Matched Distributions in Cyclotrons with Higher Order Moments of the Charge Distribution

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Outline

Task Description

[Fre15]

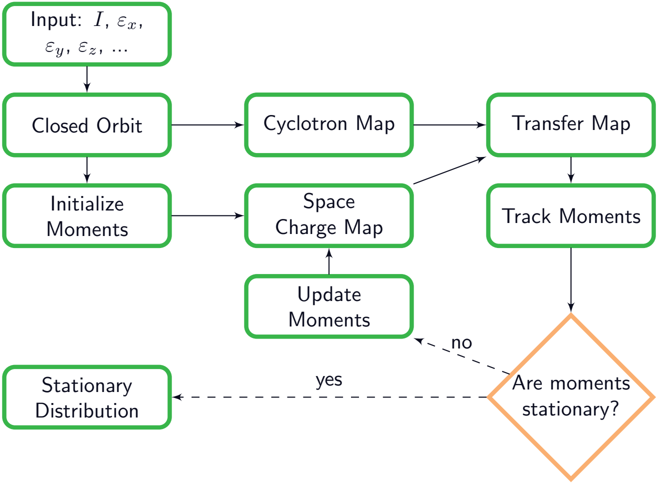
- **Goal:** Find non-linear mapping $\mathcal{M}$ and $\sigma^{(k)}$ such that

$$\begin{aligned} \sigma^{(k)}(s) &= \mathcal{M} \circ \sigma^{(k)}(0) \circ \mathcal{M}^{\mathcal{T}} \text{ and} \\ \sigma^{(k)} &= \sigma^{(k)}(s) \stackrel{!}{=} \sigma^{(k)}(0) \end{aligned} \quad (1)$$

  where $\sigma^{(k)}(t)$ are the k-th order moments of a distribution in 6-dimensional phase space and $\circ$, $\mathcal{T}$ are operations.
- **Input:** Energy, emittances, intensity, field map, higher order moments
- **Linear Theory:**
  - C. Baumgarten. Transverse-longitudinal coupling by space charge in cyclotrons. Phys. Rev. ST Accel. Beams, 14:114201, Nov 2011.
  - C. Baumgarten. Geometrical method of decoupling. Phys. Rev. ST Accel. Beams, 15:124001, Dec 2012.
- **Tests:** PSI Injector-2, PSI Ring Cyclotron (coasting beam)

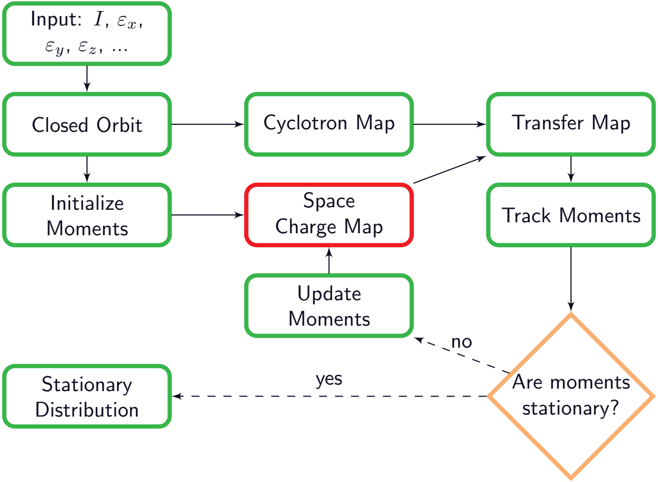
Big Picture

Process of finding a Matched Distribution



Map Generation

Context



From Hamiltonian to Map

- Hamiltonian of particle accelerators

$$H = H_{mag} + H_{sc},$$

with magnetic part H_{mag} and space charge part

$$H_{sc} = q\phi,$$

where q is the charge and ϕ the self-field potential.

