Fokker-Planck Description of Transverse Stochastic Cooling

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Classical Cooling Rate

system bandwidth

system gain

$$\frac{1}{\tau} = \frac{2W}{N}\left(gB - g^2\left(M + U\right)\right)$$

cooling rate

particle number

undesired mixing „bad“

desired mixing (Schottky noise) „good“

thermal heating (thermal noise)

Transverse Action J

J is a kind of single particle emittance

$$x = D\frac{\delta p}{p} + \sqrt{2J_x\beta_x}\,\sin(\mu_x)$$

Fokker Planck Equation

a continuity equation in action space

$$\frac{\partial \Psi\left(J_x, J_y, \delta p / p ; t\right)}{\partial t} + \nabla \bullet \Phi\left(J_x, J_y, \delta p / p ; t\right) = 0$$

flux in action space $$\Phi_m = -F_m \Psi + \frac{1}{2} D_{mn} \frac{\partial \Psi}{J_n}$$

drift (coherent cooling effect) $$F_m = \frac{1}{\tau} \left\langle \Delta J_m \right\rangle$$

diffusion (incoherent cooling effect) $$D_{mn} = \frac{1}{\tau} \left\langle \left(\Delta J_m \Delta J_n \right) \right\rangle$$

Assumptions and Simplifications

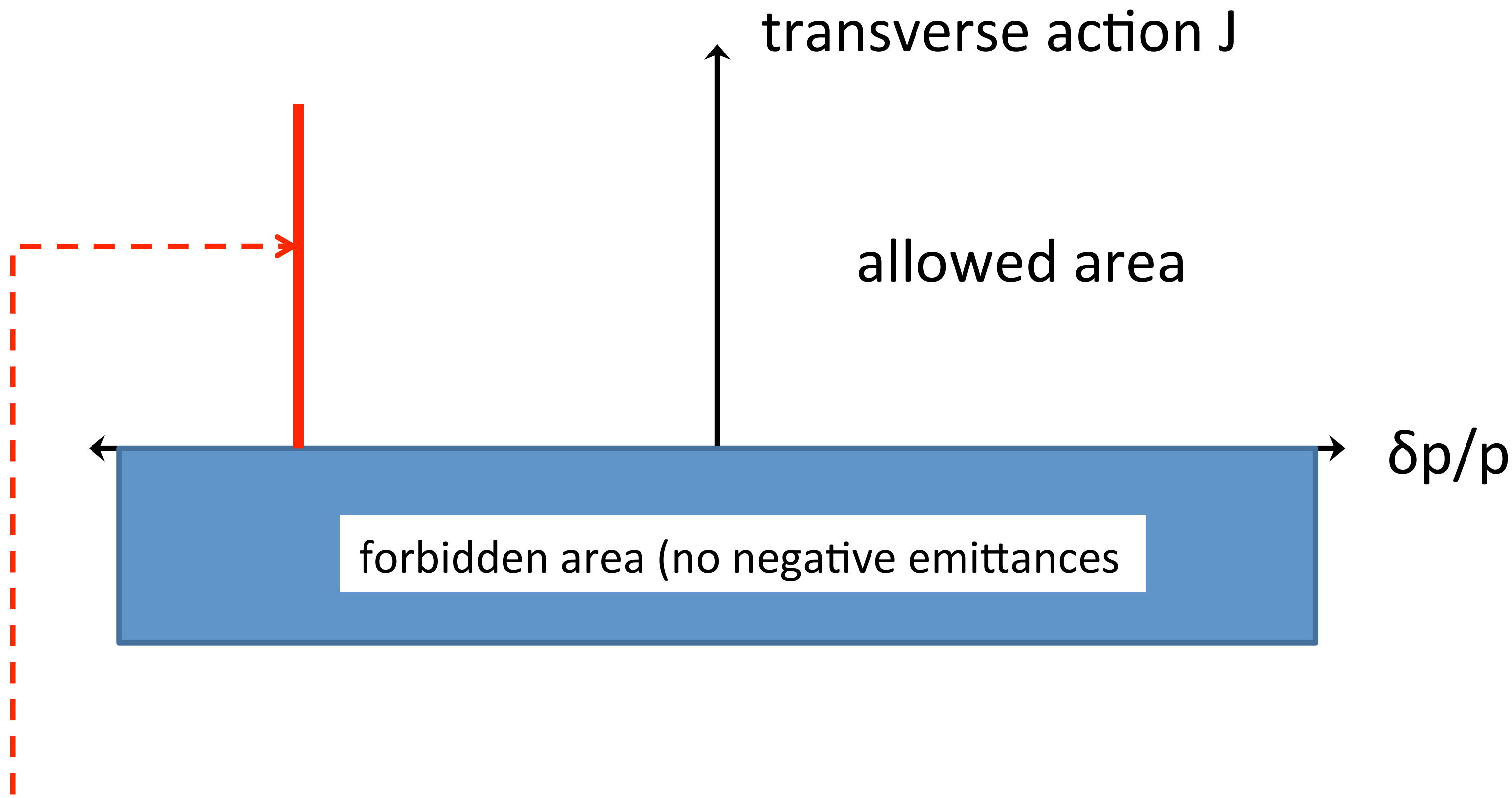
- Zero dispersion at pick-ups and kickers
- Linear response for pick-ups and kickers
- No Schottky overlap
- No Chromaticity

