FAST ALGORITHMS FOR THE TRACKING OF ELECTRON BEAMS*

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Abstract

The simulation of the motion of a particle beam in linear colliders requires fast and robust numerical algorithms. The following approach to simulate the effect of nonlinear space charge forces on the dynamics of space charge dominated electron beams has been realized in the tracking code **Q** [9]. The related tracking algorithm has to solve the relativistic equations of motion for a large number of time steps and unknowns. The underlying physical model uses a grid–based method for the field calculations which allows the consideration of far more macroparticles than other methods. The numerical convergence studies presented in this paper show the robustness of the tracking algorithm and coincide with theoretical results. Further, several techniques for the construction of a fast adapted multigrid solver for the determination of the nonlinear space charge forces are investigated.

1 INTRODUCTION

An important task in the design and operation of future linear colliders is to simulate the motion of the particle beam in phase space. The related tracking algorithm considering nonlinear space charge effects in electron beams, as it is realized in the corresponding tracking code **Q** [9], has to solve the relativistic equations of motion for a large number of time steps (~ 1000). Its overall accuracy should be linear in the number of unknowns N concerning computing time in order to enable a systematic convergence study. An electrostatic model assuming piecewise constant space charge forces has this linearity, provided that the computing time for the determination of the space charge fields is of $O(N)$. In addition the time integration step Δt can be varied independently of the spatial resolution of the beam due to static field calculation in contrast to a time dependent electromagnetic Particle-in-Cell scheme [3].

For a short description of the physical problem let N be the number of macroparticles of the beam, where the set of macroparticles represents the distribution of all particles in the beam. Further, let the i-th macroparticle ($i = 1, \ldots, N$) have the position $\vec{r}_i$ and the momentum $\vec{p}_i$. The particle itself has the rest mass m_0 and the charge q. Then the relativistic equations of motion are given by [6]

$$\begin{aligned} \frac{\partial \vec{r}_i}{\partial t} &= \frac{\vec{p}_i}{\gamma_i m_0}, \\ \frac{\partial \vec{p}_i}{\partial t} &= q(\vec{E} + \frac{\partial \vec{r}_i}{\partial t} \times \vec{B}), \qquad i = 1, \ldots, N \end{aligned}$$

with the Lorentz factor $\gamma_i = \sqrt{1 + \frac{p_i^2}{m_0^2 c^2}}$, the electric field $\vec{E}$ and the magnetic flux density $\vec{B}$. A fast method to determine nonlinear space charge forces is to compute the electrostatic potential φ' as a solution to Poisson's equation

$$-\Delta\varphi' = \frac{\varrho'}{\varepsilon_0} \qquad \text{in } \Omega \subset \mathbb{R}^3$$

in the beam's restframe from a charge density ϱ' with the dielectric constant ε_0. The computational domain Ω is considered to be longitudinally (z–direction) invariant and to have an elliptic or polygonal cross–section in the (x, y)–plane. With respect to the boundary an ideal conducting pipe is assumed transversally and an open boundary longitudinally. The necessary space charge field can then be computed by Lorentz–transformation of the electrostatic field $\vec{E}' = -\vec{\nabla}\varphi'$ into the laboratory frame.

The tracking algorithm has from the numerical point of view two essential aspects: the solution of the relativistic equations of motion and the solution of Poisson's equation. The time integration scheme chosen for solving the set of coupled first–order differential equations of motion is a fifth–order embedded Runge–Kutta method [5] with adaptive step size control. Its characteristics in phase space are investigated for single particle dynamics. The accuracy of the space charge fields entering the equations of motion is analyzed concerning its dependence on the mesh size h of the finite difference grid and the relative residual of the potential. An efficient Poisson solver with optimal convergence rates and a linear dependence of computing time on the number of unknowns is given by the application of the multigrid technique [2]. The rectangular grid generated by the tracking algorithm provides a good basis for the construction of a geometric multigrid solver. The main difficulty is the handling of the anisotropic grid stretched in longitudinal direction which is caused by the Lorentz–transformation ($h'_z = \gamma h_z$). This circumstance rapidly slows down the standard multigrid algorithm and makes adaptions to the anisotropy necessary.

