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On the Energy Limit of Compact Isochronous Cyclotrons

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Main goal of this study

- We use a general Hamiltonian approach, to find formulas for the energy limit E_{max} of compact isochronous cyclotrons.
- They will be helpful in preliminary and conceptual design studies for defining the following parameters:
  - The rotational symmetry number *N* of the magnetic sectors
  - The maximum spiral angle ξ of the sectors
  - The flutter *F*
- The limit is assumed to be determined by two competing requirements:
  - To have sufficient azimuthal field variation to garrantee stable vertical motion
  - To have a not-too-high azimuthal field variation to avoid the stopband of the linear structural resonance $2\nu_r=N$. (a resonance that can not be crossed).
- Other resonances due to magnetic field imperfections are not considered.

General remarks

- This paper pays tribute to and is inspired by my PhD thesis promotor prof. Henk Hagedoorn; it is an application and generalization of his well-know publication:
    - Ref. 1: H. L. Hagedoorn and N. F. Verster, “Orbits in an AVF Cyclotron”, in *Nuclear Instruments and Methods*, vol. 18,19, pp. 201-228, 1962.
- The derivation is too elaborate and too complex to show in detail in this presentation. We present here a strongly compressed version. The full derivation and results can be found in the following two publications:
    - Ref. 2: W. Kleeven, *Energy limit of compact isochronous cyclotrons*, 2022, arXiv:2211.01029
    - Ref. 3: W. Kleeven, *On the Energy limit of compact isochronous cyclotrons,* Accepted for publication in the Journal of Instrumentation (JINST) as contribution to a special issue of the ICFA (International Committee for Future Accelerators) newsletter #84 on: Dynamics of High Power and High Energy Cyclotrons.

Main assumptions and approximations (1)

- We solve the static motion (no acceleration) ⇒ $P=P_0$=constant
- No field errors (perfect *N*-fold rotational and median plane symmetry)

$$B(\theta,r) = \bar{B}(r) + \sum_n \mathcal{A}_n(r)\cos n\theta + \mathcal{B}_n(r)\sin n\theta$$

- Motion is solved near a radius r_0 defined by: $P_0 = qr_0\bar{B}(r_0)$ and $x = (r - r_0)/r_0$
- Reduced magnetic field: $\mu(\theta,r) = \dfrac{B(\theta,r)}{\bar{B}(r_0)} = \dfrac{B(\theta,r_0+r_0x)}{\bar{B}(r_0)} = \bar{\mu}(r) + f(\theta,r)$
- Where $f(\theta,r)$, the normalized azimuthal field variation, is considered as small:
  - First important approximation: results derived up to $O(f^2)$
- Therefore our limitation to « compact » cyclotrons (as compared to SSC's)

Main assumptions and approximations (2)

- The AVF function *f* is expanded in a Fourier series:

$$f(\theta, r) = \sum_n A_n(r)\cos n\theta + B_n(r)\sin n\theta = \sum_n C_n(r)\cos n(\theta - \varphi_n(r))$$

- And the reduced field is Taylor expanded:

$$\mu(\theta, x) = 1 + \bar{\mu}' x + \tfrac{1}{2}\bar{\mu}'' x^2 + \tfrac{1}{6}\bar{\mu}''' x^3 + \dots \\ + \sum_n (A_n + A_n' x + \tfrac{1}{2}A_n'' x^2 + \dots)\cos n\theta + \sum_n (B_n + B_n' x + \tfrac{1}{2}B_n'' x^2 + \dots)\sin n\theta$$

- with normalized average field derivatives:

