THE STRICT SOLUTION OF A RADIATION PROBLEM IN TOROIDAL CAVITY

M.A. Khojoyan, E.D. Gazazyan, YerPhi, Yerevan, Armenia
T.A. Harutyunyan, YSU, Yerevan, Armenia

Abstract

The radiation of the charged particles bunch which is moving along the axis of toroidal cavity cross section is considered. The toroidal cavity has a finite value of the quality factor and is filled with special symmetry inhomogeneous dielectric medium. The problem's solution is based on the complete set of the toroidal cavity's own modes being defined strictly for the mentioned dielectric medium the cavity is filled with. The charged particles bunch exists in the cavity during a finite time period and the charged bunch's arising and vanishing effects are examined and are taken into account as well. The toroidal cavity is considered as a convenient model to investigate the electromagnetic properties of the tokamak system, using the defined modes.

INTRODUCTION

Toroidal cavities represent interest in many aspects of physics. First of all, they are interesting as merely a radiotechnic module, then – as a simple model of a circular accelerator, as a synchrotron radiation source and, what is most important today, toroidal cavity is a convenient model of a tokamak system [1].

Toroidal cavity can be described in different systems of coordinates, particularly, in [2] the problem is considered in the natural system of coordinates, in [3-5] – in quasi-spherical and in [6-7] – in toroidal systems. But in all of these systems it is not possible to separate variables in the wave equation (only longitudinal variable is separated), and hence, can't be determined the own functions of the toroid analytically. For this reason there are different asymptotic methods to define the own frequencies and own functions of the toroid.

But in this paper we will not consider any of these methods. We will follow the papers [6-7] where the problem of eigenvalues and eigenfunctions is considered in toroidal system of coordinates and the variable separation can be achieved by filling the toroid by an inhomogeneous dielectric medium of a definite type, which has a cylindrical symmetry.

The toroidal coordinates (τ,σ,φ) are connected with the Cartesian coordinates by the following formula:

$$x=\frac{a\,\mathrm{sh}\,\tau}{\mathrm{ch}\,\tau-\cos\sigma}\cos\varphi,\quad y=\frac{a\,\mathrm{sh}\,\tau}{\mathrm{ch}\,\tau-\cos\sigma}\sin\varphi,\quad z=\frac{a\sin\sigma}{\mathrm{ch}\,\tau-\cos\sigma}. \tag{1}$$

In this system of coordinates the given toroid is defined from the equation

$$\tau=\tau_1, \tag{2}$$

where τ_1 represents toroid with the radius $R=a\,\mathrm{cth}\,\tau_1$ and the cross-section radius. $r=a/\mathrm{sh}\,\tau_1$. a is a constant that describes the toroidal system.

As it is shown in [6-7] that if we represent the electromagnetic field in the following form

$$\begin{aligned}\mathbf{E}(\tau,\sigma,\varphi,t)&=\mathbf{E}(\tau,\sigma)\exp(-i(m\varphi+\omega t)),\\ \mathbf{H}(\tau,\sigma,\varphi,t)&=\mathbf{H}(\tau,\sigma)\exp(-i(m\varphi+\omega t)),\end{aligned} \tag{3}$$

where $m=0,1,2,\ldots$, then if the field in the toroid we separate to E-type ($E_\varphi\neq 0, H_\varphi=0$) and H-type ($E_\varphi=0, H_\varphi\neq 0$) of waves, then we will get all the transverse coordinates of the toroid, expressed by the longitudinal component of electric or magnetic fields, i.e. the analytical expressions for E and H types of waves with the corresponding boundary conditions.

