A CONTROL THEORY APPROACH FOR DYNAMIC APERTURE*

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Abstract

The dynamic aperture problem dates back to the design of the first synchrotrons. Over time, both analytical and numerical methods have been pursued. In the former case mainly by applying techniques developed for celestial mechanics to rather simplified equations of motion. Over the last decade, analysis of the Poincaré map has become the method of choice. In particular, application of symplectic integrators, truncated power series algebra, and Lie series techniques has led to a complete set of tools for self-consistent numerical simulations- and analytic treatment of realistic models. Nevertheless, a control theory for the general nonlinear case remains elusive. We summarize how to apply this framework to the design of modern synchrotron light sources. Moreover, we also outline how a control theory can be formulated based on the Lie generators for the nonlinear terms.

INTRODUCTION

The modern approach [1-3] enables a straightforward pursuit of self-consistent analytical- and numerical studies of realistic models that includes the effects of engineering tolerances and radiation damping. In particular, numerical integration of the equations of motion, extraction of the corresponding Taylor maps to arbitrary order, and a decomposition of the dynamics into a linear map and a Lie generator for the nonlinear part. However, while:

- the KAM theorem [4-6] justifies a perturbative approach, roughly, the KAM-tori survive for sufficiently smooth and small perturbations away from resonances,
- and the Nekhoroshev theorem [7] shows that the confinement time grows exponentially with the inverse of the magnitude of the perturbation[1]

they are of limited use for quantitative work. In particular, for the design of a strongly nonlinear dynamical system; mainly because a rigorous treatment of stability requires very strict assumptions for the strength of the perturbation [8].

CHROMATIC CORRECTION

Traditional design strategies to introduce sextupoles for (linear) chromatic correction are:

I. Anti-symmetry: introduce pairs of sextupoles separated by modulo-π phase advance in the horizontal- and vertical planes.
II. Higher order achromat: introduce a unit cell with a cell tune such that all the sextupolar modes are cancelled over *N* cells.

However, the first approach drives higher order chromaticity and the second both amplitude dependent tune shift and higher order chromaticity.

More precisely, the Poincarè map has the (formal) Lie series representation [9]

$$\mathcal{M}^k = \left(e^{:h:}\mathcal{M}_{\text{linear}}\right)^k = \mathcal{A}^{-1}\left(e^{:h_3+h_4+\ldots:}\mathcal{R}\right)^k \mathcal{A}$$

where

$$h_3 = \sum_{\bar{J}} h_{jklmp}\left(2J_x\right)^{(j+k)/2}\left(2J_y\right)^{(l+m)/2}\delta^p e^{i\left[(j-k)\phi_x+(l-m)\phi_y\right]} + \text{c.c.}$$

$$h_{jklmp} = h^*_{kjmlp} = \sum_{n=1}^{N}\left(b_3L\right)_n \beta_{xn}^{(j+k)/2}\beta_{yn}^{(l+m)/2}\eta_x^p e^{i\left[(j-k)\mu_{xn}+(l-m)\mu_{yn}\right]}$$

$$h_4 = \frac{1}{2}\sum_{\bar{J}_1,\bar{J}_2}\left[h_{3\bar{I}}\left(2J\right)^{\bar{J}_1/2}\delta^{p_1}, h_{3\bar{I}}\left(2J\right)^{\bar{J}_2/2}\delta^{p_1}\right] + \text{c.c.}$$

Clearly, the long term stability depends on h, $\mathcal{R}$, and the initial conditions. Moreover, h depends on the cell tune as well. It is straightforward to show that h is zero for the geometric terms[2] for case I, and that the sextupolar modes[3] can be cancelled over *N* cells for case II.

By generalizing the second approach, our strategy is: given a set of sextupole families (~10) for the supercell, minimize h for N cells over a range of cell tunes and determine the dynamic aperture by tracking. To second order in the sextupole strengths, there are [10]:

- 2+3+2 chromatic terms,
- 5+8 geometric terms (modes),
- 3+3 tune shift with amplitude and momentum

i.e. a total of 26 terms. In particular:

1. For a given cell tune, calculate h and its parametric dependence (the gradient) on the sextupole strengths for $J_{x,y}$ and δ at the anticipated dynamic aperture.
2. Minimize $\|h\|$ with e.g. the steepest descent method and evaluate the dynamic aperture.
3. Change the cell tune by adjusting the quadrupoles in the matching sections and repeat steps 1-3.

The algorithm can be automated. Also, the off-momentum aperture can be included by using a weighted average for the dynamic aperture.

A robust solution is obtained by establishing a broad local optimum; for the bare lattice. A typical result for a 10×10 grid of cell tunes is shown in Figure 1. Local optimization for a select set of cell tunes can then be pursued by e.g. fine tuning weights for the various terms in the generator and including even higher order terms. A prerequisite for an effective approach is the implementation of an analytical/numerical computational

[1] But does not exclude the possibility of chaotic motion.

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[2] For thin sextupoles.

[3] Phase dependent terms.

framework that has become known as the polymorphic tracking code [11,12].

