} V^\varepsilon = \; & \left(\tfrac{1}{\varepsilon}\gamma_{-1} + \tfrac{\varepsilon}{4\gamma_{-1}}\right)\partial_t \\ & + \left(\tfrac{1}{\varepsilon}\gamma_{-1} - \tfrac{\varepsilon}{4\gamma_{-1}}\right)\partial_z + O(\varepsilon^2) \\ F^\varepsilon = \; & -(d\Phi_0 + \varepsilon d\Phi_1) \wedge dt \\ & + (d\Phi_0 + \varepsilon d\Phi_1) \wedge dz + O(\varepsilon^2) \end{aligned} \tag{18}$$

where

$$\begin{aligned} &\gamma_{-1} = \hat{\gamma}(x, y), \\ &\rho_1 = \hat{\rho}_1(z - t, x, y), \;\; \rho_2 = \hat{\rho}_2(z - t, x, y) \end{aligned} \tag{19}$$

are fixed by by their forms at an instant in t given as data. The fields Φ_0 and Φ_1 are solutions to the transverse Poisson equations

$$\begin{aligned} d_\perp \#_\perp d_\perp \Phi_0 = \gamma_{-1}\rho_1 \#_\perp 1, \\ d_\perp \#_\perp d_\perp \Phi_1 = \gamma_{-1}\rho_2 \#_\perp 1 \end{aligned} \tag{20}$$

where $\#_\perp 1$ and $d_\perp$ are the volume 2-form and exterior derivative in the (x, y) planes. The 3-velocity of the beam is along the z axis and has the subluminal Newtonian speed $1 - \frac{\varepsilon^2}{2\gamma_{-1}^2} + O(\varepsilon^3)$.

A Gaussian bunch with transverse radius R_0 travelling at constant Newtonian speed $1 - \frac{\varepsilon^2}{2b_0^2} + O(\varepsilon^2)$ is described by

$$\begin{aligned} &\hat{\gamma}_{-1}(x, y) = b_0, \\ &\hat{\rho}_1(z, x, y) = a_0 \exp\left(-\tfrac{x^2+y^2}{R_0^2}\right)\Xi(z) \end{aligned} \tag{21}$$

where a_0, R_0 and b_0 are constants and Ξ is a smooth bump function vanishing outside the interval $(-z_1, z_1)$ and $\Xi(z) = 1$ for $z \in (-z_2, z_2)$ and $z_1 > z_2 > 0$. The leading order laboratory reduced charge density $\gamma_{-1}\rho_1$ for some range of t is

$$\begin{aligned} \gamma_{-1}\rho_1 \;\; &= \hat{\gamma}_{-1}(x, y)\hat{\rho}_1(z - t, x, y) \\ &= a_0\, b_0 \exp\left(-\tfrac{x^2+y^2}{R^2}\right)\Xi(z - t). \end{aligned} \tag{22}$$

Working in the cylindrical polar coordinates (t, R, ϕ, z) where $x = R\cos\phi$ and $y = R\sin\phi$, a cylindrically symmetric solution to (20) well-behaved at $R = 0$ is

$$\Phi_0 = \left\{\int_0^R a_0\, b_0\, \frac{R_0^2}{2s}\left[1 - \exp\left(-\frac{s^2}{R_0^2}\right)\right] ds\right\}\Xi(z - t) \tag{23}$$

and the corresponding electromagnetic 2-form F_0 is

$$F_0 = a_0\, b_0\, \frac{R_0^2}{2R}\left[1 - \exp\left(-\frac{R^2}{R_0^2}\right)\right]\Xi(z-t)\, dR \wedge (-dt+dz). \tag{24}$$

The laboratory electric field is radial, the magnetic field is azimuthal and their magnitudes are equal and vanish outside of the support of Ξ.

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