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Stochastic cooling as Wiener process

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Traditional theoretical description of stochastic cooling process involves either ordinary differential equations for desired rms quantities or corresponding Fokker-Planck equations. Both approaches use different methods of derivation and seem independent, making transition from one to another quite an issue, incidentally entangling somewhat the basic physics underneath. On the other hand, description of the stochastic cooling starting from the single-particle dynamics written in the form of Langevin equation seems to bring more clarity and integrity. Present work is an attempt to develop a simple and consistent way of deriving well-known equations for stochastic cooling process description starting from Langevin equations.

Langevin equation

Consider an ensemble of *non-interacting* particles orbiting in an accelerator and undergoing a stochastic cooling. On each revolution every particle receives a correction kick from the cooling system, that is the sum of the self-signal of that particle (*coherent signal*) and some random noise signal due to signals from other particles and noises in the electronics (*incoherent signal*).

We are interested in the evolution of some parameter x (momentum spread, emittance, rms betatron amplitudes, etc.) of an arbitrary particle under influence of stochastic cooling system. Since particle parameter depends solely on its present state and we can consider step $dt \to 0$ and number of particles $N \to \infty$, the process of stochastic cooling is a *continuous Wiener process* and all related formalism could be applied for the present case.

The corresponding Langevin equation for non-constant diffusion:

$$\frac{dx}{dt} = F + \underline{\frac{1}{2}\frac{\partial D}{\partial x}} + \sqrt{D}\xi(t) \qquad \begin{aligned}\langle \xi \rangle &= 0 \\ \langle \xi(t)\xi(t') \rangle &= \delta(t-t')\end{aligned}$$

white noise

Where $F(x,t) = f_0 \Delta x_c(x,t), D(x,t) = 1/2 f_0 \Delta x_{ic}^2$ - are the usual drift and diffusion coefficients. The underlined summand is needed to compensate the effect of non-constant diffusion, a so-called *noise-induced drift* (see Fokker-Planck).

In this form equation could be used for tracking simulations in software like Betacool, in order to include different effects like IBS or electron cooling altogether in the similar fashion.

Time-averaged diffusion

We anticipate that incoherent effect for a given particle has following statistics:

$$\langle x_{ic} \rangle = 0$$
$$\langle x_{ic}^2 \rangle = \Delta x_{ic}^2$$

So for *incoherent effect* on the long-term average we expect that:

$$\frac{d}{dt}x^2 = \overline{\left(\frac{1}{2}\frac{\partial D}{\partial x} + \sqrt{D}\xi(t)\right)^2} = f_0 \Delta x_{ic}^2 = 2D$$

Thus we immediately derive the formula for the incoherent dynamics:

$$\frac{dx}{dt} = \frac{D}{x}$$

Summing up coherent and incoherent effects for single particle we the get following equation:

$$\frac{dx}{dt} = F + \frac{D}{x}$$

We can rewrite this equation for the rms-particle (at a given time):

$$\frac{1}{\tau} = -\frac{1}{x_{rms}}\frac{dx_{rms}}{dt} = -\frac{F}{x_{rms}} - \frac{D}{x_{rms}^2}$$

Oversimplifying, we then immediately derive the well-known time-domain formula, considering:

- Flat distribution of N particles
- $\Delta x_c = -\lambda x$ (coherent correction is proportional to particle parameter value)
- $\Delta x_{ic}^2 = \lambda^2 \cdot x^2 N_s + \lambda^2 \cdot Thermal\ noise$ (incoherent correction is proportional to the sum of particles in the sample $N_s = N/(2WT_0)$ and a thermal noise)

Under following assumptions the equation for rms-particle simplifies to

$$\frac{1}{\tau} = \frac{W}{N}[2g - g^2(1+U)]$$

where $g = \lambda N_s, U = Therm.noise/(x^2 N_s)$

Fokker-Planck equation

Consider we have a general Langevin equation:

$$\frac{dx}{dt} = h(x,t) + g(x,t)\xi(t)$$

We known that there is always a corresponding Fokker-Planck equation for the given problem:

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial x}\left[\left(\mathrm{h} + \underline{\frac{1}{2}g\frac{\partial g}{\partial x}}\right)\Psi\right] + \frac{\partial^2}{\partial x^2}(g^2\Psi)$$

Where $\frac{1}{2}g\frac{\partial g}{\partial x}$ is a so-called *noise-induced drift*.

In order to compensate this additional drift the additional summand was added in the original Langevin equation. Thus for our case we immediately have the Fokker-Planck equation in its traditional for stochastic cooling form:

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial x}(F\Psi) + \frac{\partial}{\partial x}\left(D\frac{\partial \Psi}{\partial x}\right)$$

While without this compensational term we would have a different and *incorrect* form of the equation:

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial x}(F\Psi) + \frac{\partial^2}{\partial x^2}(D\Psi)$$

It was quite an issue in the early days, which form of equation is suitable for the stochastic cooling.

summary

The stochastic cooling theory was formulated as continuous Wiener process. Such treatment derives the well-known equations in a clear and natural way, and besides it provides:

- Langevin equation for tracking simulations
- Usual differential equations for initial cooling time in more general form
- Explains the form of Fokker-Planck equation (at least technically)
- All three simulation approaches use same well-established drift and diffusion coefficients

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