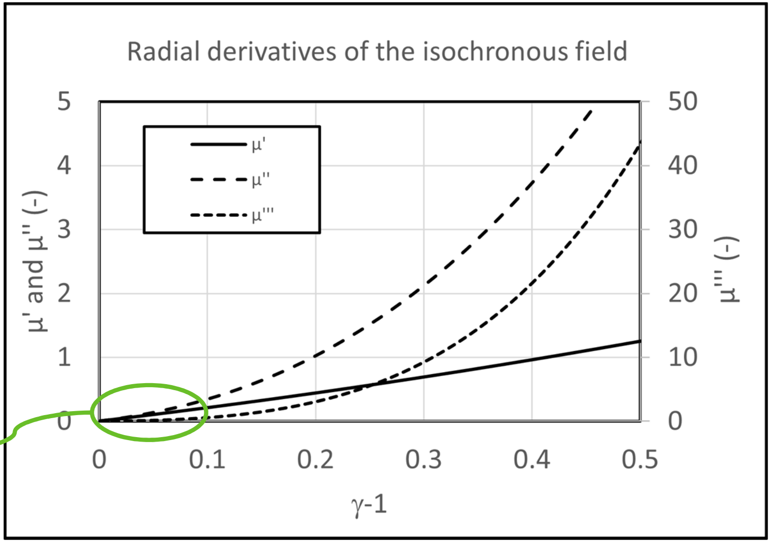
$$\bar{\mu}' = r\frac{d\bar{\mu}}{dr} \Rightarrow \text{quadrupole-term,}$$

$$\bar{\mu}'' = r^2\frac{d^2\bar{\mu}}{dr^2} \Rightarrow \text{sextupole-term, etc.}$$

- We assume that the magnetic field is **perfectly isochronous**. Then $(\bar{\mu}', \bar{\mu}'', \bar{\mu}''')$ can be expressed in the relativistic γ parameter

Main assumptions and approximations (3)

- An important generalization wrt the HV-study is that we do not consider $(\bar{\mu}', \bar{\mu}'', \bar{\mu}''')$ as small, because the half-integer resonance stopband happens at relativistic energies. We therefore have to keep cross-terms between $(\bar{\mu}', \bar{\mu}'', \bar{\mu}''')$ and the Fourier-terms A_n,B_n. This complicates the derivation considerably.



HV-non-relativistic zone

Main assumptions and approximations (4)

- The particle motion is mainly determined by the dominant Fourier term C_N. To simplify our final results we express the higher harmonics in terms of the dominant contribution, by assuming a hard-edge profile for $f(\theta,r)$, with equal hill and valley width. For this we find:

$$C_{(2k+1)N} = \frac{C_N}{2k+1}(-1)^k \,, \quad \text{for } k = 0, 1, 2, \ldots$$

- The flutter of this profile is:

$$F(r) = \frac{\langle B^2(\theta,r)\rangle - \langle B(\theta,r)\rangle^2}{\langle B(\theta,r)\rangle^2} = \frac{1}{2}\sum_n C_n^2(r) = \frac{1}{2}C_N^2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{16}C_N^2$$

- In this way, all Fourier coefficients can be expressed in the same flutter function:

$$C_n^2 = \frac{N^2}{n^2}C_N^2 = \frac{16F}{\pi^2(2k+1)^2}$$

This is the second important approximation in the derivation

Method of derivation (1)

- We start with a general form of the cyclotron Hamiltonian in polar coordinates.
- We decouple the horizontal/vertical motions by assuming that the horizontal motion is in the median plane and the projection of the vertical motion follows the Equilibrium Orbit (EO).
- This gives two separate Hamiltonians; H_x for the horizontal and H_z for the vertical motion.
- The EO is found as the periodic solution of H_x and is expressed in a Fourier series as function of the independent variable θ.
- New canonical variables wrt the EO are introduced and the Hamiltonian H_x is Taylor expanded with respect to these new variables
- Only terms up to 2nd degree in the canonical variables need to be kept. This corresponds to linear motion and is sufficient as the half-integer resonance $2\nu_r=N$ is linear.
- By 2nd canonical transformations, H_x and H_z are brought to the normal form:

$$H(p,x,\theta) = \tfrac{1}{2}p^2 + \tfrac{1}{2}(\nu_0^2 + f_x(\theta))x^2 \qquad \text{(A)}$$

Method of derivation (2)

