COUPLING IMPEDANCES OF SMALL DISCONTINUITIES FOR NON-ULTRARELATIVISTIC BEAMS*

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Abstract

The beam coupling impedances of small discontinuities of an accelerator vacuum chamber have been calculated (e.g., [1]) for ultrarelativistic beams using Bethe's diffraction theory. Here we extend the results for an arbitrary beam velocity. The vacuum chamber is assumed to have an arbitrary, but fixed, cross section. The impedance dependence on the beam velocity exhibits some unusual features. For example, the reactive impedance, which dominates in the ultrarelativistic limit, can vanish at a certain beam velocity, or its magnitude can exceed the ultrarelativistic value many times.

INTRODUCTION

A general analytical approach for calculating the beam coupling impedances of small discontinuities on the walls of an accelerator vacuum chamber has been developed in [2, 1] for ultrarelativistic beams. The method is based on the Bethe theory of diffraction by small holes [3], according to which the fields diffracted by a hole can be found as those radiated by effective electric and magnetic dipoles. The Bethe idea of effective dipoles, applicable for wavelengths large compared to the typical hole size h, was first used to calculate the coupling impedances of pumping holes in a circular waveguide for ultrarelativistic beams [4, 5]. We extend the analytical approach [2, 1] to non-ultrarelativistic beams. The earlier results related to the subject [6, 7] were restricted to an axisymmetric vacuum chamber. Our study treats a more general case of an arbitrary simply-connected chamber cross section.

FIELDS

Let us consider an infinite cylindrical pipe with an arbitrary cross section S and perfectly conducting walls. The z axis is directed along the pipe axis, a small discontinuity (e.g., a hole) is located in the cross section $z = 0$ at the point $(\vec{b}, 0)$, and a typical hole size h satisfies $h \ll b$. The discontinuity is considered small when its size is much smaller than the wavelength of interest. To evaluate the coupling impedance one has to calculate the fields induced in the chamber by a given current. Consider a charge q that moves on or parallel to the chamber axis with velocity $v = \beta c$ and has a "pancake" charge distribution $\rho(\vec{r}, z; t) = q f(\vec{r})\delta(z - \beta ct)$, where $f(\vec{r})$ is a normalized transverse charge density, $\int_S d\vec{r} f(\vec{r}) = 1$.

We will use eigenvalues k_g^2 and orthonormalized eigenfunctions (EFs) $e_g(\vec{r})$ of the Dirichlet boundary problem in S: $\left(\nabla^2 + k_g^2\right) e_g = 0;\ e_g\big|_{\partial S} = 0$, where $g = \{n, m\}$ is a generalized 2D index. The fields harmonics $\vec{E}, \vec{H}$ produced by at the location $(\vec{b}, z)$ on the chamber wall without hole can be expressed in terms of EFs as

$$
\begin{aligned}
E_\nu(\vec{b}, z; \omega) &= Z_0 H_\tau(\vec{b}, z; \omega)/\beta \qquad (1)\\
&= -\frac{Z_0 q}{\beta} \exp\left(i\frac{\omega z}{\beta c}\right) \sum_g \frac{f_g \nabla_\nu e_g(\vec{b})}{k_g^2 + \kappa^2},
\end{aligned}
$$

where $\kappa \equiv \omega/(\beta\gamma c)$. Here $Z_0 = \sqrt{\mu_0/\varepsilon_0} = 120\pi\ \Omega$, $\hat{\nu}$ is an outward normal unit vector and $\hat{\tau}$ is a unit vector tangent to the boundary ∂S of the chamber cross section S, $\vec{\nabla}$ is the 2D gradient in plane S, $\nabla_\nu \equiv \vec{\nabla} \cdot \hat{\nu}$, and $\{\hat{\nu}, \hat{\tau}, \hat{z}\}$ form a right-handed basis. In Eq. (1) f_g are the coefficients of EF expansion $f(\vec{r}) = \sum_g f_g e_g(\vec{r})$; they are given by $f_g = \int_S d\vec{r} f(\vec{r}) e_g(\vec{r})$. For the case of a point charge with the transverse offset $\vec{s}$ from the axis, we have $f_g = e_g(\vec{s})$.