- **Assumptions:**
  - No collisions, no residual gas, no walls
  - Linear magnetic force ($\rightarrow H_{mag}$ 2nd order)

From Hamiltonian to Map

- Lie Algebra to obtain transfer maps in

$$\mathcal{M} = e^{-s\,:H:} \quad \text{with} \quad :H: = \frac{\partial H}{\partial \vec{q}}\frac{\partial}{\partial \vec{p}} - \frac{\partial H}{\partial \vec{p}}\frac{\partial}{\partial \vec{q}},$$

where $(\vec{q}, \vec{p})$ are dynamical variables.

- Truncated-Power-Series-Algebra [1]

[1] Differential Algebra package used in OPAL

Computation of Non-Linear Space Charge Map

- **Ansatz:** Compute non-linear $\mathcal{M}_{sc}$ starting from self-field potential of ellipsoid [Glu86]

$$\phi(x,y,z) = \frac{Q a_x a_y a_z}{4\varepsilon_0} \int_0^\infty \frac{1}{\sqrt{\left(a_x^2+u\right)\left(a_y^2+u\right)\left(a_z^2+u\right)}} \int_{S(u)}^\infty f(s) ds du,$$

where

$$S(u) = \frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u},$$

with semi-principal axes a_x, a_y and a_z.

Computation of Non-Linear Space Charge Map

- Since we defined

$$H_{sc} = q\phi,$$

we get

$$\mathcal{M}_{sc} = e^{-s\,:H_{sc}:} = e^{-sq\,:\phi:} = 1 - sq\,:\phi: + \frac{(sq)^2}{2}\,:\phi:^2 - \frac{(sq)^3}{3!}\,:\phi:^3 + \ldots,$$

where

$$:\phi: = \frac{\partial \phi}{\partial x}\frac{\partial}{\partial p_x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial p_y} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial p_z} - \underbrace{\frac{\partial \phi}{\partial p_x}}_{0}\frac{\partial}{\partial x} - \underbrace{\frac{\partial \phi}{\partial p_y}}_{0}\frac{\partial}{\partial y} - \underbrace{\frac{\partial \phi}{\partial p_z}}_{0}\frac{\partial}{\partial z}$$

$$= \frac{\partial \phi}{\partial x}\frac{\partial}{\partial p_x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial p_y} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial p_z}$$

$$:\phi: = \boxed{-E_x \frac{\partial}{\partial p_x} - E_y \frac{\partial}{\partial p_y} - E_z \frac{\partial}{\partial p_z}}$$

Computation of Non-Linear Space Charge Map

Expansion of Electric Field

- Since $\vec{E}$ independent of momenta $\implies :\phi:^{j} \equiv 0 \; \forall j > 1$, thus

$$\mathcal{M}_{sc} = e^{-sq\,:\phi:} = 1 - sq\,:\phi: = 1 + sq\left(E_x \frac{\partial}{\partial p_x} + E_y \frac{\partial}{\partial p_y} + E_z \frac{\partial}{\partial p_z}\right)$$

Idea: Expanding electric field

$$E_w \approx \sum_{i=0}^{n} c_{w,i} w^i,$$

where $w \in \{x, y, z\}$.

Computation of Non-Linear Space Charge Map

Expansion of Electric Field

- General form of space charge map:

$$\mathcal{M}_{sc}(s) = \begin{pmatrix} 1 \\ 1 + sqE_x \\ 1 \\ 1 + sqE_y \\ 1 \\ 1 + sq\gamma^2 E_z \end{pmatrix} \approx \begin{pmatrix} 1 \\ 1 + sq\sum_{i=0}^{n} c_{x,i}x^i \\ 1 \\ 1 + sq\sum_{i=0}^{n} c_{y,i}y^i \\ 1 \\ 1 + sq\gamma^2 \sum_{i=0}^{n} c_{z,i}z^i \end{pmatrix},$$

  with variable ordering (x, p_x, y, p_y, z, p_z).

