DYNAMICS OF THE ELECTRON PINCH AND INCOHERENT TUNE SHIFT INDUCED BY ELECTRON CLOUD

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Abstract

When a proton bunch passes through an electron cloud, the cloud electrons are attracted by the beam electric field and their density strongly increases near the beam centre. This gives rise to an incoherent proton tune shift, which depends on the longitudinal and radial position within the bunch. We present an analytical description of the 'electron pinch' and the resulting proton tune shift for a circular symmetry and a Gaussian cloud, considering a linear transverse force and various longitudinal beam profiles. Benchmarking and extending the results by computer simulations, we can also explore the effects of a non-linear transverse force.

INTRODUCTION

During the passage of a proton (or positron) bunch through an electron cloud, the electrons are accumulated around the beam center. This pinch effect produces an incoherent tune shift and a tune spread in the bunch that could cause a slow emittance growth over successive turns. We compute the electron-cloud density evolution during a bunch passage and from this we infer the tune shift of individual beam particles, for a cylindrically symmetric model.

We first solve the equations of motion of a single electron in the bunch potential under the simplifying approximation of a linear transverse force. Next, assuming an initially Gaussian electron distribution of finite temperature in transverse phase space, we compute the evolution of the electron density during the bunch passage, using Liouville's theorem. Finally, from the electron distribution, we calculate the tune shift experienced by individual protons as a function of their transverse and longitudinal position. An explicit analytical solution is derived for an arbitrary longitudinal profile, under the assumption of a linear transverse force. Approximations for low electron temperature are discussed. In the second part of this paper we employ a computer simulation to extend the analysis to a non-linear transverse force for a Gaussian transverse beam profile. From the simulation result, we estimate the incoherent tune spread in the LHC at injection.

ELECTRON DENSITY EVOLUTION, APPROXIMATION OF LINEAR FORCE

We start from the electron distribution in the four-dimensional transverse phase space. In the linear force approximation, the horizontal and vertical planes are uncoupled. We thus factorize the electron density distribution and the spatial density as follows:

$$\rho(x,\dot{x},y,\dot{y},t) = \rho_x(x,\dot{x},t)\ \rho_y(y,\dot{y},t) \qquad (1)$$

$$n_e(r,t) \equiv n_e(x,y,t) = n_x(x,t)\ n_y(y,t) \qquad (2)$$

where the projected spatial densities are obtained by integrating the projected phase-space densities over the electron velocities:

$$n_x(x,t) = \int d\dot{x}\rho_x(x,\dot{x},t)\ . \qquad (3)$$

By our symmetry assumption n_e depends on x and y only in terms of the radius $r \equiv \sqrt{x^2+y^2}$.

From Liouville's theorem, we know that the electron density in the phase space is locally preserved. Hence, with the hypothesis of an initially Gaussian distribution for the electrons in their transverse phase space, we can write for the horizontal distribution

$$\rho_x(x,\dot{x},t) = \rho_x(x_0,\dot{x}_0,0) = \frac{\sqrt{\lambda_e}}{2\pi\sigma_0\dot{\sigma}_0}\ e^{-\frac{x_0^2}{2\sigma_0^2}}\ e^{-\frac{\dot{x}_0^2}{2\dot{\sigma}_0^2}}\ , \qquad (4)$$

Here, the parameters σ_0 and $\dot{\sigma}_0$ denote the horizontal rms size of the initial electron distribution and its horizontal rms velocity, respectively. For the circularly symmetric problem that we consider here, the vertical density has the same form with identical rms size and velocity. We will later obtain some approximate compact expressions for the special case that the initial velocities of the electrons are small compared with the (correlated) velocities acquired in the beam potential, i.e. $\dot{\sigma}_0 \ll \omega_e\sigma_0$.

If we are able to solve and invert the equation of motion of a single electron in the bunch potential, we can express $(x_0,\dot{x}_0)$ as a function of $(x,\dot{x},t)$ and insert the resulting expressions on the right-hand side of (4) in order to obtain the electron density at the time t [1].