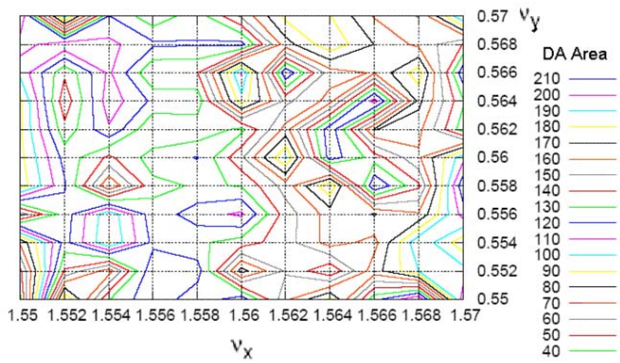

Figure 1: Tune Scan of Normalized Dynamic Aperture for the Original TBA-24 Cell.

APPLICATION TO TBA- AND DBA LATTICES

In general, the dynamic aperture is a complex structure in phase space; known as a Cantor set [13]. So, while short term tracking may give a rough outline of the boundary between stable- and unstable motion, it provides little insight into the dynamics inside the presumably stable region. At PAC05, we reported on preliminary work for a TBA-24 lattice [14]. While the prototype lattice had the desirable natural horizontal emittance, (linear) chromatic correction led to rather strong chromatic sextupoles. An in-depth attempt was therefore made to control the nonlinear dynamics with 9 sextupole families and by e.g. including amplitude dependent tune shifts[4] to cubic terms in $J_{x,y}$; leading to corrections with $(b_3 L)$ up to ~8.6 m^{-2}.

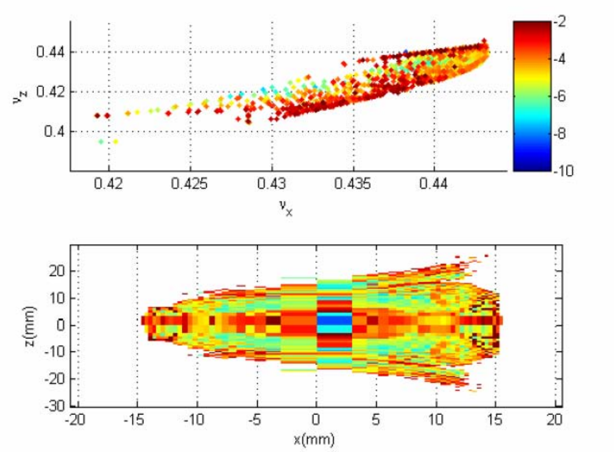

Figure 2: Frequency Map[5] for the Original TBA-24 Cell ($\beta_x = 3.0\,\mathrm{m}, \beta_y = 5.5\,\mathrm{m}$).

[4] To 6th order in the sextupole strength.

[5] Amplitude dependent tune shift and diffusion map.

However, shortly thereafter[6] a frequency map [15] was generated; see Figure 2, which led us to conclude that the optics had to be relaxed [16]. In particular, by using the guideline:

- a horizontal chromaticity per cell of ~3,
- and a peak dispersion of ~0.3 m

for a more realistic approach.

The algorithm outlined above has been used to evaluate other candidate lattices as well. In particular, a DBA-30 option [17,18]. A comparison of the magnitude of the residual Lie generators is shown in Table 1 and the corresponding dynamic apertures in Figure 3. The 10 first- and 16 second order terms can be controlled with ~10 free parameters largely because the latter appears due to cross terms of the former.

Not surprisingly, the amplitude dependent tune shift is the main limiting factor for the original TBA-24 cell. Note also that the h_{20001} term is excited systematically in the TBA structure since the horizontal cell tune is ~1.5. Similarly, h_{10002} is intrinsic to the DBA structure because the horizontal tune of the supercell is ~1.0. The former generates momentum dependence of the horizontal beta function and the latter second order horizontal dispersion. Combined with the strong sextupoles, this leads to significant second- and higher order chromaticity in the horizontal plane [10,18]. Presumably, h_{10002} could be reduced/controlled if it was included in the linear optics optimizations. Also, the magnitude of the residual Lie operators provide a guideline for acceptable magnitudes of other nonlinearietes, e.g. from insertion devices [19].

Due to the scaling of the generators mentioned earlier, there are basically only two groups of terms: sextupolar modes[7]- and tune shifts. Since the former are phase dependent, they may be cancelled over the *N* cells in the achromat, whereas the latter grows systematically. Hence, a scale factor is introduced to take this into account. It appears that, after it has been determining for some specific lattice, little is gained by adjusting it or introducing individual weights for the other terms; even when third- and higher order terms are included. To summarize, the outlined algorithm is generic and apparently does not require the introduction of numerous heuristic parameters. Regardless, the frequency map may be found useful as a diagnostic tool for further fine tuning.

CONCLUSION

We have developed an algorithm for dynamic aperture optimization of TBA- and DBA lattices. In particular, for a given cell tune, the magnitude of the Lie generator for the nonlinear dynamics is minimized at the dynamic aperture, and the dynamic aperture evaluated by tracking. Moreover, the cell tune is optimized by repeating the process over a range of cell tunes. The algorithm has

[6] In collaboration with L. Nadolski at the SOLEIL project.