accelerating voltage: $U_k(\Omega) = S_k(x, y, \Omega) V_k(\Omega)$

pick-up response: $V_p(\Omega) = \frac{Qe\omega}{2} S_p(x, y, \Omega) j_p(x, y, \Omega)$

transverse sensitivity: $S(x, y, \Omega) = x S'(0, 0, \Omega)$

2D Phase space



In the following we shall study the 1D distribution function *Ψ(J)* of transverse actions at a given δp/p

Pick-up Signal for Transverse Cooling

$$V_p(\Omega) \propto \frac{QeZ_l\sqrt{2J_x\beta_x}\,\partial S_p/\partial x}{4i}$$

at the frequencies $\omega_{m,\pm} = (m \pm Q_x)\omega_{rev}$

in the following: $\frac{\partial S_{pk}}{\partial x} := S'_{pk}$

Voltage Power Densities (1)

The voltage spectral power density $C(\Omega)$ is by definition the Fourier transform of the voltage autocorrelation $R(t) = \langle V(\tau) V(t+\tau) \rangle$.

In a coasting beam it does not depend on τ,

i.e. $V(t)$ is a *stationary process*.

The dimension of $C(\Omega)$ is V^2s.

Voltage Power Densities (2)

Voltage power density due to Schottky noise:

$$C_p(\Omega) = \frac{(QeZ_l)^2 \omega \beta_x |S_p'|^2}{16\pi |m\eta|} \langle J \rangle \psi(\delta p / p)$$

average emittance

$$\langle J \rangle = \frac{\int_0^\infty J_x \Psi_x \, dJ}{\int_0^\infty \Psi_x \, dJ}$$

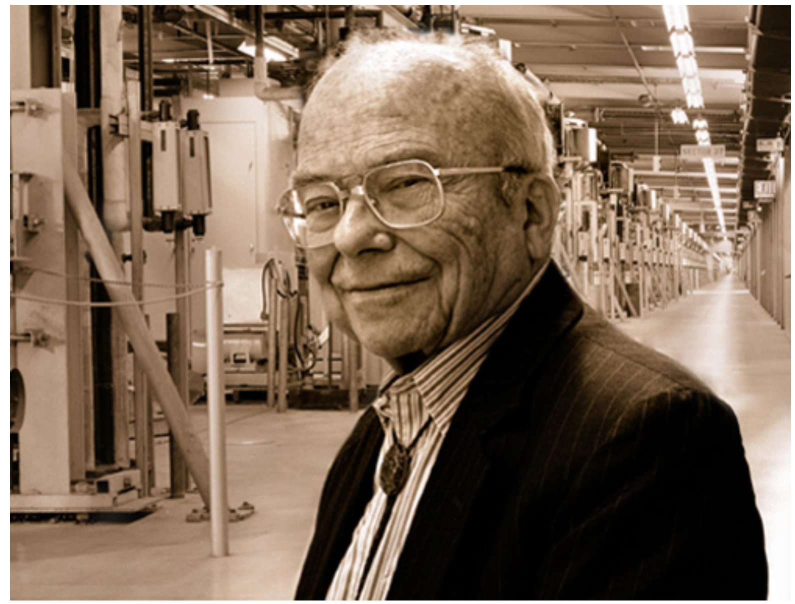
Voltage Power Densities (3)

Voltage power density due to thermal noise:

$$C_n(\Omega) = \frac{1}{2} Z_l k_B T_{eff}$$

Transverse Kicks

The transverse kicks are described by the Panofsky-Wenzel theorem:

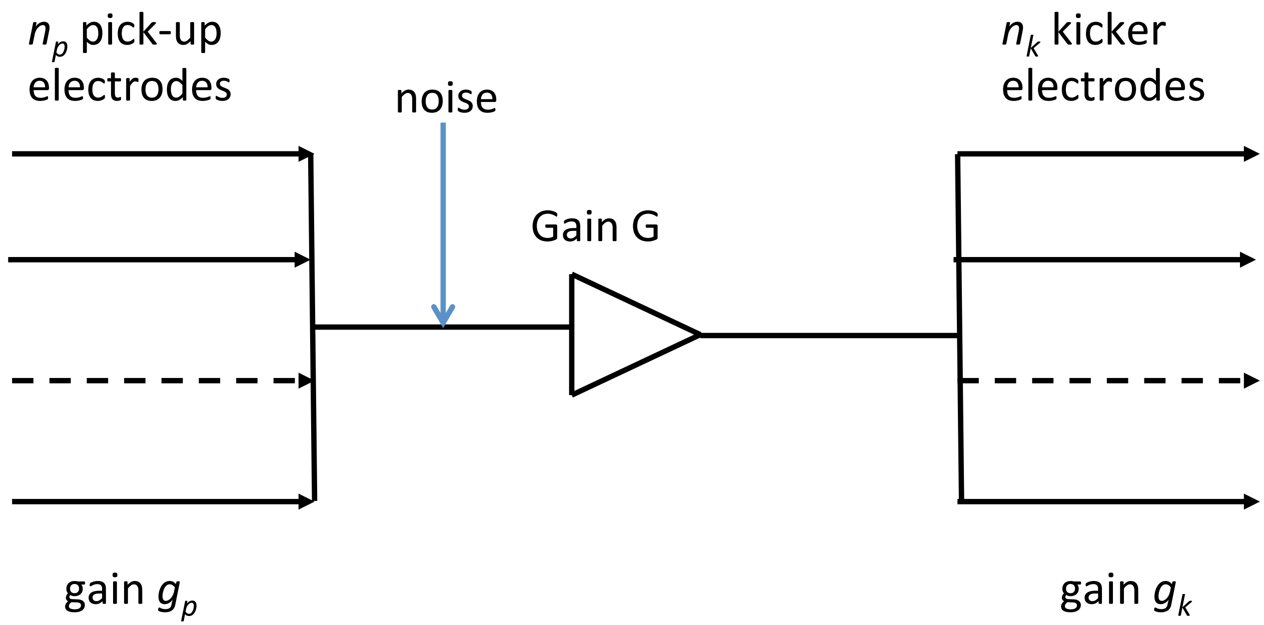


$$\delta p_x = \frac{Qe}{i\Omega} S_k^{'}(\Omega) V_k(\Omega)$$

The kick strength is

- proportional to the transverse gradient of the longitudinal electric field
- inversely proportional to frequency
- max at the zero transition of the electric field

Gain Factors



$$g_{pk} = \sqrt{n_{pk}}$$

Fokker Planck flux

$$\Phi = -F\,\Psi + \frac{1}{2} D \frac{\partial \Psi}{J}$$

$$F = -C\,J$$

$$D = \left(S\left\langle J\right\rangle + H\right) J$$

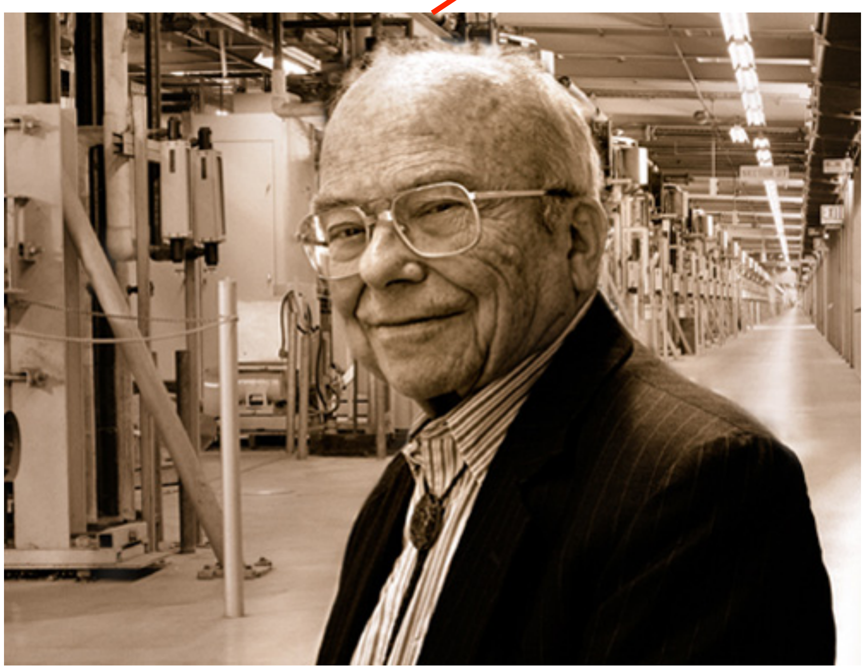
The terms C (cooling), S (Schottky noise), and H (thermal noise) do not depend on J.