- Here the parameters ν_{0x} and $f_x(\theta)$ depend only on the reduced magnetic field quantities.
- The term ν_{0x} is $O(f^0)$ and the term $f_x(\theta)$ is an oscillating term of $O(f^1)$ (zero average)
- We first solve the above Hamiltonian (A) in a general way.
- For that we design a third linear canonical transformation, that moves the oscillating part $f_x(\theta)$ to new terms of the next higher order $O(f^2)$. Within our required level of approximation, we only have to keep the average of these new terms.
- This transformation solves the motion (upto $O(f^2)$) as H becomes independent of θ
- The betatron tunes of the (horizontal or vertical) motion become:

$$\nu_x^2 = \nu_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{c_n^2}{n^2 - 4\nu_0^2} .$$

- Here c_n are the Fourier amplitude coefficients of $f_x(\theta)$.

Method of derivation (3)

- For the derivation of the $2\nu_r = N$ resonance we study the motion by introducing action-angle variables in a phase space that rotates with frequency *N/2.* and then move the oscillating part to the next higher order $O(f^2)$.
- The lower and upper stopband limits are obtained as:

$$\nu_0(1,2) = \frac{N}{2} \mp \frac{c_N}{2N} - \frac{3}{8}\frac{c_N^2}{N^3} - \frac{1}{2N}\sum_{n>N}\frac{c_n^2}{n^2 - N^2},$$

- Here ν_{0x} and c_n must be expressed in terms of the reduced magnetic field parameters.
- The $O(f^2)$-parts in ν_{0x} and $\nu_0(1,2)$ have the following general form:

$$R^{(2)} = \sum_n \alpha_n(\bar{\mu}', \bar{\mu}'', \ldots)C_n^2 + \beta_n(\bar{\mu}', \bar{\mu}'', \ldots)C_n^2\varphi_n'^2 + \gamma_n(\bar{\mu}', \bar{\mu}'', \ldots)C_nC_n' + \delta_n(\bar{\mu}', \bar{\mu}'', \ldots)C_n'^2$$

Method of derivation (4)

- We simplify the above expression by the following steps:

1. Assume that the harmonic phase derivatives φ'_n do not depend on *n* (valid for at least the first five harmonics) $\Rightarrow$ φ' can be taken out of the summation.
2. Express Fourier coefficients C_n in terms of the flutter *F* and take out of summation.
3. Express average field derivatives μ', μ'', μ''' in terms of relativistic γ (perfect isochronism)
4. Sum up the series analytically

- The O(f²)-parts are thus transformed to the simpler form:

$$R^{(2)} \approx F\left(a_N(\gamma) + b_N(\gamma)\varphi'^2 + c_N(\gamma)\frac{F'}{F} + d_N(\gamma)\left(\frac{F'}{F}\right)^2\right) \qquad \left(F' = r\frac{dF}{dr}\right)$$

Results (horizontal tune)

- Horizontal tune:

$$\nu_x^2 = 1 + \bar{\mu}'_{rel} + \frac{8N^2F}{\pi^2}\left[\tilde{a}_N + \tilde{b}_N\varphi'^2 + \tilde{c}_N\frac{F'}{F} + \tilde{d}_N\left(\frac{F'}{F}\right)^2\right].$$

$$1 + \bar{\mu}'_{rel} = \gamma^2$$

$$\tilde{a}_N(\gamma) = \tilde{q}_1 \tan\left(\frac{\pi\gamma}{2N}\right) + \tilde{q}_2 \tan\left(\frac{\pi\gamma}{N}\right) + \tilde{q}_3\left(1+\tan^2\left(\frac{\pi\gamma}{2N}\right)\right) + \tilde{q}_4 + \tilde{q}_5 \tan\left(\frac{\pi\gamma}{2N}\right)\left(1+\tan^2\left(\frac{\pi\gamma}{2N}\right)\right),$$

$$\tilde{b}_N(\gamma) = \frac{\pi}{8\gamma N}\left[\tan\left(\frac{\pi\gamma}{N}\right) - 2\tan\left(\frac{\pi\gamma}{2N}\right)\right], \quad \tilde{d}_N(\gamma) = \frac{\pi}{128\gamma^3 N}\left[\frac{3\pi\gamma}{N} + \tan\left(\frac{\pi\gamma}{N}\right) - 8\tan\left(\frac{\pi\gamma}{2N}\right)\right]$$