For the longitudinal component we get an equation in which the variables can't be separated analytically. But if we consider that the toroid is filled by an inhomogeneous medium of the form

$$\varepsilon=\varepsilon(\tau,\sigma)=\frac{h^2}{sh^2\tau}, \tag{4}$$

where $h=h(\tau,\sigma)=ch\tau-\cos\sigma$, then the variables can be separated and we get the own functions of the toroid:

$$E_\varphi(\tau,\sigma,\varphi)=A(\mathrm{ch}\,\tau-\cos\sigma)^{1/2}f_E(\tau)\sin(p\sigma)\exp(-im\varphi), \tag{5a}$$

$$H_\varphi(\tau,\sigma,\varphi)=B\sin\frac{\sigma}{2}(\mathrm{ch}\,\tau-\cos\sigma)^{3/2}f_H(\tau)f_2(\sigma)\exp(-im\varphi). \tag{5b}$$

In these solutions the functions $f_E(\tau)$ and $f_H(\tau)$ satisfy differential equations of the second order and when $ka=0$, they both become the equation of the torus functions [8]. These equations can't be solved analytically and hence, one must look for digital methods of the solutions.

As it is shown in [7], both the functions E_φ and H_φ are orthogonal. The function E_φ is orthogonal with a weight ε, where ε is the dielectric medium the toroid is filled with, and the function H_φ is orthogonal with a weight 1.

As the own functions of the toroid are orthogonal, it means that one can expand any physical amount in the toroid by a series of these functions. Using this property we will discuss the radiation problem of a charged particle in the toroid in next chapter.

THE RADIATION PROBLEM IN THE TOROIDAL CAVITY

Let the charged particle (electron) moves along the cross-section axis of the toroid at a constant velocity $\mathrm{v} = \mathrm{v}_\varphi$. We consider again that the toroid is filled with inhomogeneous medium (4). The external magnetic field H_0 keeps the particle on the given orbit with the Larmor frequency

$$\Omega = \frac{\mathrm{v}}{R} = \frac{eH_0}{m_e c\gamma}, \tag{6}$$

where e and m_e are the charge and the mass of the particle respectively, and γ is the Lorenz-factor.

The radius of the orbit coincides with the radius of the toroid $R = a\,\mathrm{cth}\,\tau_1$. If the particle is injected the toroid at the moment of time $t = 0$ to the point with the coordinates $(\tau_0, \sigma_0, \varphi_0)$, then the current j_φ will have the form

$$\begin{aligned} j_\varphi(\tau,\sigma,\varphi,t) &= \frac{h^3}{a^3\,\mathrm{sh}\,\tau} e\mathrm{v}\times \\ &\times\delta(\tau-\tau_0)\delta(\sigma-\sigma_0)\delta(\varphi-\Omega t)\times \\ &\times\left[\theta(t)+(1-\theta(t-T))\right], \end{aligned} \tag{7}$$

where

$$\theta(t) = \begin{cases} 1, & t \ge 0, \\ 0, & t < 0. \end{cases} \tag{8}$$

In this case the function will have the following form:

$$j_\varphi(\tau,\sigma,\varphi,\omega) = \frac{1}{2\pi}\frac{h^3 e\mathrm{v}}{a^3\,\mathrm{sh}\,\tau}\delta(\tau-\tau_0)\delta(\sigma-\sigma_0)\times$$

$$\times\frac{1}{\Omega}\exp(i\omega\frac{\varphi}{\Omega}). \tag{9}$$

From the Maxwell equations one can get the equation for the function in the toroidal system of coordinates:

$$\nabla^2 E_\varphi(\omega) + \varepsilon\frac{\omega^2}{c^2}E_\varphi(\omega) + \frac{1}{\varepsilon}\mathrm{grad}_\varphi(\mathbf{E}\cdot\mathrm{grad}\,\varepsilon) =$$

$$= -\frac{4\pi i\omega}{c^2}\left(1-\frac{c^2}{a^2\Omega^2}\right)j_\varphi(\omega). \tag{10}$$

The solution of this equation will be done expanding the field and the current to a series by the own functions:

$$E_\varphi(\tau,\sigma,\varphi,\omega)\sum_{p,m} E_\varphi^{pm}(\omega)\psi_E^{pm}(\tau,\sigma,\varphi), \tag{11a}$$

$$j_\varphi(\tau,\sigma,\varphi,\omega)\sum_{p,m} j_\varphi^{pm}(\omega)\psi_E^{pm}(\tau,\sigma,\varphi). \tag{11b}$$