2 TIME INTEGRATION SCHEME

The characteristics of the Runge–Kutta time integration scheme concerning the conservation of observables such as the particles' momentum and energy is investigated on the basis of single particle dynamics (no space charge forces), yielding the possibility to compare the numerical results to analytically known exact solutions in position, momentum and energy. The fifth–order Runge–Kutta method is momentum and energy conserving in the absence of forces. Given a homogeneous electric field according to a constant

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acceleration the error in position is $O(\Delta t^{5.6\pm0.4})$ as shown in Fig. 1. It is exact in momentum for one time step, limited by round off errors only. From this result it can be concluded that the exactness in momentum holds also for the electrostatic multi particle problem under homogeneous external electric fields. The method is limited concerning energy conservation under constant acceleration with an error in energy of $O(\Delta t^{0.95\pm0.02})$.

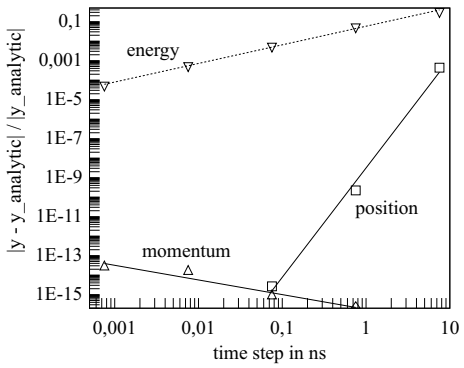

Figure 1: Relative error in space, momentum and energy of a single particle after acceleration within a homogeneous electric field of $E_z = 250\frac{\mathrm{kV}}{\mathrm{m}}$ over a distance of 1 m.

3 FINITE DIFFERENCE SPACE CHARGE CALCULATION

We use space–centered finite differences for discretizing Poisson's equation. In a momentum conserving particle mesh scheme it is in addition necessary and sufficient to choose identical charge assignment and force interpolation functions [3], in our model linear (Cloud–in–Cell).

The error relative to the numerically exact solution of the electrostatic potential of a point charge within a cube with perfectly conducting boundaries as a function of the mesh size is shown in Fig. 2. It is of $O(h^{2.11\pm0.04})$, which is in agreement to theoretical functions [3]. The value of the error of the electric field at the location of 32 randomly distributed particles within a spherical homogeneous charge distribution at rest is of $O(h^{1.38\pm0.05})$ as expected from the fact, that the randomized particle positions are not centered between gridpoints. Fig. 3 shows the dependence of the field error taken at 640 particle positions on the residual of the potential. It is of $O(r^{1.56\pm0.06})$. Already a residual of 10^{-2} yields an error in the electric field of $4 \cdot 10^{-4}$, which leads to a considerable saving of computing time.

4 ADAPTED MULTIGRID AS FAST POISSON SOLVER

The tracking algorithm involves the solution of Poisson's equation in every time step. The rectangular grid generated in the particle mesh scheme is used for the discretization of the Laplacian, where the standard finite difference scheme

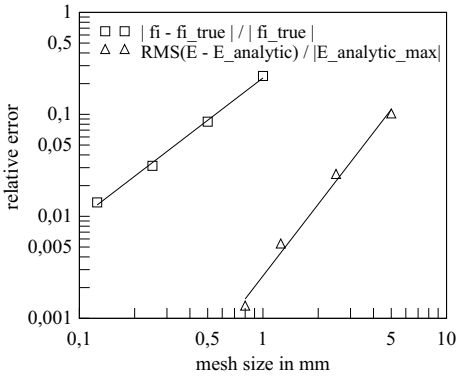

Figure 2: Local discretization error of the potential and the electric field as a function of the mesh size.

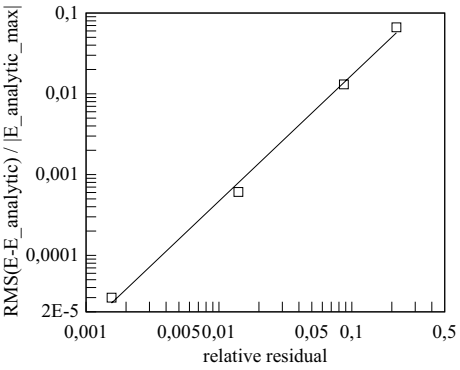

Figure 3: Error of the electric field as a function of the residual.

(seven–point stencil) is applied. The dimension of the resulting system of equations with ~ 1 million unknowns demands the construction of an efficient solver.

