HIGH-ORDER EFFECTS AND MODELING OF THE TEVATRON

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Abstract

The role and degree of nonlinear contributions to machine performance is a controversial topic in current collider operations and in the design of future colliders. A high-order model has been developed of the Tevatron in COSY Infinity [1], which includes the strongest sources of nonlinearities. Signatures of nonlinear behavior are studied and compared with performance data.

INTRODUCTION

The linear tune is one of the most important characteristics of the linear dynamics of the particles in the accelerator. In the similar way, the nonlinear tune and the nonlinear tune shifts with amplitude and parameters are crucial characteristics of the nonlinear dynamics. In this article we obtain the nonlinear tune shifts with amplitude for the recent model of the Tevatron machine [2] using the transformation to the normal form coordinates and compare them with the results of the BPM (beam position monitors) measurements. This comparison yields that by using only the measurement data it is possible in many cases to recover the nonlinear tune shifts which agree to the large extent with the model predictions.

NONLINEAR TUNE SHIFT CALCULATION

To calculate the nonlinear tune shifts with amplitude it is necessary to employ the normal form transformation algorithm derived in details in [3]. In this article we will only briefly outline the steps of the algorithm necessary to understand the nonlinear tune shifts with amplitude and their connection to the measured BPM data.

Normal Form Algorithm

We want to find such a nonlinear change of variables that the motion in the new variables, up to a certain order is circular with an amplitude-dependent frequency.

Consider the nonlinear transfer map [3–5] of a particle optical system

$$\vec{z}_f = \mathcal{M}(\vec{z}_i), \tag{1}$$

where $\vec{z}$ is the vector of $2v$ phase space coordinates.

The transformation to the normal form coordinates consists of a sequence of nonlinear coordinate transformations of the form

$$\mathcal{N} = \mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1}. \tag{2}$$

Assume that the transformation matrix satisfies the following relation:

$$\mathcal{M}(\vec{0}) = \vec{0}. \tag{3}$$

If this does not hold, we can find the fixed point and transform the map to this fixed point, therefore the new transfer map satisfies (3). To be able to do this we assume that none of the eigenvalues of the linear part of the map is 1, which is always the case for stable multi-turn systems.

The first step of the transformation to the normal form coordinates is the diagonalization of the linear part of the map. This can only be done with the assumption that all the $2v$ eigenvalues are different. For most of the cyclic accelerators this does not resemble a limitation.

If we limit ourselves to the case of stable symplectic systems, all the eigenvalues can be written as complex conjugate pairs $r_j e^{\pm i\mu_j}$, where $r_j = 1$ and μ_j are real, $j = \overline{1, v}$. In the basis of the complex conjugate vectors $\vec{v}_j^{\pm}$ corresponding to the complex conjugate eigenvalues the linear part of the transfer map has the following form:

$$\mathcal{R} = \begin{pmatrix} r_1 e^{i\mu_1} & 0 & \cdots & 0 & 0 \\ 0 & r_j e^{-i\mu_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_v e^{i\mu_v} & 0 \\ 0 & 0 & \cdots & 0 & r_v e^{-i\mu_v} \end{pmatrix}. \tag{4}$$

After the diagonal form of the linear transfer map is found, we proceed to a sequence of order-by-order transformations, each of which affects only one order of nonlinearities. The aim of the transformation is to simplify the nonlinear part of the transfer map as much as possible, ideally removing all the nonlinear elements up to a certain order.

At the mth step we divide the transfer map into the linear part $\mathcal{R}$ and the part $\mathcal{S}_m$, containing all the nonlinearities. If the previous step was successful, the matrix $\mathcal{S}_m$ contains only the terms of order m and higher.