$$\tilde{c}_N(\gamma) = \frac{\pi}{96\gamma^3 N}\left[(11-9\gamma^2)\frac{3\pi\gamma}{N} + 3(\gamma^2+1)\tan\left(\frac{\pi\gamma}{N}\right) + 24(2\gamma^2-3)\tan\left(\frac{\pi\gamma}{2N}\right) - \frac{12\pi\gamma}{N}(\gamma^2-1)\tan^2\left(\frac{\pi\gamma}{2N}\right)\right].$$

The coefficients $\tilde{q}_i$ are defined as:

$$\tilde{q}_1(\gamma) = \frac{-\pi}{8N\gamma^3}\left[6-(\gamma^2+1)^2+15\tilde{q}_0\right], \quad \tilde{q}_2(\gamma) = \frac{+\pi}{32N\gamma^3}(\gamma^2+1)^2,$$

$$\tilde{q}_3(\gamma) = \frac{\pi^2}{16N^2\gamma^2}\left[4-(\gamma^2+1)^2+7\tilde{q}_0\right], \quad \tilde{q}_4(\gamma) = \frac{+\pi^2}{32N^2\gamma^2}\left[4-(\gamma^2+1)^2+16\tilde{q}_0\right],$$

$$\tilde{q}_5(\gamma) = \frac{-\pi^3\tilde{q}_0}{16N^3\gamma}, \quad \tilde{q}_0(\gamma) = \frac{1}{4\gamma^4}\left(4+(\gamma^2-1)(\gamma^2+10)\right)(\gamma^2-1)^2.$$

- A cyclotron design template was made in excel. The first 3 columns pasted in by the user:
  1. radius r
  2. flutter F
  3. the azimuth φ of the center of the magnetic sector
- And three constants:
  1. the number of sectors N
  2. the particle revolution frequency f_{rev}
  3. the central field B_0
- The template calculates all results derived here:
  - Tunes, resonance stopband, B_{iso}, relation $\gamma=\gamma(r)$

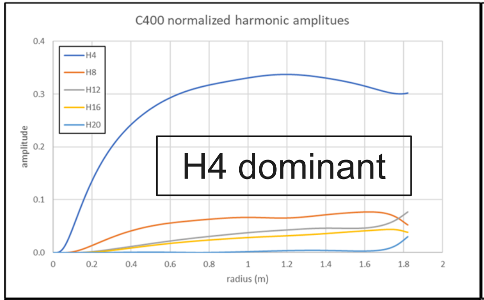
Benchmark against IBA/NHa C400 carbon therapy cyclotron (1)