Therefore the flux is proportional to J.

No flux towards J<0!

Transverse System Gain

$$g_{\perp} = \frac{(Qe)^2 \omega Z_l}{8\pi\, p\, \Omega} \sqrt{\beta_p \beta_k} S_k' g_k G g_p S_p'$$



frequency dependent!

Cooling Coefficient

$$C = \frac{\omega}{2\pi} \sum_{m,\pm} \pm g_{\perp} \exp\left[i\left(\mu_k - \mu_p\right) - \left(m \pm Q\right)\omega T_{pk}\eta_{pk}\frac{\delta p}{p} \right]$$

betatron phase advance

undesired mixing

Schottky Noise Coefficient

Schottky power density depends on longitudinal distribution

$$S = \frac{\omega \psi\left(\delta p / p\right)}{2\pi} \sum_{m,\pm} \left| \frac{g_{\perp}^{2}}{m\eta} \right|$$

Schottky power density decreases with harmonics

Thermal Heating Coefficient

$$H = \frac{4k_B T_{eff}}{(Qe)^2 Z_l \beta_p \left|g_p\right|^2} \sum_{m,\pm} \left| \frac{g_\perp^2}{S_p'} \right|$$

Equilibrium Distribution

condition: $\dfrac{\partial \Psi}{\partial t} = 0$

The equilibrium distribution turns out to be an exponential:

$$\Psi(J) = \Psi_0 \exp\left(- \frac{2C - S}{H} J \right)$$

with average emittance $\langle J \rangle_\infty = \dfrac{H}{2C - S}$

Time Dependent Solution

$$\Psi(J,t) = \alpha(t)\exp(-\alpha(t)J)$$

It turns out that this ansatz is a time-dependent solution to the transverse Fokker-Planck equation if

$$\frac{\dot{\alpha}}{\alpha} + \left(-C + \frac{S + \alpha H}{2}\right) = 0$$

If C, S, and H are constant in time, then

$$\langle J\rangle(t) = \left(\langle J\rangle_0 - \langle J\rangle_\infty\right)\exp\left(-\frac{t}{\tau}\right) + \langle J\rangle_\infty$$

with $\tau = C - \dfrac{S}{2}$ not the instantaneous cooling rate!

Instantaneous Cooling Rate

$$\frac{1}{\tau_\perp} = C - \frac{S}{2} - \frac{H}{2\langle J \rangle}$$

$$g_\perp = \frac{(Qe)^2 \omega Z_l}{8\pi\, p\, \Omega} \sqrt{\beta_p \beta_k} S'_k g_k\, G\, g_p\, S'_p$$

$$C = \frac{\omega}{2\pi} \sum_{m,\pm} \pm g_\perp \exp\left[i(\mu_k - \mu_p) - \eta_{pk} (m \pm Q) \omega T_{pk} \frac{\delta p}{p} \right]$$

$$\frac{S}{2} = \frac{\omega \psi(\delta p / p)}{4\pi} \sum_{m,\pm} \left| \frac{g_\perp^2}{m\eta} \right|$$

$$\frac{H}{2\langle J \rangle} = \frac{2 k_B T_{eff}}{(Qe)^2 Z_l \beta_p |g_p|^2 \langle J \rangle} \sum_{m,\pm} \left| \frac{g_\perp^2}{S'_p} \right|$$

Naive Approximation

- constant system gain:

  replace $\omega \sum_{m,\pm}$ by *4W*

- rectangular longitudinal ditribution:

  replace $\psi\left(\delta p/p\right)$ by $\frac{N}{\Delta p/p}$

  full momentum width

Almost Textbook Cooling Rate

cooling rate: $\frac{1}{\tau_\perp} = 2W\left[2Bg_\perp - (M\,N + h)|g_\perp|^2\right]$

undesired mixing: $B \approx \cos\left(m_c \omega T_{pk}\, \eta \frac{\delta p}{p}\right)$

desired mixing: $M \approx \left(m_c |\eta| \frac{\delta p}{p}\right)^{-1}$

thermal heating: $h \approx \frac{8k_B T_{eff}}{(Qe)^2 \omega Z_l \beta_p |g_p S'_p|^2 \langle J \rangle}$

What‘s „new“?

- Equilibrium distribution is exponential.
- Exponential initial distribution remains exponential during the cooling process.
- In that case, a cooling rate equation for each frequency in the cooling band can be derived.
- If this can be made frequency-independent somehow, one gets to the standard van der Meer/Möhl description.
- Why not modify it (particle number *N*)?