Some factors determining the beam quality of A.V.F. cyclotrons

H. L. Hagedoorn
Technological University, Eindhoven, Netherlands

N. Hazewindus and J. M. van Nieuwland
Philips Research Laboratories, Eindhoven, Netherlands

Presented by H. L. Hagedoorn

ABSTRACT

In this paper some factors which have a large influence on the beam properties of A.V.F. cyclotrons are discussed. The connection between injected beam and extracted beam is considered without going into detailed orbit dynamics. This results in a qualitative prediction of the time, energy and geometry structure of the external beam as a function of the injected beam parameters.

Remarks on the beam dynamics of the first few turns are given, using our general orbit theory. Some numerical calculations on the axial motion are presented, taking into account space charge effects.

1. INTRODUCTION

A particle in an A.V.F. cyclotron can be represented in a three dimensional space of which two co-ordinate axes belong to the radial motion and one to the rf phase angle [the (A, θ, ϕ)-space].

A beam can be represented by a part of this space or of a subspace. In first approximation there is no need to describe the energy of the beam as it is strongly coupled to the rf phase.

The coupling between radial and axial motion will be neglected. To further simplify the reasoning in Sections 2, 3, and 4 only a linear radial motion is considered.

In Section 2 the volume in the (A, θ, ϕ)-space yielded by the ion source with or without diaphragms will be discussed. Then in Section 3 the conditions for single turn and multiturn acceleration are shown. The influence of slits in the external beam handling system on the volume in the (A, θ, ϕ)-space occupied by

CYC69D02

the beam is given in Section 4. This results in conclusions about the external beam properties.

Sections 5 and 6 deal with some remarks on slits and grids in the cyclotron centre in connection with axial focusing. In Section 7 numerical results about the axial beam envelope in the cyclotron centre are given, taking into account space charge effects.

2. THE INJECTED BEAM

Due to the rf acceleration in an A.V.F. cyclotron there is a very strong coupling between the starting phase at which the ions leave the internal ion source and their energy.[1] This coupling persists throughout the whole acceleration. Only when particles are injected through an axial injection system with a large energy spread—comparable to the energy gain per turn—may this coupling be made unobservable in the external beam.

The radial phase space area delivered by an ion source after the first gap crossing may be a square or may have an elongated shape for particles starting at the same rf phase. This is shown in Fig. 1, where orbit centre co-ordinates are used. Mallory and Blosser[2] have found radial phase space area equal to 300 mm mrad for protons of 35 keV with a d.c. current of 2·7 mA. [Their latest results show a smaller value at the same current and energy[3] (30 mm mrad).] These qualities can be expressed in orbit centre co-ordinates. Then for an orbit radius of 20 mm an area of 15 mm^2 corresponds to 300 mm mrad. Due to the special ion

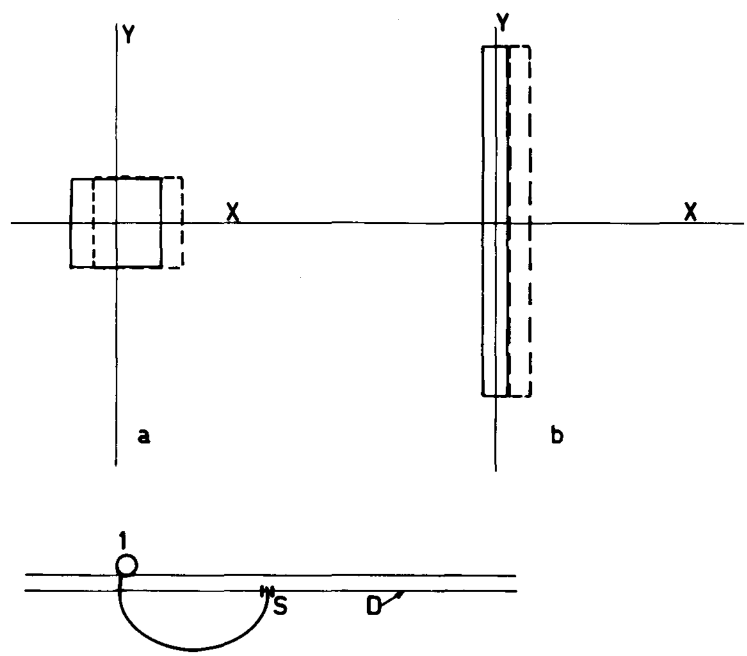

Fig. 1 The radial phase space in orbit centre co-ordinates emitted by an ion source (a) without and (b) with a diaphragm. The dashed figures show a shift due to a different starting energy. The phase space figures belong to particles at azimuthal positions just after the diaphragm (S). I *is the ion source and* D *the dee*