- **Question:** How do we get coefficients $c_{x,i}$, $c_{y,i}$, $c_{z,i}$?
  $\rightarrow$ Least squares minimization (LSM)

Computation of Non-Linear Space Charge Map

Expansion of Electric Field

- LSM yields to a linear system of equations:

$$\underbrace{\begin{pmatrix} \langle 1\rangle & \langle w\rangle & \langle w^2\rangle & \cdots & \langle w^n\rangle \\ \langle w\rangle & \langle w^2\rangle & \langle w^3\rangle & \cdots & \langle w^{n+1}\rangle \\ \langle w^2\rangle & \langle w^3\rangle & \ddots & & \vdots \\ \vdots & \vdots & & \langle w^{2n-2}\rangle & \langle w^{2n-1}\rangle \\ \langle w^n\rangle & \langle w^{n+1}\rangle & \dots & \langle w^{2n-1}\rangle & \langle w^{2n}\rangle \end{pmatrix}}_{\text{symmetric, only monomials}} \cdot \begin{pmatrix} c_{w,0} \\ c_{w,1} \\ \vdots \\ c_{w,n-1} \\ c_{w,n} \end{pmatrix} = \underbrace{\begin{pmatrix} \langle E_w\rangle \\ \langle E_w w\rangle \\ \vdots \\ \langle E_w w^{n-1}\rangle \\ \langle E_w w^n\rangle \end{pmatrix}}_{\text{E-field moments}}$$

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Example: Linear Expansion of Electric Field

- **Assumption:**
  - Centered beam, i.e. $\langle x \rangle = \langle y \rangle = \langle z \rangle = 0$
  - $E_w \approx c_{w,0} + c_{w,1} w$
- **Result:**

$$\begin{pmatrix} 1 & 0 \\ 0 & \langle w^2 \rangle \end{pmatrix} \cdot \begin{pmatrix} c_{w,0} \\ c_{w,1} \end{pmatrix} = \begin{pmatrix} 0 \\ \langle E_w w \rangle \end{pmatrix} \Longrightarrow E_w \approx \frac{\langle w E_w \rangle}{\langle w^2 \rangle} w$$

where $w \in \{x, y, z\}$ and moments $\langle \cdot \rangle$. Thus, we get

$$M_{sc}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ q\frac{\langle x E_x \rangle}{\langle x^2 \rangle} s & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & q\frac{\langle y E_y \rangle}{\langle y^2 \rangle} s & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & q\gamma^2 \frac{\langle z E_z \rangle}{\langle z^2 \rangle} s & 1 \end{pmatrix}.$$

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Linear Space Charge Map

- Space charge Hamiltonian

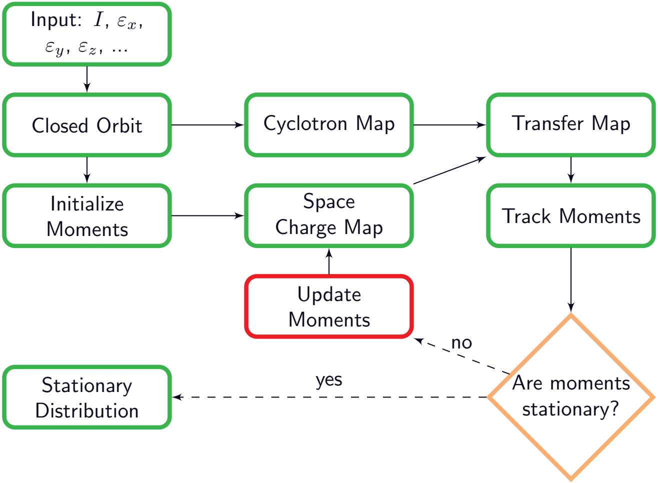
$$H_{sc} = -\frac{K_x}{2}x^2 - \frac{K_y}{2}y^2 - \frac{\gamma^2 K_z}{2}z^2,$$

with space charge strengths K_x, K_y and K_z. This leads to

$$M_{sc}(s) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ K_x s & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & K_y s & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & K_z \gamma^2 s & 1 \end{pmatrix}.$$

Updating Moments

Context



Updating Moments - Minimization Problem

Goal:

$$\langle x^i p_x^j y^k p_y^l z^m p_z^n \rangle(s) \equiv \langle x^i p_x^j y^k p_y^l z^m p_z^n \rangle(s+L) \quad \forall s,$$

where L is the length of the orbit and $\langle x^i p_x^j y^k p_y^l z^m p_z^n \rangle$ specifies a general moment.