The desired transformation has the form

$$\mathcal{A}_m = \mathcal{I} + \mathcal{T}_m, \tag{5}$$

where $\mathcal{T}_m$ vanishes to order $m-1$. As the linear part of $\mathcal{A}$ is invertible, the map $\mathcal{A}$ is invertible [3], and the inverse to order m is

$$\mathcal{A}_m^{-1} =_m \mathcal{I} - \mathcal{T}_m. \tag{6}$$

After the transformation we have up to order m:

$$\begin{aligned} & \mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1} =_m \\ =_m \quad & (\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{R} + \mathcal{S}_m) \circ (\mathcal{I} - \mathcal{T}_m) =_m \\ =_m \quad & (\mathcal{I} + \mathcal{T}_m) \circ (\mathcal{R} - \mathcal{R} \circ \mathcal{T}_m + \mathcal{S}_m) =_m \\ =_m \quad & \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m + \mathcal{S}_m + \mathcal{T}_m \circ \mathcal{R} =_m \\ =_m \quad & \mathcal{R} + \mathcal{S}_m + (\mathcal{T}_m \circ \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m). \end{aligned} \tag{7}$$

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Our goal is to simplify the terms of order m of $\mathcal{M}$, in other words, we need to find $\mathcal{T}_m$ such that $\mathcal{S}_m = -\mathcal{C}_m$, where $\mathcal{C}_m = (\mathcal{T}_m \circ \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m)$. If the range of $\mathcal{C}_m$ is the full space, $\mathcal{S}_m$ can be removed entirely. However, there are circumstances, limiting the range of $\mathcal{C}_m$.

Nonlinear Tune Shift with Amplitude

In the case of the stable symplectic system the limiting condition can be written as

$$\vec{\mu} \cdot (\vec{k}^+ - \vec{k}^-) = \pm\mu_j \mod 2\pi, \tag{8}$$

where the sign on the right hand side corresponds to $\vec{v}_j^{\pm}$.

Assume the equation $\vec{\mu} \cdot \vec{n} = 0 \mod 2\pi$ does not have any nontrivial solutions. In this case the only way (8) can hold is

$$k_l^+ = k_l^- \;\forall\, l \neq j,\; k_j^+ = k_j^- \pm 1. \tag{9}$$

A somewhat involved argument yields that in this case the elements of the transfer map in the new coordinates $s_1^+, s_1^-, \ldots, s_v^+, s_v^-$ corresponding to the pairs of the complex conjugate eigenvectors $\vec{v}_1^{\pm} \ldots \vec{v}_v^{\pm}$ has the form

$$\left\{ \begin{array}{l} \mathcal{M}_j^+ = s_j^+ \cdot f_j(s_1^+ s_1^-, \ldots, s_v^+ s_v^-) \\ \mathcal{M}_j^- = s_j^- \cdot \overline{f_j}(s_1^+ s_1^-, \ldots, s_v^+ s_v^-) \end{array} \right. . \tag{10}$$

It is not convenient to work with the variables $s_j^{\pm}$, therefore we express the resulting map in terms of the variables $t_j^{\pm}$, such that

$$\left\{ \begin{array}{l} t_j^+ = (s_j^+ + s_j^-)/2 \\ t_j^- = (s_j^+ - s_j^-)/2i \end{array} \right. , \tag{11}$$

then

$$\left\{ \begin{array}{l} s_j^+ = t_j^+ + i t_j^- \\ s_j^- = t_j^+ - i t_j^- \end{array} \right. , \tag{12}$$

and $s_j^+ \cdot s_j^- = (t_j^+)^2 + (t_j^-)^2$.

For the complex conjugate $s_j^{\pm}$ the values of $t_j^{\pm}$ are purely real, besides,