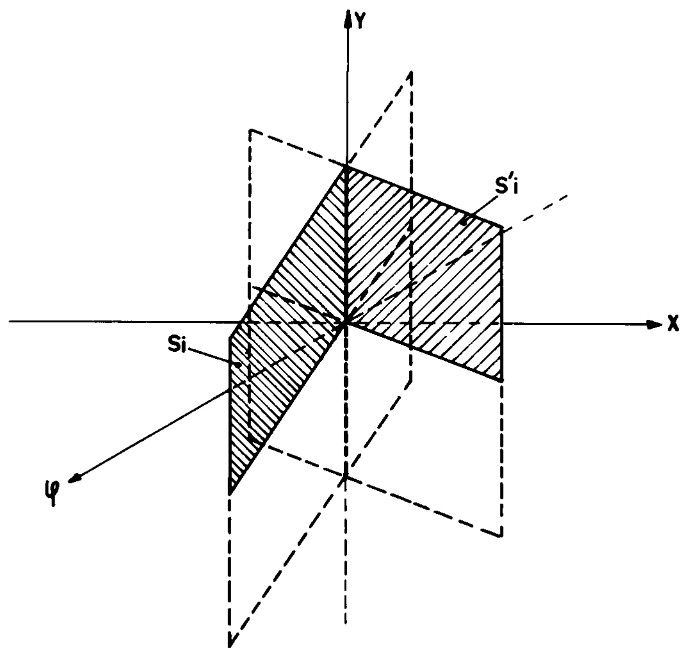

Fig. 2. The two surfaces S_i *and* S_i' *are shown in the* (X_i, Y_i, ϕ)*- or* (A, θ_i, ϕ)*-space*

source geometry they found a rather elongated figure which gives in the orbit centre co-ordinate space an area with the dimensions 1×15 mm^2.

If now the ions start at a different rf phase the extraction voltage is changed. Then the orbit centres will shift away and will take new positions indicated by the dashed lines in Fig. 1.[4] In Fig. 1(a) the two areas still overlap, in Fig.1(b) they do not. This means that in the first case an area in phase space can be given which contains particles within a relatively large rf phase and energy range. Thus a beam in the cyclotron is represented by a volume in the (X, Y, ϕ)-space, where X and Y are the orbit centre co-ordinates and ϕ the rf phase of the particles. Instead of X, Y co-ordinates it is more convenient to use the polar co-ordinates A, θ, where A equals the radial oscillation amplitude and θ the radial oscillation phase. In the second case [Fig. 1 (b)] each area has a reasonably well defined rf phase and energy so that all ions will lie in a thin sheet in the (A, θ, ϕ)-space. The thickness of this sheet depends linearly on the width of the phase space area. Also a sheet is formed when a diaphragm is positioned 180° away from the ion source yielding an area shown in Fig. 1(a). The thickness of the sheet then depends linearly on the aperture of the diaphragm, which may be as small as 0·1 mm.

A sheet will be represented by a surface S_i:

$$S_i = S_i(A, \theta_i, \phi) = 0$$

where the suffix i means the ion source with or without a diaphragm. A second diaphragm placed at a different aximuthal and radial position yields a second sheet S_i^1, when transformed back into (A, θ, ϕ)-space given above. Now only the line cross-section L_i of the two sheets will be accelerated in the cyclotron. When

$\nu_r = 1$ and when the azimuthal position of the second diaphragm differs by 180° from the position of the first one, the line L_i coincides with a line of constant ϕ in each surface. Then we have a situation which is well suited for single turn extraction (Fig. 2).

It must be noted that due to space charges—at enhanced beam currents—the radial oscillation frequency ν_r may deviate from unity.[5] Then for single turn extraction the azimuthal positions of the diaphragms have to differ from the above given values.

Besides surfaces arising from diaphragms in the cyclotron centre, surfaces corresponding to limitations in the extraction system and beam handling system can be defined in the initial (A, θ_i, ϕ)-space. This will be considered in Section 4.