Invited talk THBI1
by Jerome Mandrillon

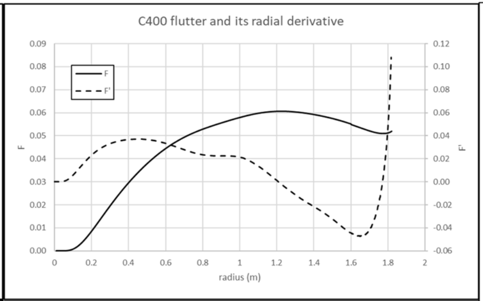
Harmonic content of the C400 cyclotron

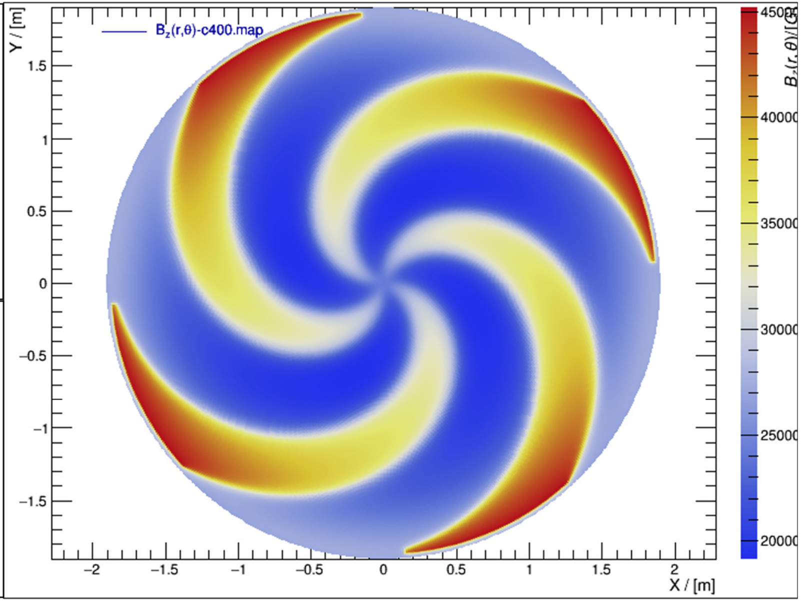
Harmonic amplitudes

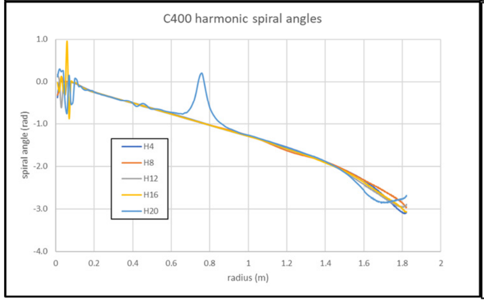
Flutter *F* and *F’*

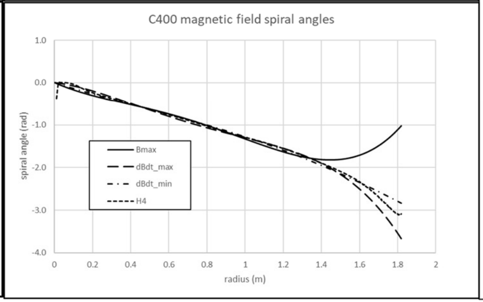


H4 dominant









Phase-derivatives $r d\varphi_n/dr$ almost independent on n

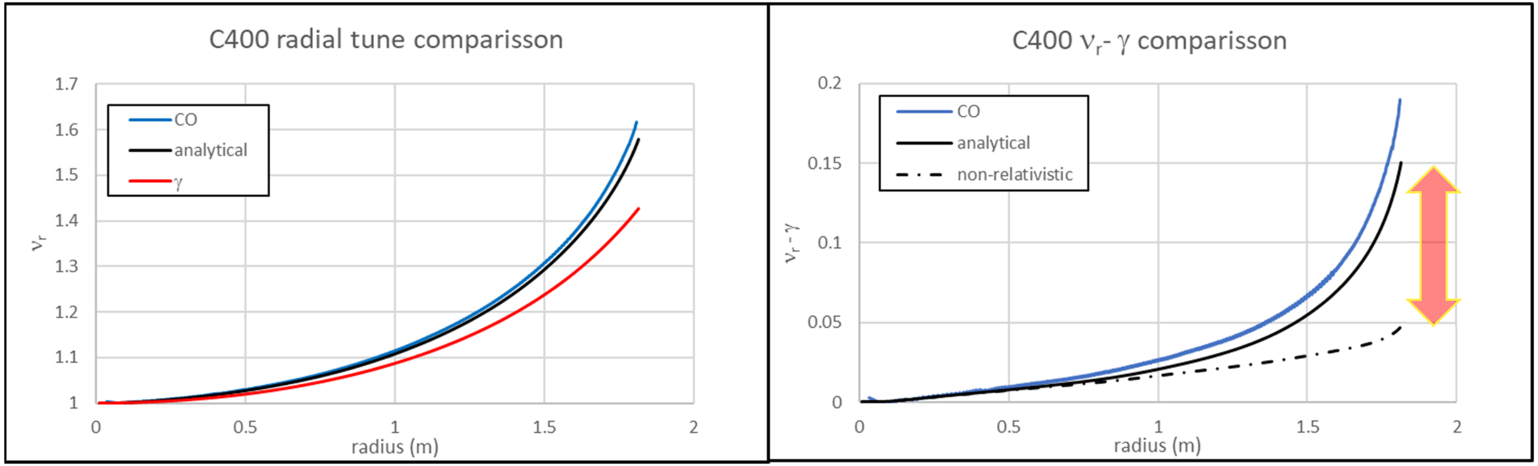
4 alternatives for $\varphi' = r d\varphi/dr$