Approximation of Linear Force

Under the linear approximation (strictly valid for $r \ll \sigma_r$) the motion of an electron in the bunch potential is decoupled for the two transverse planes. The equation in the horizontal plane (a similar expression holds for the vertical) is [2]:

$$\ddot{x} + \omega_e^2(t)x = 0 \qquad (5)$$

where:

$$\omega_e^2(t) = \lambda_b(t)r_ec^2/\sigma_r^2\ , \qquad (6)$$

c is the velocity of light, $r_e \equiv e^2/(4\pi\varepsilon_0 m_e c^2)$ is the classical electron radius, σ_r is the transverse beam size (namely $\sigma_r = \sigma_x = \sigma_y$) and $\lambda_b(t)$ is the beam longitudinal profile as a function of time $t = (n\sigma_z - z)/c$. We define $t = 0$ as the moment when the bunch enters the cloud (we will use $n = 3$) and z is the longitudinal distance from the bunch center.

With the linear approximation, it is possible to solve the equation of motion (5) and invert the solution, yielding

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[1] A similar method was used in [1] to compute the beam density evolution under the influence of nonlinear field errors.

$(x_0, \dot{x}_0)$ as a function of $(x, \dot{x})$ in the form:

$$\begin{aligned} x_0 &= a(t)x + b(t)\dot{x} \\ \dot{x}_0 &= c(t)x + d(t)\dot{x} \ , \end{aligned} \tag{7}$$

where the coefficients $a(t)$,...,$d(t)$ depend on the longitudinal distribution and for a conservative system $(ad - bc) = 1$. The electron distribution in phase space is computed by inserting (7) into (4) and the spatial electron density evolution is obtained by integrating the distribution function over the velocities as in (3).

Tune Shift

From the electron density, we can compute the electric field acting on the protons of the bunch (see again [2] for details) and the incoherent tune shift induced on the beam over one turn around the ring. The tune shift in the horizontal plane is given by

$$\Delta Q_x = \frac{1}{4\pi} \oint_C ds\, \beta(s) \Delta k_x \tag{8}$$

$$\Delta k_x = -\frac{e}{\gamma m_p c^2} \frac{\partial \tilde{E}_{e,x}(r,z)}{\partial x} \ , \tag{9}$$

where $\tilde{E}_{e,x}(r,z)$ is the field experienced by a proton at position (r, z), and is equal to:

$$\Delta Q_x(r,z) = \oint_C ds\, \beta(s) \frac{r_p}{\gamma} \left[\tilde{n}_e(r,z) - \frac{1}{r^2} \int \tilde{n}_e(r',z) r' dr' \right] \ . \tag{10}$$

In principle, the proton beam size depends on the beta functions and, thus, also the electron density $\tilde{n}_e(r,z)$ depends on the position around the ring s. In the following we will use the smooth focusing approximation, (i.e. $\beta(s) = \bar{\beta} = const$) and we also assume a constant electron-cloud density, so that the integrand becomes independent of s.

We now derive the tune shift for two specific longitudinal bunch profiles and for the general case.

Uniform Bunch Profile

In the case of $\lambda_b(t) = \overline{\lambda_b} = const$ the equation of motion reduces to the harmonic oscillator, whose solution can be written in terms of $C = \cos(\omega_e t)$ and $S = \sin(\omega_e t)$ The tune shift (8) for a particle at position r and z in the bunch is

$$\begin{aligned} \Delta Q_x(r,z) \approx\ & \frac{\bar{\beta} L \lambda_e r_p}{4\pi\gamma C^2 \sigma_0^2} \frac{1}{1 + \frac{S^2 \dot{\sigma}_0^2}{C^2 \omega_e^2 \sigma_0^2}} \\ & \left[1 - \frac{\omega_e^2 r^2}{2(C^2 \omega_e^2 \sigma_0^2 + S^2 \dot{\sigma}_0^2)} + O\left((\omega_e r)^4 \right) \right] \end{aligned}$$

where L is the circumference of the ring and the smooth focusing assumption has been invoked. The tune shift depends on the longitudinal position with respect to the bunch center and it decreases parabolically with transverse distance r. We note that for $\dot{\sigma}_0 \ll \sigma_0 \omega_e$, the tune shift becomes maximum at periodic intervals along the bunch, when $C = 0$.