If we consider that the particle moves in the toroid for a period, we will get:

$$j_\varphi^{pm}(\omega) = -\frac{i\Omega Ce\mathrm{v}}{\omega-m\Omega}(\exp[i(\omega-m\Omega)T]-1). \tag{12}$$

From the equations (11)-(12) we get:

$$E_\varphi^{pm}(\omega) = -\frac{4\pi Ce\mathrm{v}(\beta^2-1)}{\varepsilon\beta^2}\times$$

$$\times\frac{\omega}{(\omega-\omega_{pm})(\omega+\omega_{pm})(\omega-m\Omega)}(\exp[i(\omega-m\Omega)T]-1). \tag{13}$$

If the toroidal cavity had ideal conducting walls, then its quality factor would be infinite, and the own oscillations would have infinite narrow width of frequencies. But each real system has a finite conductivity of walls, i.e. the spectral lines of the frequencies have a finite width $\Delta\omega_{pm} = \omega_{pm}/Q_{pm}$, where p and m are the numbers of modes, ω_{pm} are the own frequencies, Q_{pm} - the quality factor of the cavity corresponding to a given mode. In this kind of cavity the structure of the field will not change, i.e. will be described by the expressions of an ideal cavity [9].

We need to define the time dependence of the field. Replacing (13) to integral

$$E_\varphi^{pm}(t) = \frac{1}{2\pi}\int E_\varphi^{pm}(\omega)\mathrm{e}^{-i\omega t}\,d\omega, \tag{14}$$

we see that when ,the amplitude of the field diverges conditioned by choice of ideal cavity, whereas in reality, the amplitude of the field must increase, staying finite. In this case only modes p and m in the cavity will excited. The existence of the quality factor will bring to the broadening of the spectral lines. Because of the fading the frequencies will have an imaginary part, expressed by the quality factor:

$$\omega_{pm}(1-i/2Q) \equiv \omega_{pm} - i\eta, \quad \eta = \frac{\omega_{pm}}{2Q} > 0. \tag{15}$$

In this case (13) will have the following form:

$$E_\varphi^{pm}(\omega)=\frac{4\pi Ce\,\mathrm{v}(\beta^2-1)}{\varepsilon\beta^2}\times$$
$$\times\frac{\omega[\mathrm{e}^{i(\omega-m\Omega)T}-1]}{(\omega-\omega_{pm}+i\eta)(\omega+\omega_{pm}-i\eta)(\omega-m\Omega)}. \quad (16)$$

Inserting (16) into (14) one can see that the function under the integral has singularity in the form of simple poles on the frequencies $\omega=\omega_{pm}-i\eta$, $\omega=-\omega_{pm}+i\eta$, $\omega=m\Omega$. The contour of the integration will be chosen in the lower half plane. In that case in the integral of Koshi only the poles $\omega=\omega_{pm}-i\eta$ and $\omega=m\Omega$ will contribute.

Calculating the integral one can get the following solution, when $\omega_{pm}\neq m\Omega$:

$$E_\varphi^{pm}(t)=\frac{4\pi iCe\,\mathrm{v}(\beta^2-1)}{\varepsilon\beta^2}\times$$
$$\times\frac{\mathrm{e}^{-i\omega_{pm}(t-T)}\mathrm{e}^{-\eta(t-T)}\mathrm{e}^{-im\Omega T}-\mathrm{e}^{-i\omega_{pm}t}\mathrm{e}^{-\eta t}}{\omega_{pm}-m\Omega-i\eta}. \quad (17)$$

When the rotation frequency of the particle coincides with the one of the own frequencies ($\omega_{pm}\to m\Omega$), we will get:

$$\int E_\varphi^{pm}(\omega)e^{-i\omega t}d\omega=-\frac{\pi}{\eta}\cdot(e^{-i\omega_{pm}t}e^{-\eta t}(e^{\eta T}-1)). \quad (18)$$