3. THE SINGLE AND MULTITURN MODES OF OPERATION

Single turn extraction has been adequately described by Gordon in a paper at the Gatlinburg conference.[1] Experimental material has been given by Blosser.[6] Details about multiturn extraction are given in reference 7. We shall make here a few remarks valid for single turn and multiturn extraction without going into details.

The energy and rf phase of the particles are so strongly coupled that in first approximation the energy is only a function of ϕ, with the turn number as a discontinuous parameter. This approximation is valid as long as we assume very small radial and axial oscillation amplitudes[1] and negligible space charge effects.[8]

If now by means of an analysing system the energy spread of the external beam is cut down to a very small value (much less than the energy gain per turn) one will observe a discrete rf structure in the beam.[1,6] The smaller the energy spread of the analysed beam the larger a part of the total intensity will be found at the region with smallest phase excursions (i.e. around $\phi = 0$). The rf phase of particles with minimum turn-number and the right energy is put equal to $\phi = 0$ (see Fig. 3) at injection.

The discrete rf phase regions come closer together when the number of turns in the cyclotron becomes larger (i.e. for a low dee voltage). The distances between these regions become smaller for larger ϕ values. Both statements follow from an equation given by Gordon.[1] If we assume a homogeneous distribution of the particle density in the (A, θ_i, ϕ)-space, we shall observe that the intensity per unit phase interval is proportional to $\delta E/\Delta E$, where ΔE is the energy increase per turn and δE the energy width of the analysing system. The intensity of the particles in the region $\phi \approx 0$ (i.e. the top of the parabola) is proportional to $(\delta E/\Delta E)^{1/2}$. Therefore decreasing δE appears to favour relatively these central phase particles.

The maximum distance between two rf phase regions occurs for the phases $\phi = 0$ and $\phi = \phi_1$ and is given by $\Delta\phi_m = (2/N)^{1/2}$, with N the number of turns, assuming $\delta E = 0$. This corresponds to a time interval of 0·5 ns in a cyclotron with $N = 400$ and a frequency of 24 MHz. If δE is not negligible we find

$$\Delta\phi_m = \sqrt{2/N}\left[1 - (\delta E/\Delta E)^{1/2} - \frac{1}{2}(\delta E/\Delta E)\right]$$

For $\delta E/\Delta E \approx 0{\cdot}1$, $\Delta\phi_m$ corresponds to 0·3 ns.

If the resolving time of the coincidence circuits used for nuclear physics is smaller than the time intervals corresponding to the distances between successive phase regions then no rf phase structure will be observed. Therefore