Arbitrary Longitudinal Profile

If the longitudinal distribution of the beam, $\lambda_b(t)$, is not a constant, but the change is adiabatic so that: $|3\dot{\omega}_e/2\omega_e|$, $|\ddot{\omega}_e/\dot{\omega}_e| \ll 2\omega_e$, we can apply the WKB approximation [3]. Then the general solution, dropping small terms, is

$$x(t) = \frac{c_1}{\sqrt{\omega_e(t)}} \cos S(t) + \frac{c_2}{\sqrt{\omega_e(t)}} \sin S(t) \tag{11}$$

$$S(t) = \int_0^t \omega_e(t)\, dt \tag{12}$$

where c_1 and c_2 are determined from the initial conditions x_0 and $\dot{x}_0$. In this general case – but as before for a linear transverse force –, we can still invert the solution and determine $(x_0, \dot{x}_0)$ as a function of $(x, \dot{x})$, as in Eq.(7), and insert the result into the expression of the electron distribution in phase space (4). The density $n_e(r,t)$ obtained by integrating over the velocities becomes

$$n_e(r,t) = \frac{\lambda_e}{2\pi D(t)} e^{-\frac{r^2}{2D(t)}} \tag{13}$$

with

$$D(t) = d(t)^2 \sigma_0^2 + b(t)^2 \dot{\sigma}_0^2 \tag{14}$$

which depends on the longitudinal profile of the bunch.

Again assuming the smooth focusing approximation, the tune shift in the horizontal or vertical plane has the general form:

$$\Delta Q_{x,y}(r,z) = \frac{\lambda_e r_p \bar{\beta} L}{2\pi\gamma\, r^2} \left(1 - \frac{e^{-\frac{r^2}{2D}} \left(r^2 + D \right)}{D} \right) \ . \tag{15}$$

Expanding and keeping only the lowest-order terms in r^2/D, this simplifies to

$$\Delta Q_x(r,z) \approx \frac{\bar{\beta} L \lambda_e r_p}{4\pi\gamma D} \left(1 - \frac{3}{4} \frac{r^2}{D} \right) \tag{16}$$

Gaussian Longitudinal Profile

In the case of a bunch with a Gaussian longitudinal shape

$$\tilde{\lambda}_b(z) = \frac{N_b}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} \quad ; \quad z \in (-\infty, +\infty) \tag{17}$$

we have:

$$\begin{aligned} \omega_e(t) &= \Omega\, e^{-\frac{(n\sigma_z - ct)^2}{4\sigma_z^2}} \\ S(t) &= \Omega \frac{\sigma_z \sqrt{\pi}}{c} \left\{ \mathrm{Erf}\left(\frac{n}{2} \right) + \mathrm{Erf}\left[\frac{1}{2} \left(\frac{ct}{\sigma_z} - n \right) \right] \right\} \\ \Omega &= \sqrt{\frac{r_e N_b c^2}{\sigma_r^2 \sigma_z \sqrt{2\pi}}} \ . \end{aligned}$$

The coefficients b and d in (14) are

$$\begin{aligned} b(t) &= -e^{\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{8} - \frac{n\, \tilde{z}}{4} \right)} \frac{1}{\Omega} \sin S(t) \\ d(t) &= e^{\left(\frac{\tilde{z}^2}{8} - \frac{n}{4}\tilde{z} \right)} \cos S(t) \\ & \quad + e^{\left(\frac{n^2}{4} + \frac{\tilde{z}^2}{8} - \frac{n\, \tilde{z}}{4} \right)} \frac{n}{4\sigma_r} \frac{c}{\Omega} \sin S(t) \end{aligned}$$

with $\tilde{z} = ct/\sigma_z$.

The tune shift at the start of the bunch ($\tilde{z} = 0$) is

$$\Delta Q_x(r, z) \approx \frac{\bar{\beta}\tilde{C}\lambda_e r_p}{4\pi\gamma\sigma_0^2} , \tag{18}$$

as expected for the unperturbed initial cloud density [4].