- **Minimization Problem:**

$$\arg\min \frac{1}{2} \left|\left| \langle x^i p_x^j y^k p_y^l z^m p_z^n \rangle(s+L) - \langle x^i p_x^j y^k p_y^l z^m p_z^n \rangle(s) \right|\right|_2^2 .$$

($\arg\min f(x) \hat{=}$ find x for which $f(x)$ is minimal)

- **Matrix form:**

$$\arg\min \frac{1}{2} \left|\left| A\vec{x}_{s+L} - \vec{x}_s \right|\right|_2^2 .$$

Updating Moments - Constraining the System

- **Matrix form:**

$$\arg\min \frac{1}{2} \left|\left| A\vec{x}_{s+L} - \vec{x}_s \right|\right|_2^2 .$$

Attention: Matched distribution $\rightarrow \vec{x}_{s+L} \equiv \vec{x}_s$

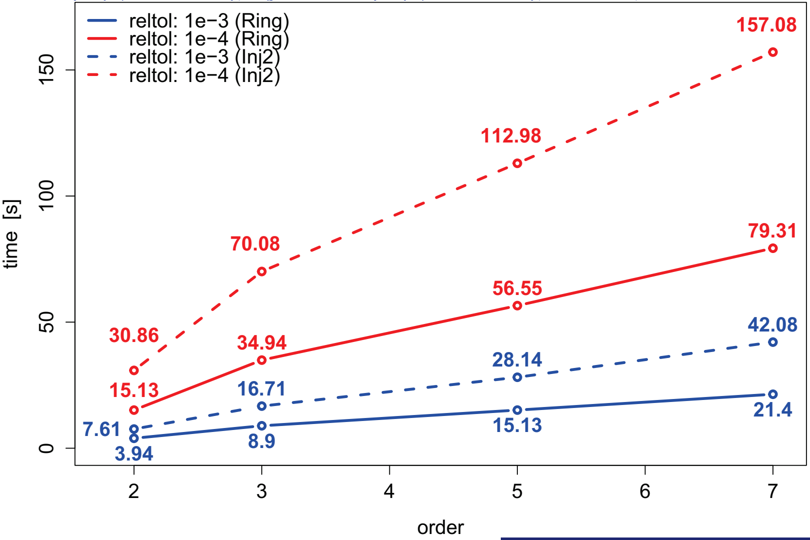
- **Questions:**
  - How do we avoid non-physical results?
  - How do we incorporate emittances?

  $\rightarrow$ Lagrange multipliers
  - How do we solve that problem?

  $\rightarrow$ Fixed point computation, e.g. Newton-Raphson method

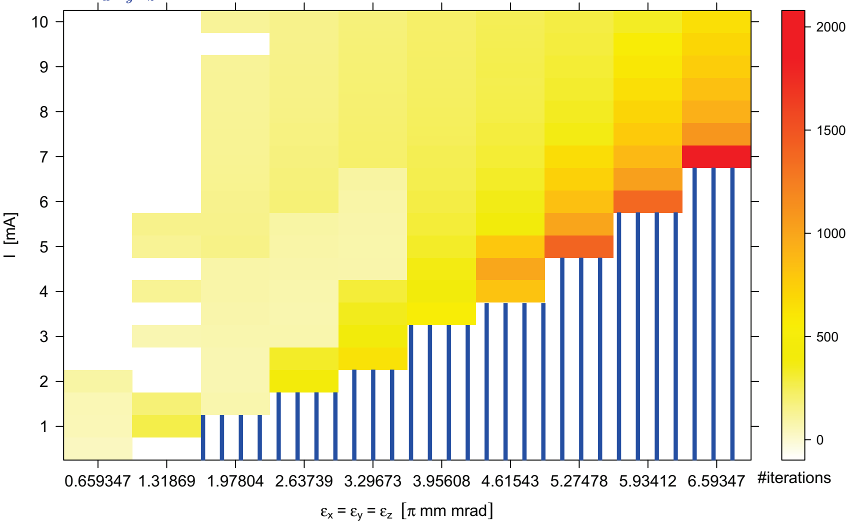
Results: Space Charge Map Constructions $\mathcal{M}_{sc}$