Replacing the form of η, one can get:

$$\int E_\varphi^{pm}(\omega)e^{-i\omega t}d\omega=$$
$$=-\frac{2\pi Q}{\omega_{pm}}(e^{-[\omega_{pm}(t-T)]/2Q}-e^{-\omega_{pm}t/2Q})e^{-i\omega_{pm}t}. \quad (19)$$

As it is seen from (19), when $\omega_{pm}\sim m\Omega$, the amplitude of the field increases, proportionally to a quality factor Q. If the energy is injected to a cavity form of a monochromatic wave with a frequency coinciding with one of the own frequencies $(m\Omega=\omega_{pm})$, then the voltage of the field, corresponding to that frequency, first of all, when $t\ll 2Q/\omega_{pm}$, will increase with a linear rule, proportional to a time t, and when $t\gg 2Q/\omega_{pm}$, it will increase, proportionally to a quality factor Q [10]:

$$\int E_{\varphi\omega}^{pm}\mathrm{e}^{-i\omega t}d\omega=-\frac{2\pi Q}{\omega_{pm}}(1-\mathrm{e}^{-\omega_{pm}T/2Q})\mathrm{e}^{-i\omega_{pm}T}. \quad (20)$$

A dynamic balance will be created in the cavity, i.e. the further gain of the voltage will be compensated by the loss in the walls. In the cavities with ideal conducting walls the voltage of the field will increase infinitely during the infinite period of time, which is impossible from the point of view of physics.

At the time $t=T$ the particle flies from the cavity, hence as it can be seen from (20), at the times $t\gg T$ the voltage of the field will decrease and will fade:

$$\int E_\varphi^{pm}(\omega)\mathrm{e}^{-i\omega t}d\omega\to 0. \quad (21)$$

The authors would like to express his gratitude to Dr. M. Ivanyan and PhD V. Kocharyan for their valuable comments.

REFERENCES

[1] Huysmans G. Magneto-Hydro-Dynamics in Tokamak Plasmas. Principles of Magnetohydrodynamics from 21 to 24 Mar. 2005 – St-Paul-lez-Durance., 2005.

[2] Harutyunyan T.A., Kalantaryan D.K. Electromagnetic oscillations in the toroidal cavities. Proc. of National Academiy of Science of Armenia, Physics vol. 40 pp.244-247 (2005).

[3] Lileg.L., Schizer B., Keil R. Perturbation Theoretic Computation of Toroidal Uniform Modes with in Empty Torus // AEU, V.37, pp.359-365 (1983).

[4] Cap.F., Deutsch R. Toroidal Resonators for Electromagnetic Waves // IEEE Trans. V. MTT-28, N7.- pp.700-703. 1980

[5] Kocharyan V.G., Synchrotron, Transition and Cherenkov Radiations in Cavities and Waveguides, PhD Dissertation, Yerevan Physics Institute After A.Alikhanyan, Yerevan 2003,

[6] E.D.Gazazyan, M.I.Ivanyan, A.D.Ter-Poghosyan, Scalar and Vector Toroidal Oscillations in Big Torus Approximation. Proc.of National Academiy of Science of Armenia, Physics vol.29, pp.141-151 (1994).

[7] E.D.Gazazyan, T.A.Harutyunyan, The Classification of Electromagnetic Oscillations in Toroidal Cavity. Proc. of National Academiy of Science of Armenia, Physics vol. 41 pp.91-98 (2006).

[8] M.Morse, H.Feshbach. Methods of Theoretical Physics, Part II, New-York , Mc Craw Hill Book Company, (1953)

[9] E.D.Gazazyan, G.G. Oxuzyan, A.D.ter-Poghosyan. The Excitation of the Cavity with Finite Q Factor. Proc. of National Academiy of Science of Armenia, Physics vol. 39 pp.87-92 (2004).

[10] E.D.Gazazyan, T.A.Harutyunyan. The Magnetic Bremsstrahlung Effect in Toroidal Cavity.An International Journal of Electromagnetic Waves and Electronic Systems, V.11, pp.6-11, (2006).