[7] Which contribute to the amplitude dependent tune shift as well, i.e. in the map normal form.

been automated by extending the analytical- and numerical framework originally prototyped at the CBP, LBNL. The approach has essentially only one free parameter. During the development, we have found the frequency map analysis to be a very useful diagnostic tool.

Table 1: Normalized Lie Generators to Second Order in the Sextupole Strength.

Lie Generator	Effect	Original TBA-24	Relaxed TBA-24	DBA-30
$\lvert h_{11001}\rvert$	$\partial\nu_x/\partial\delta$	5.5e-9	1.4e-8	1.6e-11
$\lvert h_{00111}\rvert$	$\partial\nu_y/\partial\delta$	6.2e-10	3.0e-9	1.3e-12
$\lvert h_{10002}\rvert$	$\partial\eta_x/\partial\delta$	8.3e-8	6.1e-8	3.3e-6
$\lvert h_{20001}\rvert$	$\nu_x \pm \nu_s$	1.5e-5	3.4e-6	6.0e-7
$\lvert h_{00201}\rvert$	$\nu_y \pm \nu_s$	4.6e-7	1.2e-7	3.5e-8
$\lvert h_{21000}\rvert$	ν_x	4.9e-6	2.0e-6	6.4e-7
$\lvert h_{10110}\rvert$	ν_x	1.1e-6	2.4e-7	1.5e-7
$\lvert h_{30000}\rvert$	$3\nu_x$	3.2e-6	1.3e-6	2.1e-8
$\lvert h_{10020}\rvert$	$\nu_x - 2\nu_y$	1.2e-6	6.6e-7	5.4e-8
$\lvert h_{10200}\rvert$	$\nu_x + 2\nu_y$	3.9e-6	1.0e-6	7.1e-7
$\lvert h_{20110}\rvert$	$2\nu_x$	6.6e-6	5.3e-7	2.2e-8
$\lvert h_{31000}\rvert$	$2\nu_x$	3.6e-5	1.8e-6	8.8e-8
$\lvert h_{40000}\rvert$	$4\nu_x$	3.4e-7	1.6e-7	7.4e-9
$\lvert h_{20020}\rvert$	$2\nu_x - 2\nu_y$	6.7e-6	1.5e-8	3.4e-7
$\lvert h_{20200}\rvert$	$2\nu_x + 2\nu_y$	2.9e-8	3.3e-7	1.3e-9
$\lvert h_{11200}\rvert$	$2\nu_y$	1.2e-6	3.3e-8	5.5e-8
$\lvert h_{00310}\rvert$	$2\nu_y$	9.4e-9	7.9e-8	1.5e-7
$\lvert h_{00400}\rvert$	$4\nu_y$	1.4e-8	1.3e-7	2.3e-8
$\lvert h_{22000}\rvert$	$\partial\nu_x/\partial J_x$	3.5e-5	1.9e-6	1.1e-6
$\lvert h_{11110}\rvert$	$\partial\nu_{x,y}/\partial J_{y,x}$	1/1e-5	9.1e-7	1.3e-7
$\lvert h_{00220}\rvert$	$\partial\nu_y/\partial J_y$	2.7e-6	5.6e-7	6.9e-7
$\lvert h_{22001}\rvert$	$\partial^2\nu_x/\partial J_x\partial\delta$	5.4e-6	1.2e-6	7.0e-7
$\lvert h_{11111}\rvert$	$\partial^2\nu_{x,y}/\partial J_{y,x}\partial\delta$	3.3e-6	8.0e-7	3.7e-7
$\lvert h_{00221}\rvert$	$\partial^2\nu_y/\partial J_y\partial\delta$	6.3e-7	3.0e-7	7.6e-8
$\lvert h_{11002}\rvert$	$\partial^2\nu_x/\partial\delta^2$	1.1e-6	1.7e-6	3.9e-6
$\lvert h_{00112}\rvert$	$\partial^2\nu_y/\partial\delta^2$	1.8e-7	5.2e-8	1.1e-7

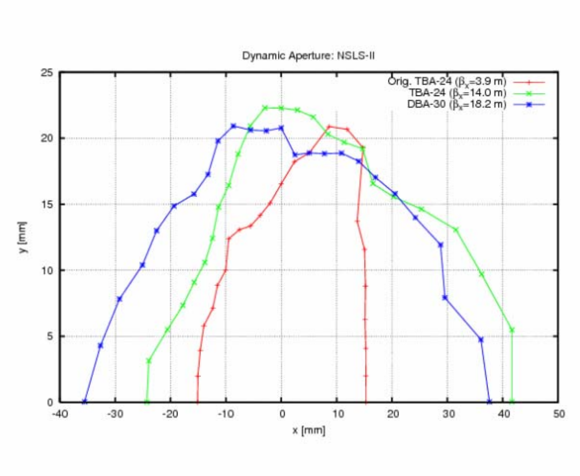

Figure 3: Dynamic Aperture (normalized with $\sqrt{\beta_x\beta_y}$) for the Original TBA-24, TBA-24, and DBA-30 Options.

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