Ring: 180 maps (1/8 machine), Inj2: 360 maps (1/4 machine); Gaussian, 16 cores



Results: Linear Stability PSI Injector-2

Uniform ($f = \frac{1}{8a_x a_y a_z}$), $E = 2$ MeV



Results: Non-Linear Matching

- **Goal:**

$$\sigma^{(3)}(s) = \mathcal{M} \circ \sigma^{(3)}(0) \circ \mathcal{M}^{\mathcal{T}} \text{ and}$$
$$\sigma^{(3)} = \sigma^{(3)}(s) \stackrel{!}{=} \sigma^{(3)}(0)$$

- **Assumptions:**
  - beam centered, i.e. $\langle x \rangle = \langle y \rangle = \langle z \rangle = 0$
  - symmetric, $\langle E_x \rangle = \langle E_y \rangle = \langle E_z \rangle = 0$
  - moments of order > 3 equal zero
- **Arbitrary distribution perturbation:**

$$f(x, y, z) = f_{Gaussian}(x, y, z) \cdot \exp\left[-k\langle x^3 \rangle \frac{x^3}{\langle x^2 \rangle^{3/2}}\right]$$

with constant $k = 0.1$.

- Initially 3rd order moments uniformly in $[-0.01, 0.01]$

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Non-Linear Matching

- L_2-error: $9.8451 \cdot 10^{-4}$

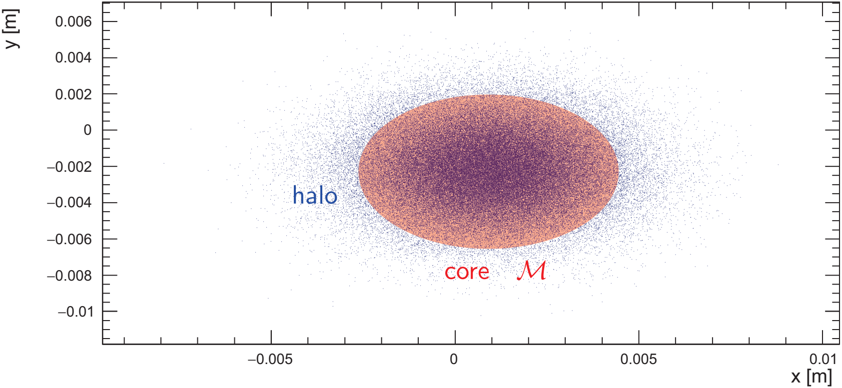
$\langle x^3 \rangle$	$= -0.1946$,	$\langle x^2 p_x \rangle$	$= -7.5797 \cdot 10^{-5}$,	$\langle x^2 y \rangle$	$= 0.0954$,	$\langle x^2 p_y \rangle$	$= -0.0136$,
$\langle x^2 z \rangle$	$= -0.0786$,	$\langle x^2 p_z \rangle$	$= -0.0393$,	$\langle x p_x^2 \rangle$	$= -0.0131$,	$\langle x p_x y \rangle$	$= 0.0042$,
$\langle x p_x p_y \rangle$	$= 0.0095$,	$\langle x p_x z \rangle$	$= -0.0231$,	$\langle x p_x p_z \rangle$	$= 5.1948 \cdot 1$[illegible]$^{-5}$,	$\langle x y^2 \rangle$	$= 0.0070$,
$\langle x y p_y \rangle$	$= 0.0012$,	$\langle x y z \rangle$	$= 0.0399$,	$\langle x y p_z \rangle$	$= 0.0263$,	$\langle x p_y^2 \rangle$	$= 0.0006$,
$\langle x p_y z \rangle$	$= 0.0108$,	$\langle x p_y p_z \rangle$	$= -0.0035$,	$\langle x z^2 \rangle$	$= -$[illegible]$.015$[illegible],	$\langle x z p_z \rangle$	$= -0.0285$,
$\langle x p_z^2 \rangle$	$= -0.0029$,	$\langle p_x^3 \rangle$	$= 0.0004$,	$\langle p_x^2 y \rangle$	$= 0.$[illegible]004,	$\langle p_x^2 p_y \rangle$	$= -0.0007$,
$\langle p_x^2 z \rangle$	$= -0.0051$,	$\langle p_x^2 p_z \rangle$	$= -0.0072$,	$\langle p_x y^2 \rangle$	$=$ [illegible]0007,	$\langle p_x y p_y \rangle$	$= 0.0004$,