$$\begin{aligned} \mathcal{M}_j^{\pm} &= \left(\begin{array}{c} s_j^+ \cdot f_j(s_1^+ s_1^-, \ldots, s_v^+ s_v^-) \\ s_j^- \cdot \overline{f_j}(s_1^+ s_1^-, \ldots, s_v^+ s_v^-) \end{array} \right) = \\ &= \frac{1}{2} \left(\begin{array}{cc} 1 & 1 \\ -i & i \end{array} \right) \cdot \\ &= \left(\begin{array}{c} f_j((t_1^+)^2 + (t_1^-)^2, \ldots, (t_v^+)^2 + (t_v^-)^2) \\ \overline{f_j}((t_1^+)^2 + (t_1^-)^2, \ldots, (t_v^+)^2 + (t_v^-)^2) \end{array} \right) \cdot \\ &\cdot \left(\begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right) \left(\begin{array}{c} t_j^+ \\ t_j^- \end{array} \right) = \\ &= a_j \left(\begin{array}{cc} \cos(\phi_j) & -\sin(\phi_j) \\ \sin(\phi_j) & \cos(\phi_j) \end{array} \right) \left(\begin{array}{c} t_j^+ \\ t_j^- \end{array} \right), \end{aligned} \tag{13}$$

where $f_j = a_j \cdot e^{i\phi_j((t_1^+)^2 + (t_1^-)^2, \ldots, (t_v^+)^2 + (t_v^-)^2)}$ and $a_j = \mathrm{const}$, as we consider the symplectic motion; $\phi_j = \phi_j((t_1^+)^2 + (t_1^-)^2, \ldots, (t_v^+)^2 + (t_v^-)^2)$ depends on the rotationally invariant quantity.

Therefore, in the new coordinates the dynamics of the particles is given by a rotation the frequency of which depends only on the amplitudes $(t_j^+)^2 + (t_j^-)^2$. The functions ϕ_j for $j = \overline{1, v}$ give the tunes of the nonlinear motion.

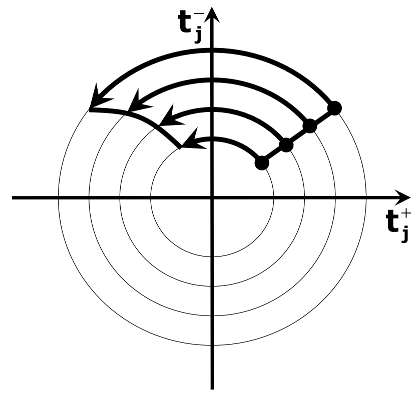

Figure 1: The behavior of the particles in the normal form coordinates

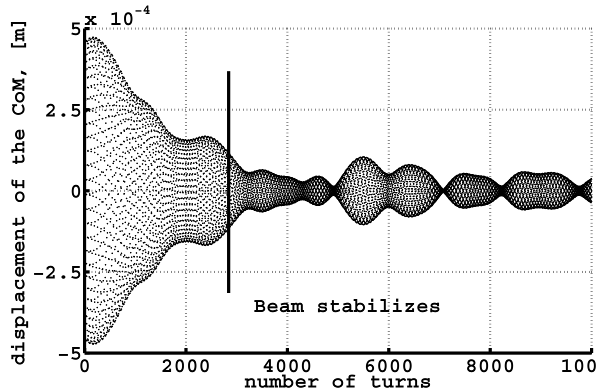

Figure 2: Displacement of the beam center of mass after a single horizontal kick in normal form coordinates

COMPUTED DATA AND MEASURED DATA

As the motion in the normal form coordinates resembles a rotation with amplitude dependent phases, that allows us to establish a strong connection between the value of the nonlinear tune shift for some particular amplitude and the behavior of the beam. Fig.1 shows the phase portraits of four particles launched along a straight line in the normal form coordinates after a certain number of turns. The particles cannot leave their respective circles, but the frequencies of the rotations are different for different amplitudes. Assume the outer particles move faster than the inner particles. In this case after a number of turns the outermost particle will be 2π ahead in phase compared to the innermost particle, and as the phase dependence on the amplitudes is continuous, all the particles in between the two will have $0 < \Delta\phi_j < 2\pi$ relative to the innermost particle. That means the center of mass of the system of particles moves toward the origin. When the outermost particle has $\Delta\phi_j \geq 2\pi$, the beam can be considered scattered all around the normal form phase space, and since then the center of mass oscillates in the neighborhood of the origin with a constant oscillation amplitude.