Benchmark against IBA/NHa C400 carbon therapy cyclotron (2)

radial tune

Total: 1+μ'+*F*-terms

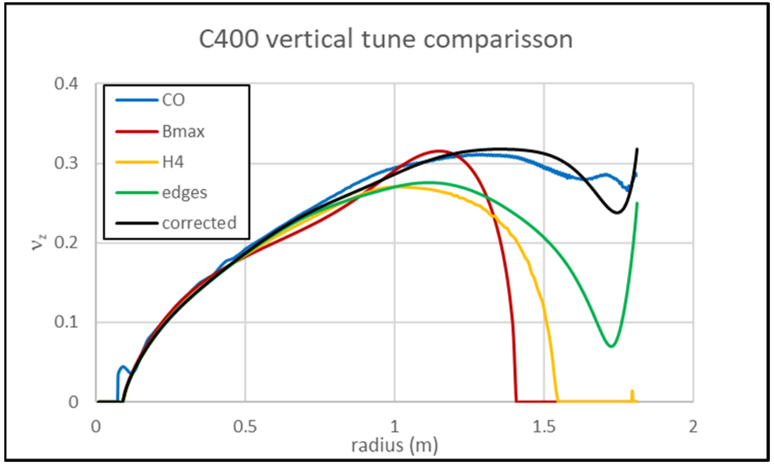
F-terms only



- Good agreement between closed orbit code (CO) and analytical results for the « F-terms only » case on the right give confidence in correctness of our approach
- The non-relativistic approach (as used in HV-paper) is not precise at higher energies and therefore not good enough to predict the $2\nu_r$=*N* resonance stopband accurately

Results (vertical tune)

- The vertical tune is very sensitive to the precise definition of the harmonic phase derivative φ'= *rdφ/dr*. This is due to the fact that the b_N term dominates at higher energies and that ν_z is the difference between two « large » numbers.



$$\nu_z^2 = -\bar{\mu}'_{\rm rel} + \frac{8N^2F}{\pi^2}\left[\hat{a}_N + \hat{b}_N\varphi'^2 + \hat{c}_N\frac{F'}{F} + \hat{d}_N\left(\frac{F'}{F}\right)^2\right].$$

Blue = closed orbit (CO)
Red=azimuth where B is maximum
Orange= phase of harmonic 4
Green = average of sector edges

$$\left(\frac{dB}{d\theta}\Big|_{max} + \frac{dB}{d\theta}\Big|_{min}\right)/2$$

The EO is not a circle: this changes the pole face angles at the sector entrance and exit. We can correct for this.

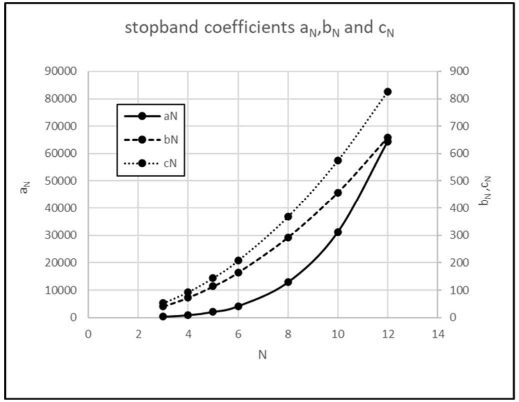
$$\varphi'_{corr} = \varphi'\left(1 + \frac{\pi^2 F}{4N^2}(1+\varphi'^2)\right) + O(f^4)$$