State of the art is the application of multigrid techniques [2]. The general idea of multigrid is the combination of the approximate solution of the system of equations obtained by a few steps of a relaxation method (e.g. Gauss–Seidel relaxation) with a coarse grid correction computed on a coarser grid. While this method has optimal convergence rates in standard situations, it is very slow for anisotropic problems. As mentioned in the introduction, the Lorentz–transformation causes an anisotropic grid which requires an adaption of the standard algorithm. Three methods for the handling of anisotropies recommended in literature will be investigated in the present paper. The first one is the technique of *semi–coarsening* [2, 10] which doesn't coarse the mesh of all coordinate directions. A rule for the semi–coarsening in the tracking context is: do not coarsen the mesh of the longitudinal direction, if the mesh size is still two times larger than the mesh size of the transversal direction [4]. As relaxation method the standard red–black Gauss–Seidel iteration is used. A second geometric multigrid algorithm keeps the standard coarsening strategy and uses an *adapted relaxation scheme*, where in our case a plane relaxation in the (x, y)–plane is chosen [1]. The *algebraic multigrid algo-*

rithm (AMG) [7] which is investigated third does not use the geometrical data of the problem for the construction of the coarser levels but the matrix entries of the resulting system of equations. Since AMG distinguishes between strong and week coupled variables, the adaption to the anisotropy is performed automatically. Thus, it can be used as black box solver [8].

Numerical experiments have been performed with the data of a 10 MeV beam. The charge density has Gaussian distribution on a cylindrical pipe. The first example (see Fig. 4) is computed on a grid with $33 \times 33 \times 129$ (=140 481) mesh points. The related grid has a small anisotropy of $h_z \approx 3h_x \approx 3h_y$. The second example (see Fig. 5) has a large anisotropy with $h_z \approx 25h_x \approx 25h_y$ and is computed on a grid with $65 \times 65 \times 65$ (=274 626) mesh points.

The semi–coarsening strategy comes up with the best results (solid line): the computing time increases only linearly with the number of unknowns. It is of $O(N^{1.12\pm0.16})$ in agreement to a value of $O(N)$ in literature [5]. This is not the case with the method of plane relaxation. A further advantage of the semi–coarsening algorithm is the simple implementation. The algebraic multigrid method (AMG) has the same convergence speed as the geometric multigrid with semi–coarsening, but the setup–phase, i.e. the determination of the coarser levels and their interaction, takes a lot of time — even more in the case of the small anisotropy, because more levels are built. The results of the *standard multigrid method (MG)* are given as reference (dotted line).

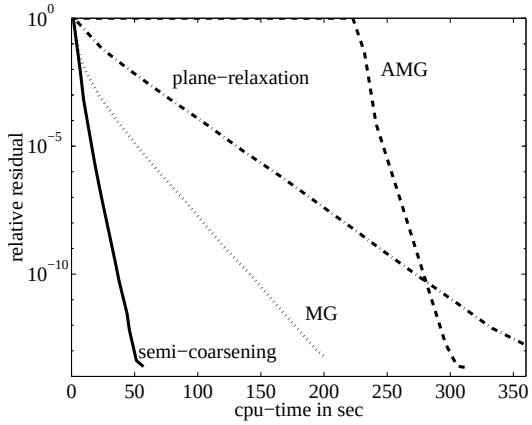

Figure 4: Several mutigrid strategies for a grid with small anisotropy.

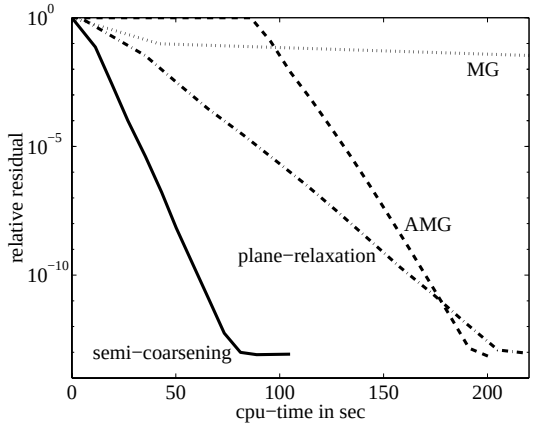

Figure 5: Several mutigrid strategies for a grid with large anisotropy.

5 CONCLUSION

An electrostatic algorithm for the investigation of nonlinear space charge effects in electron beams has been presented. It has proven exact in momentum under homogeneous electric fields using a fifth–order Runge–Kutta scheme. The accuracy of the space charge forces determined by the finite difference particle mesh scheme is of $O(h^{1.38\pm0.05})$ in the mesh size and of $O(r^{1.56\pm0.06})$ in the residual of the potential. In order to deal with the longitudinally strongly anisotropic grid in the beam's restframe, which is stretched due to Lorentz–transformation, several adapted multigrid techniques have been investigated. A multigrid strategy using semi–coarsening turned out to be linear concerning computing time in the number of unknowns and proved to be the adequate way to transfer the robustness and speed characterizing the standard multigrid method to the field of application of relativistic space charge calculation.

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