Next we have to calculate the fields scattered into the vacuum chamber by the discontinuity. According to the Bethe theory, the fields radiated by a small discontinuity (hole) into the pipe are equal to those produced by effective electric and magnetic dipoles [3] $P_\nu = \alpha_e \varepsilon_0 E_\nu^h$; $M_\tau = \alpha_m H_\tau^h$, where α_e, α_m are the hole polarizabilities, and 'h' means that the beam fields (1) are taken at the hole location $(\vec{b}, 0)$. For a circular hole of radius h in a thin wall $\alpha_m = 4h^3/3$ and $\alpha_e = -2h^3/3$ [3]; for other shapes see [8].

The static values of α_e, α_m can only be used when the beam fields (1) do not change significantly from one point of the discontinuity to another, i.e. when $\omega h/(\beta c) \ll 1$, which gives the applicability condition for our results. For most small discontinuities, it is satisfied at frequencies of interest. The frequency range can be extended even further if we include frequency corrections to the static polarizabilities, e.g. see [9] for elliptic holes.

After the effective dipoles are found, we calculate the scattered fields as a sum of waveguide eigenmodes excited in the chamber by the dipoles. This approach has been carried out for ultrarelativistic beams in a circular pipe [4], and in an arbitrary chamber [2]. It works in exactly the same way for non-ultrarelativistic beams, see [10] for details.

BEAM COUPLING IMPEDANCES

The **longitudinal impedance** is defined as

$$
Z(\omega) = -\frac{1}{q} \int_{-\infty}^{\infty} dz\, e^{-i\frac{\omega z}{\beta c}} \int_S d\vec{r}\, t(\vec{r}) E_z(\vec{r}, z; \omega), \qquad (2)
$$

where $t(\vec{r})$ is the test-charge transverse distribution normalized by $\int_S d\vec{r}\, t(\vec{r}) = 1$. We include into E_z only the discontinuity contribution. Performing integration leads to

$$
Z(\omega) = -iZ_0\, \omega/c\, e_\nu(f; \kappa) e_\nu(t; \kappa) \left(\alpha_m + \alpha_e/\beta^2\right), \qquad (3)
$$

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where we introduced the following notation

$$e_\nu(f;\kappa) \equiv -\sum_g f_g \nabla_\nu e_g^h / \left(k_g^2+\kappa^2\right) . \quad (4)$$

The normalized transverse field (4) is related to the transverse harmonics of the beam field at the hole location (1):

$$e_\nu(f;\kappa) = \beta E_\nu(\vec{b},0;\omega)/(Z_0 q) = H_\tau(\vec{b},0;\omega)/q . \quad (5)$$

Since the lowest eigenvalue k_g is of the order of $1/b$, for $\omega b/(\beta\gamma c) = \kappa b \ll 1$ Eq. (4) becomes frequency- and velocity-independent. The condition $\omega b/(\beta\gamma c) \ll 1$ includes two cases: (i) ultrarelativistic limit, $\gamma \to \infty$; and (ii) long-wavelength (or low-frequency) limit, when the wavelength $\lambda = 2\pi/k = 2\pi\beta c/\omega$ is large compared to the typical cross-section size b. Equation (4) with $\kappa \to 0$ gives a solution for a 2D electrostatic field created on the chamber wall in the cross section S by a uniform in z charge, equal to ε_0 per unit length of z, that has the transverse distribution $f(\vec{r})$. From the Gauss law, it satisfies the normalization condition $\oint_{\partial S} dl\, e_\nu(f;0) = 1$, where integration goes along the cross-section boundary ∂S. For a simple particular case of a circular cross section with radius b, and an axisymmetric charge distribution $f(\vec{r}) = f(r)$ — it includes an on-axis point charge, — the solution, due to the problem symmetry, is $e_\nu(f;0) = 1/(2\pi b)$, cf. [4, 2].