This gives the black curve, which agrees very well with CO

Results ($2\nu_r = N$ stopband)

$$\gamma_{1,2} = \frac{N}{2} \mp \frac{2\sqrt{F}}{\pi N}\sqrt{\left(1+\frac{N^2}{4}+\frac{F'}{2F}\right)^2 + N^2\varphi_N'^2} - \frac{F}{\pi^2 N^3}\left(\bar{a}_N - \bar{b}_N\varphi'^2 - \bar{c}_N\frac{F'}{F} + \bar{d}_N\left(\frac{F'}{F}\right)^2\right),$$

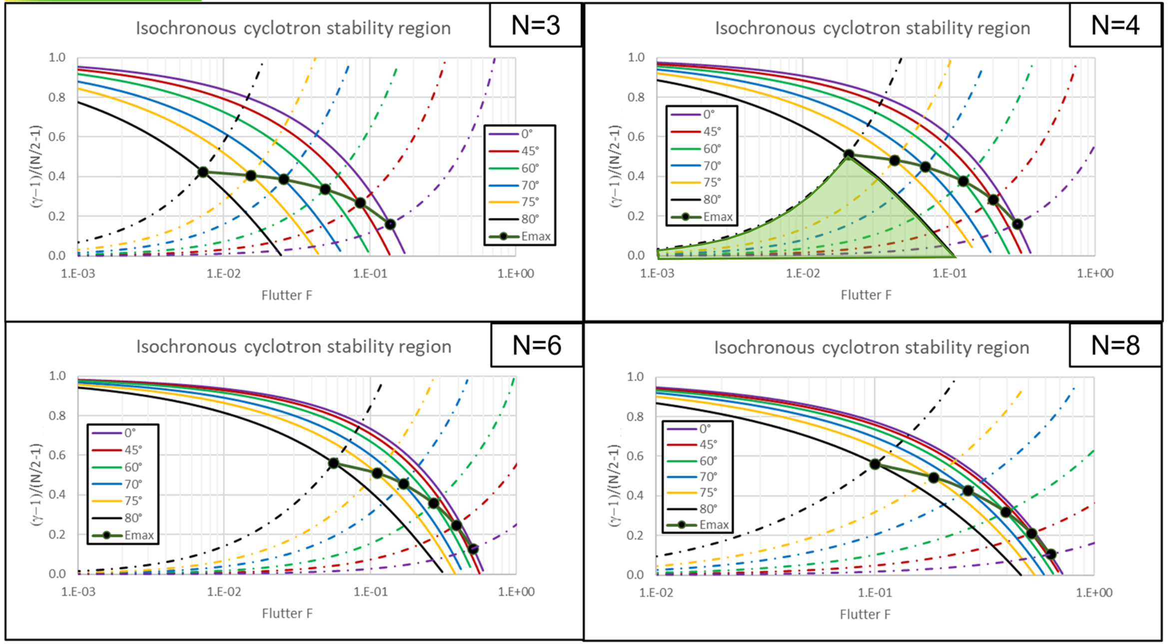
$O(f^0)$ $O(f^1)$ $O(f^2)$



N	a_N	b_N	c_N	d_N
3	312.4	41.1	52.8	1.164
4	877.2	73.1	92.9	1.164
5	2043.8	114.2	144.4	1.164
6	4145.6	164.4	207.4	1.164
8	12856.6	292.2	367.6	1.164
10	31149.8	456.6	573.7	1.164
12	64346.3	657.6	825.6	1.164

Stopband coefficients depend only on *N*

Cyclotron stability region



- Vertical focusing limit (ν_z=0, dashed curve) and resonance limit (γ_1, solid curve) as function of flutter *F* (logaritmic scale) for a number of the sector spiral angles ξ

$$\tan \xi = \varphi'$$

- Focusing limit monotonically increases with *F* and ξ
- $2\nu_r = N$ limit monotonically decreases with *F* and ξ
- Stable area is area below both limits (green area in *N*=4, ξ=80° plot)