$\langle p_x y z \rangle$	$= 0.0061$,	$\langle p_x y p_z \rangle$	$= 0.0023$,	$\langle p_x p_y^2 \rangle$	$=$ [illegible]$.0004$,	$\langle p_x p_y z \rangle$	$= 0.0049$,
$\langle p_x p_y p_z \rangle$	$= 0.0052$,	$\langle p_x z^2 \rangle$	$= -$[illegible]$.020$[illegible],	$\langle p_x z p_z \rangle$	$= -0.0124$,	$\langle p_x p_z^2 \rangle$	$= 0.0003$,
$\langle y^3 \rangle$	$= -0.0017$,	$\langle y^2 p_y \rangle$	$= -0.0$[illegible]15,	$\langle y^2 z \rangle$	$= -0.0310$,	$\langle y^2 p_z \rangle$	$= 0.0033$,
$\langle y p_y^2 \rangle$	$= -0.$[illegible]001,	$\langle y p_y z \rangle$	$=$ [illegible]$.0009$,	$\langle y p_y p_z \rangle$	$= -1.1216 \cdot 10^{-5}$,	$\langle y z^2 \rangle$	$= 0.0185$,
$\langle y z p_z \rangle$	$= 0.$[illegible]55,	$\langle y p_z^2 \rangle$	$= 0.0009$,	$\langle p_y^3 \rangle$	$= -0.0006$,	$\langle p_y^2 z \rangle$	$= -0.0054$,
$\langle p_y^2 p_z \rangle$	$= 0.0007$,	$\langle p_y z^2 \rangle$	$= 0.0185$,	$\langle p_y z p_z \rangle$	$= 0.0086$,	$\langle p_y p_z^2 \rangle$	$= -0.0009$,
$\langle z^3 \rangle$	$= -0.1298$,	$\langle z^2 p_z \rangle$	$= -0.0073$,	$\langle z p_z^2 \rangle$	$= -0.0154$,	$\langle p_z^3 \rangle$	$= -0.0011$

Converged!

Conclusion

- **Conclusions:**
  - TPSA based
  - Good agreement of linear theory
  - Lagrange method allows extension of constraints
  - Non-linear matching is working
- **Open Questions:**
  - What is the physical relevance of non-linear matching?
  - Linear Stability behaviour? Why do we observe gaps?
  - System constraints? Higher order invariants?
  - Relative error at peak of max 15 % in the field expansion for 9th order (spherical Gaussian).
- **Improvements of the presented method:**
  - Convergence rate
  - Computation time (parallelisation?)
  - Initialisation of moments
- **Take home points:**
  - Non-linear matching
  - Non-linear maps

Outlook: Non Linear Particle Core Model



- Particles outside feel non-linear forces in existing particle core models
- Higher order maps for the core → particles passing the core feel non-linear forces

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References

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[Glu86] R. L. Gluckstern. Scalar potential for charge distributions with ellipsoidal symmetry. Fermilab Internal Report TM-1402, May 1986.