Hence, if we slightly displace the beam by using the horizontal or vertical kick (as it is done for the BPM orbit measurements), the amplitude of the center of the displaced distribution in the normal coordinates will show damping until the stable amplitude is reached. Once the amplitude of the center of mass is stable, we know that the outermost particle of the distribution is 2π ahead in phase relative to the innermost particle.

The algorithm of the normal form transformation allows us to calculate the tune shift with amplitude explicitly up to a desired order. Knowing the amplitudes $(t_j^+)^2 + (t_j^-)^2$ of the outermost particle, we can plug them into the expression for the nonlinear tune shift and compare the result with the number of turns before the beam stabilizes, using the formula

$$\Delta\mu_j = 2\pi/N_j, \quad (14)$$

where N_j is the observed number of turns (see Fig.2). The comparison for one of the latest Tevatron models [2] shows a very good agreement between the two values. For the instant horizontal kick of 25 MV/m at one point along the ring the number of turns before the horizontal coordinate of the center of mass stabilizes can be estimated as $2900 - 3100$ turns, which corresponds to the tune shift of $2.0268 \cdot 10^{-3} \ldots 2.1666 \cdot 10^{-3}$, while the calculated value using the explicit formula for the tune shift obtained by the normal form transformation is $2.059 \cdot 10^{-3}$, or 3050 turns.

This result stays valid for the conventional particle optical coordinates, as the value of the nonlinear tune is the same for both the normal form coordinates and the particle optical coordinates. The transformation to the normal form coordinates merely allows a clear and explicit calculation of the value of the tune shift, and yields the formula suitable for various amplitudes.

If the model is a good approximation to the real machine, the value of the nonlinear tune shift obtained by using the number of turns necessary for the beam to stabilize after the kick, and by the direct calculation using the model and the normal form transformation should be very close. Moreover, if the two values do not agree, there is a strong reason to double check, whether the model, or the measurements are not precise enough. In the case of the Tevatron, the model is in good agreement with the measured data. The measurement result for the same horizontal kick of 25 MV/m is shown in Fig.3. The estimated number of turns before the motion stabilizes is $2600 - 3200$ which corresponds to the tune shift of $1.9635 \cdot 10^{-3} \ldots 2.4166 \cdot 10^{-3}$.

Even if we do not have a precise model, but the measurements are believed to be trustful, we can make the conclusion about the estimated nonlinear tune of the machine, and the value so obtained is justified to be close to the real nonlinear tune by the normal form transformation algorithm.

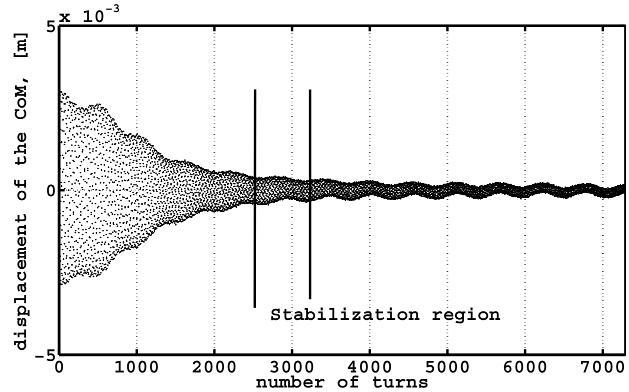

Figure 3: Displacement of the beam center of mass after a single horizontal kick in conventional coordinates, measured data

SUMMARY

Summarizing the results presented in this article, we established a connection between the calculated value of the tune shift with amplitude and the results of the beam position measurements. The connection is given by the equation (14). To calculate the value of the nonlinear tune shift, the normal form transformation algorithm implemented in COSY Infinity beam dynamics calculation and optimization code was employed. The main steps of the algorithm itself could be found in this article, for more details see [3].

The comparison of the measured and calculated data strongly suggests that even if there is no model of the machine reliable enough at hand to calculate the exact value of the nonlinear tune shift, the approximate value can be obtained by analyzing the behavior of the measured data.

The method of finding the approximate value of the nonlinear tune shift using the performance data was tested on the recent Tevatron accelerator beam position measurements and showed the validity of the proposed technique for this particular machine.

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