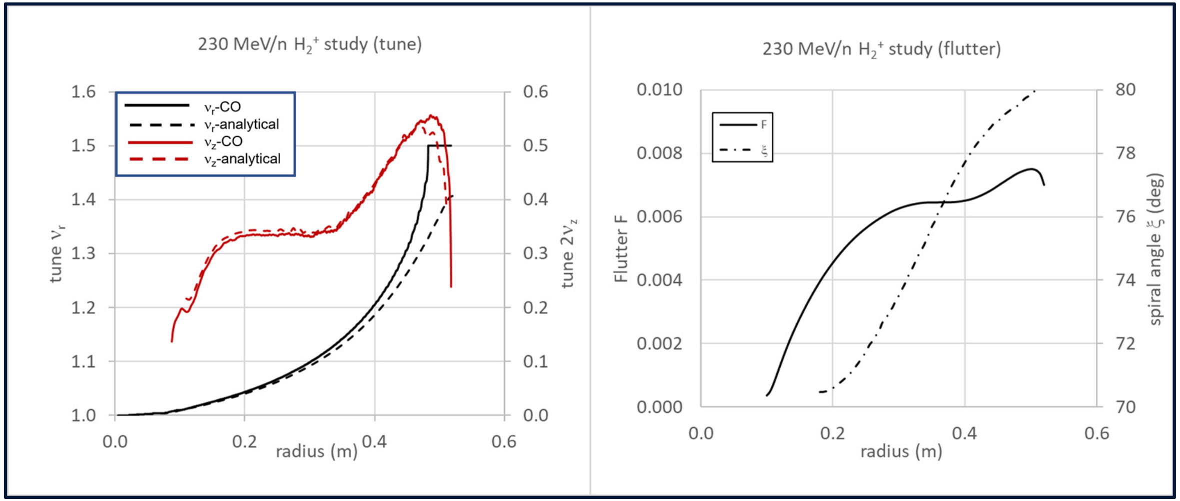
- The intersections between focusing and resonance limits represents the energy limit for a given symmetry number *N* and a given spiral angle ξ
- These are the solid dots in the plot.

Energy limits

	N=3		N=4		N=6	
ξ (deg)	F	T (MeV/u)	F	T (MeV/u)	F	T (MeV/u)
0	0.1384	75.7	0.2934	151	0.5063	243
45	0.0858	125	0.1995	263	0.387	464
60	0.0495	157	0.1245	352	0.272	668
70	0.0257	180	0.0686	418	0.168	850
75	0.0153	190	0.0424	450	0.111	950
80	0.0072	198	0.0204	477	0.056	1045
	N=8		N=10		N=12	
0	0.6324	294	0.7135	326	0.7697	348
45	0.5180	587	0.6085	673	0.6741	732
60	0.3945	891	0.4880	1049	0.5607	1170
70	0.2653	1193	0.3510	1465	0.4250	1677
75	0.1852	1380	0.2571	1738	0.3231	2034
80	0.1000	1572	0.1488	2047	0.1975	2471

- Our result for N = 3 agrees very well with the result found by King and Walkinshaw (Ref. 4) who report, in their study on the conversion of the Harwell synchrocyclotron, a proton energy limit of 190 MeV for a spiral angle of about 75°.
- For N=4 and a spiral angle of 75° an energy limit of about 450 MeV/u is predicted

Example for a 230 MeV/u H_2^+ magnet study with *N*=3



tunes:
ν_r = **black**
$2\nu_z$=red
solid = closed orbit
dashed=analytical

field modulation:
solid = flutter *F*
dashed=spiral angle ξ

- Here the resonance sets in at E= 187.5 MeV/u (r=48.2 cm)
- Spiral angle ξ=79.5° and *F*=0.0074 with ν_z=0.27
- Our table shows an energy limit of 198 MeV and a flutter of 0.0072
- Correcting for the non-zero vertical tune

$$\Delta\gamma \approx -\nu_z^2/2\gamma$$

- We find a stopband energy limit of 186.5 MeV/u, very close to the numerical value

Conclusion

1. This work determines the minimum number of sectors that are needed for obtaining a given top energy with a compact isochronous cyclotron.
2. It also provides the best tradeoff between spiral angle and flutter.
3. The provided equations, graphs and tables should be useful especially for the conceptual and/or preliminary design of a new cyclotron.
4. More numerical benchmarks are foreseen, to explore the applicability of the model