The usual *monopole longitudinal impedance* is obtained from Eq. (3) with the point charges on axis, i.e. when $f_g = t_g = e_g(0)$, so that

$$Z(\omega) = -iZ_0\,(\omega/c)\, e_\nu^2(0;\kappa)\,(\alpha_m + \alpha_e/\beta^2) . \quad (6)$$

In the ultrarelativistic limit, $\beta \to 1$, $\gamma \to \infty$, Eq. (6) coincides with the known result for a small discontinuity [2, 1].

For simple cross sections S, one can obtain explicit expressions of the normalized field (4) and the longitudinal impedance (6), see [10]. For a circular cross section of radius b, Eq. (6) takes form

$$Z(\omega) = -iZ_0 \frac{\omega}{c} \frac{\alpha_m + \beta^{-2}\alpha_e}{4\pi^2 b^2} \left[I_0(\kappa b)\right]^{-2} , \quad (7)$$

which coincides, up to notations, with the result in [7]. Here $I_0(x)$ is the modified Bessel function of the first kind.

For a rectangular cross section $a \times b$, assuming that the hole is located on the side wall at $x = a$, $y = y_h$, the longitudinal impedance (6) is

$$\begin{aligned} Z(\omega) &= -iZ_0\, \omega/c\, (\alpha_m + \beta^{-2}\alpha_e)/b^2 \times \\ &\times \left[\sum_{p=0}^{\infty} \frac{(-1)^p \sin[\pi(2p+1)y_h/b]}{\cosh(\pi u_{2p+1}/2)}\right]^2 , \quad (8) \end{aligned}$$

where $u_m = a\sqrt{m^2/b^2 + \kappa^2/\pi^2}$. In the ultrarelativistic limit, Eq. (8) coincides with the result in [1].

For an axisymmetric small obstacle on the wall of a circular beam pipe — like a small enlargement (cavity) or an iris — the longitudinal impedance is similar to Eq. (7):

$$Z(\omega) = -iZ_0 \frac{\omega}{c} \frac{\tilde{\alpha}_m + \beta^{-2}\tilde{\alpha}_e}{2\pi b} \left[I_0(\kappa b)\right]^{-2} , \quad (9)$$

where the effective polarizabilities $\tilde{\alpha}_m$ and $\tilde{\alpha}_m$ are now defined per unit length of the circumference $2\pi b$ of the chamber cross section (circle) S. In the ultrarelativistic limit Eq. (9) coincides with the previous results [11, 12]. For a small axisymmetric enlargement with area A of the longitudinal cross section, $\tilde{\alpha}_m = A$, while for an axisymmetric protrusion (iris) of the same cross-section area $\tilde{\alpha}_m = -A$. The electric polarizability $\tilde{\alpha}_e$ can be found by solving a 2D electrostatic problem, see [11, 12]. It is positive for protrusions and negative for enlargements, so that in both cases $\tilde{\alpha}_m$ and $\tilde{\alpha}_e$ have opposite signs.

The longitudinal impedance depend on the beam velocity in two ways: via $e_\nu(\vec{s};\kappa)$ and in the combination of polarizabilities $(\alpha_m + \alpha_e/\beta^2)$. The first dependence enters via the parameter $\kappa b = \omega b/(\beta\gamma c)$, cf. Eqs. (4), (7)-(9). For $\kappa b \ll 1$ the factor $e_\nu^2(\vec{s};\kappa)$ is close to its ultrarelativistic limit, while at $\kappa b > 1$ it decreases exponentially to zero. We should emphasize that for $\beta < 1$ the monopole longitudinal impedance depends on the beam position in the chamber cross section, unlike its ultrarelativistic counterpart. For a circular cross section this dependence takes a particularly simple form as an additional factor of $I_0^2(\kappa t)$ in (7) and (9), where t is the beam transverse displacement from the chamber axis.