EXTENSION TO NON-LINEAR TRANSVERSE FORCE

Via a simple tracking code, we extended the analysis to electrons moving in the potential of a transverse Gaussian beam. For the simulations we took the parametres for LHC at injection, listed in Table 1.

Table 1: Parameters used in the simulations for LHC at injection

electron cloud density	ρ_e	6×10^{11} m^{-3}
bunch population	N_b	1.1×10^{11}
rms bunch length	σ_z	0.115 m
rms beam size	σ_b	0.884 mm
nominal tunes	$Q_{x,y}$	64.28, 59.31
electron cloud size	σ_0	10 σ_b
electron initial velocity	$\dot{\sigma}_0$	$\omega_e\sigma_0/100$

The top row of Fig. 1 shows the electron density evolution at the centre of the pipe, during the passage of a bunch, computed with the linear force approximation (left) and for the Gaussian beam profile (right). The simulation with the linear force acting on the electrons is consistent with the analytical prediction (dotted green line). The small shift in the position of the peaks depends on the initial condition and the slicing in our simulation. On the other hand, if the electrons move in the potential of a transversely Gaussian beam, the modulation almost disappears after a quarter oscillation, from when on the density stays about constant. In the case of the non-linear force, in fact, the electrons do not reach the centre of the bunch simultaneously, but their oscillation frequency depends on the initial amplitude. The bottom row displays snapshots of the radial distribution of the electrons at different times during the bunch passage both for the linear force approximation and for the Gaussian potential.

As can be seen in Fig. 1 the density enhancement at the center of the bunch, for a Gaussian transverse beam distribution, is about a factor 50. This allows us to roughly estimate the tune spread via

$$\Delta Q \approx \frac{\bar{\beta}\tilde{C}r_p}{2\gamma} n_e , \tag{19}$$

where n_e denotes the enhanced electron density. For the example of the LHC, this gives the value $\Delta Q \approx 0.13$, if the initial unperturbed electron cloud density is 6×10^{11} m^{-3}. A frequency map analysis [5] from HEADTAIL simulations [6] in a frozen-field approximation gave a tune spread of ≈ 0.05 at $z = +2\sigma_z$. This tune spread corresponds to a density enhancement of a factor 20 [7], in nearly perfect agreement with the value at $+2\sigma_z$ in the top right picture of Fig. 1.

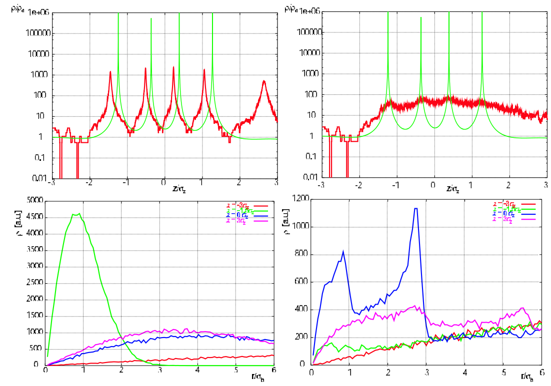

Figure 1: Top: Electron density vs. time at the centre of the pipe, during the passage of a bunch. In red the simulated density evolution and dotted in green the analytical results. Bottom: Snap shots of radial distribution ($\rho \times r$) at 4 different times during the bunch passage. The pictures are obtained assuming a linear transverse force (left) and for a Gaussian transverse beam distribution (right). A Gaussian bunch profile is assumed in z.

SUMMARY

We presented an analytical approach to compute the incoherent tune shift caused by the electron pinch during the passage of a bunch through the electron cloud. An expression for the electron density evolution was derived for any longitudinal bunch profile, a linear transverse force, and circular symmetry. From the pinched electron distribution, the incoherent tune shift has been computed as a function of the radial and longitudinal position inside the bunch.

Via a simple tracking code, we extended this study to electrons moving in the nonlinear field of a beam with Gaussian transverse profile. In this case, the electrons do not reach the centre of the bunch simultaneously, and after a quarter oscillation the density at the center of the bunch stays roughly constant. It is easy to estimate the tune shift from the value of this stationary density enhancement.

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