In the combination $\alpha_m + \alpha_e/\beta^2$ the electric contribution is enhanced as the beam velocity decreases. This sum (more exactly, this difference, because α_m and α_e always have opposite signs) can either vanish for some values of β, or become much larger than its ultrarelativistic limit $\alpha_m + \alpha_e$. It vanishes when $0 < \beta = \sqrt{-\alpha_e/\alpha_m} < 1$. This situation occurs only for discontinuities like holes or chamber enlargements (small cavities), since for them $\alpha_m > |\alpha_e|$ [2]. For a circular hole $\sqrt{-\alpha_e/\alpha_m} = 1/\sqrt{2}$. The impedance (6) of the hole, which is inductive for relativistic beams, changes its sign for $\beta < 1/\sqrt{2}$ becoming a "negative inductance". On the other hand, for protrusions and irises the impedance remains inductive for any beam velocity because of $\alpha_e > |\alpha_m|$, cf. [12, 11]. For example, a semi-spherical protrusion (bump) of radius a on the wall has polarizabilities $\alpha_e = 2\pi a^3$ and $\alpha_m = -\pi a^3$ [12].

For all small discontinuities the impedance vanishes at very slow beam velocities, when $\beta \to 0$, since the fast decrease of the factor $e_\nu^2(\vec{s};\kappa)$ suppresses the growth due to α_e/β^2. This behavior is illustrated in Fig. 1 for a hole and in Fig. 2 for a protrusion, both in a circular cylindrical chamber with the radius b of its cross section. For other cross sections the impedance behavior is similar: the impedance magnitude can exceed the ultrarelativistic value many times. In fact, the ratio $Z(\beta)/Z(1)$ for $\omega b/c = 0.1$ in Fig. 1 reaches -83.3 at $\beta = 0.062$, and in Fig. 2 its maximum is 167.5, well outside the shown range. The extremes become even larger for lower frequencies. It is worthwhile to note that the results in Figs. 1-2 are independent of the discontinuity size h provided that $h \ll b$ and, of course, the applicability condition is satisfied, i.e. $\omega h \ll \beta c$.

For long elliptic slots parallel to the chamber axis, when the ellipse semi-axes w, l satisfy $w \ll l \ll b$, the leading

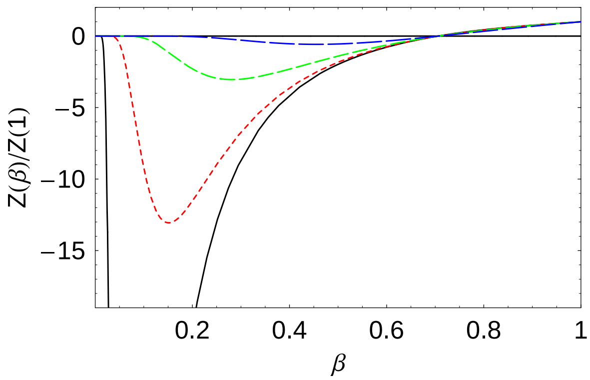

Figure 1: Ratio of impedance (7) to its relativistic value for a circular hole versus β for $\omega b/c = 0.1, 0.25, 0.5, 1$ (solid, short-dashed, dashed, and long-dashed curves).

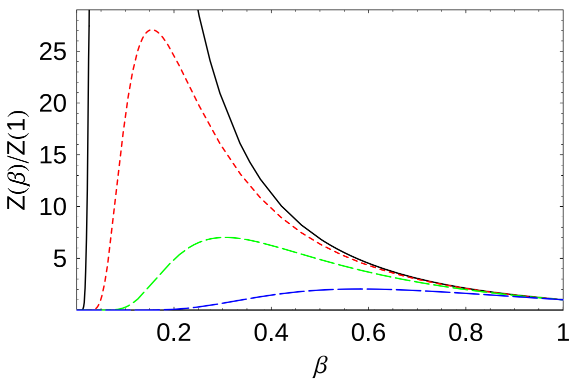

Figure 2: The same, for a small semi-spherical protrusion.

terms ($\propto w^2 l$) of the static polarizabilities α_m and α_e for a thin wall cancel each other [4, 2] in the ultrarelativistic limit. For $\beta < 1$ there is no such cancellation:

$$\alpha_m + \frac{\alpha_e}{\beta^2} \approx \frac{\pi w^2 l}{3}\left[\frac{w^2}{l^2}\left(\frac{1+\beta^2}{2\beta^2}\ln\frac{4l}{w} - \frac{1}{4\beta^2} - \frac{3}{4}\right) - \right. \\ \left. -\frac{1}{5}\left(2-\frac{1}{\beta^2}\right)\left(\frac{\omega l}{\beta c}\right)^2 - \frac{1}{\beta^2\gamma^2}\right]. \quad (10)$$

Here we used the frequency corrections to the static polarizabilities α_e, α_m from [9]. The first term in (10) is the next-to-leading order static term, while the last term is the leading static contribution for $\beta < 1$. The frequency correction, the second term in (10), vanishes when $\beta = 1/\sqrt{2}$ and changes sign at lower beam velocities. For small βs, Eq. (10) has the opposite sign and larger magnitude compared to its relativistic limit.

The **transverse dipole impedance** of a small discontinuity (see derivation in [10]) is

$$\vec{Z}_\perp(\omega) = -iZ_0\beta(\alpha_m + \beta^{-2}\alpha_e)[\hat{s}\cdot\vec{d}(\kappa)]\,\vec{d}(\kappa), \quad (11)$$

where $\hat{s} = \vec{s}/s$ is a unit vector in the direction of the beam deflection from the chamber axis, and

$$\vec{d}(\kappa) \equiv \vec{\nabla} e_\nu(0,\kappa) = -\sum_g \frac{\vec{\nabla} e_g(0)\nabla_\nu e_g^h}{k_g^2+\kappa^2}. \quad (12)$$

The impedance dependence on the discontinuity shape is obviously the same as for the longitudinal impedance. The direction of the vector of the transverse impedance (11) gives the direction of the deflecting force acting on a displaced beam. As one can see from Eq. (11), this direction is defined by vector $\vec{d}$, Eq. (12), while the force magnitude varies depending on the relative direction of the beam displacement $\vec{s}$ with respect to $\vec{d}$ via the scalar product $\hat{s}\cdot\vec{d}$.

For a circular beam pipe, Eq. (11) can be simplified:

$$\vec{Z}_\perp(\omega) = -iZ_0\frac{\beta\alpha_m + \alpha_e/\beta}{\pi^2 b^4}\left[\frac{\kappa b}{2I_1(\kappa b)}\right]^2(\hat{s}\cdot\hat{h})\,\hat{h}\,, \quad (13)$$

where $\hat{h} = \vec{b}/b$ is a unit vector in the chamber cross section S directed from the axis to the hole (discontinuity). In this case, one can rewrite the dot product in a more conventional form, $\hat{s}\cdot\hat{h} = \cos(\varphi_s - \varphi_h)$, where φ_s is the azimuthal angle of the beam position in the cross-section plane, and φ_h is the azimuthal angle in the direction to the hole. Equation (13) agrees, up to notations, with the result obtained in [7].

In a general case, it is sometimes convenient to rewrite the dipole transverse impedance (11) as

$$\vec{Z}_\perp(\omega) = -iZ_0\beta(\alpha_m + \frac{\alpha_e}{\beta^2})\,d^2(\kappa)\,\hat{d}\cos(\varphi_s - \varphi_d)\,. \quad (14)$$

In particular, this form is more convenient for the rectangular chamber, see [10]. Here $d = \sqrt{d_x^2 + d_y^2}$ and $\hat{d} = \vec{d}/d$. The angle φ_d — the azimuthal angle of $\vec{d}$ in S — shows the direction of the beam-deflecting force. The magnitude of $Z_\perp$ is maximal when the beam is deflected along this direction and vanishes when the beam offset is perpendicular to it. In a circular pipe, $\varphi_d = \varphi_h$. For a general cross section, this is not the case even in the relativistic limit [2].

More results, as well as details of the impedance derivation, can be found in the recent paper [10].

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