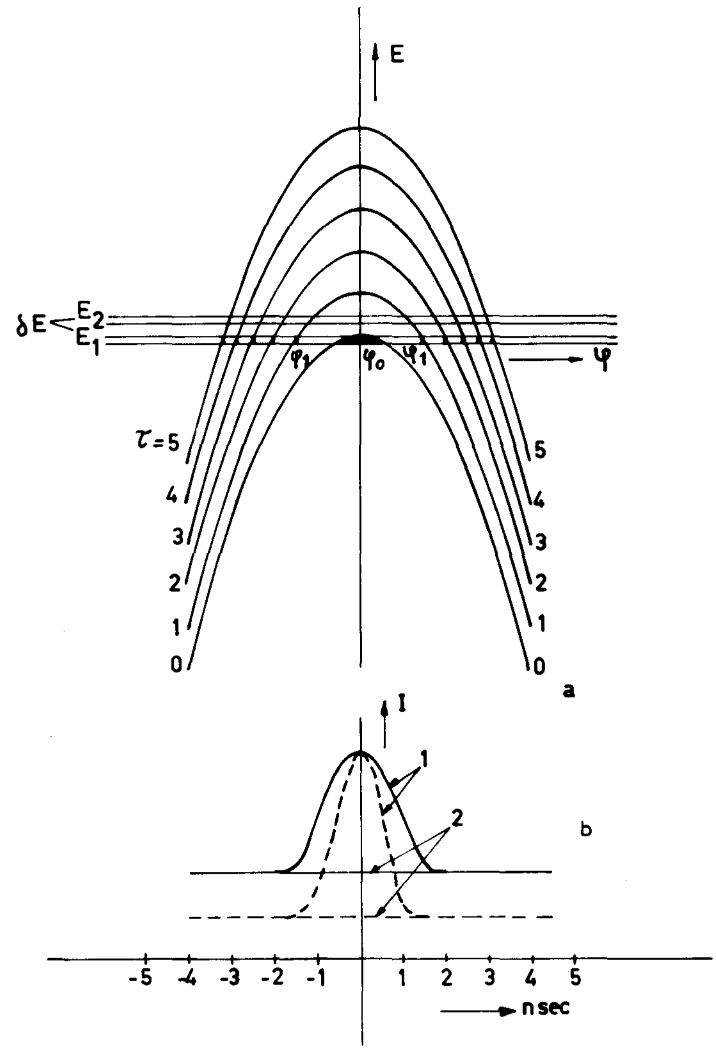

Fig. 3. The particle energy at extraction as a function of initial rf phase for different turn numbers. (a) The smoothed beam intensity per time interval as a function of time. (b) Arbitrary units are used. The full lines give the intensities which follow from the two different energies in Fig. 3(a). The broken lines represent a case in which δE *is diminished by a factor of 4*

in that case we can smooth away the discrete structure. We find an intensity per unit time interval as shown in Fig. 3(b). In this figure an enhanced intensity is found at $\phi \approx 0$. If we cut the E, ϕ-parabola at a slightly different energy this intensity distribution becomes flat. The difference in intensity between these two cases decreases as δE increases.

When operating the cyclotron in a multiturn mode contributions will be extracted from many parabolas. (In a usual multiturn mode 20 or more revolutions contribute to the external beam.) This fact results in a high total intensity and a broad rf phase or time interval.

In the single turn mode only a small rf phase interval is selected by means of slits. Then only particles belonging to one parabola will be extracted at the same time. In this case the single turn mode shows a very high energy resolution. In the multiturn mode no slits should be placed in the cyclotron centre as they may retain useful particles.

The smoothed time structure in Fig. 3 will become flatter when δE becomes larger or when the energy and rf phase are not absolutely coupled. Due to axial and radial oscillations[1] and due to space charge effects[5]—in case of high intensities —the E, ϕ-curves are smeared out. Then instead of lines in Fig. 3 bands should be drafted. The width of these bands can approximately be characterised by an interval δE_{osc}. This effect, which is not always negligible, has the same influence on the time structure of the external beam as the energy width δE of the analysing system. If $\delta E_{\mathrm{osc}} \gg \delta E$ the time structure of the beam is no longer defined by the analysing system. In this case the difference in intensity distribution between the two cases 1 and 2 in Fig. 3 will become small. A slight change in the adjustment of the cyclotron or the analysing system energy hardly affects the intensity and time structure.

Several effects may cause an rf phase deviation. For radial oscillation amplitudes of 3 mm in a 30 MeV cyclotron with 400 revolutions a phase deviation of 1°–2° may occur. This value depends quadratically on the oscillation amplitude and is strongest in the places in the magnetic field where second derivatives are large. Also due to an unbalance in the time of arrival at the two accelerating gaps (in a one dee cyclotron) for particles with radial oscillations a value of $\delta E_{\mathrm{osc}} \approx (1/2)\Delta E$ may be acquired.

In order to obtain this effect an elongated area in the radial phase space should be extracted and transported through the analysing system. For the useful particles δE_{osc} is not negligible in this case. A multiturn mode of operation will result which gives us the advantages of great stability between successive external beam pulses, broad time interval (high duty cycle) and insensitivity to small changes in adjustments, combined with high energy definition and beam quality.

In the multiturn mode many particles are accelerated which will not be used, having the wrong energy and the wrong position in radial phase space. This may lead to currents far above 100 μA in the cyclotron. A part of this current hits the septum and causes heat dissipation. At the moment this does not seem to be a limitation.

If the analysing system defines both the energy and the geometry of the beam in such a way that it picks out of radial phase space only a small portion around the equilibrium orbit we get $\delta E_{\mathrm{osc}} \approx 0$ and a beam results with high energy definition, high beam quality but small rf phase width or time interval. This case corresponds to a single turn mode of operation. Further, it will accelerate exclusively useful particles if slits are placed in the cyclotron centre.

4. THE IMAGE OF THE SLITS OF THE BEAM HANDLING SYSTEM IN THE RADIAL PHASE PLANE AT INJECTION

The energy and radius of a particle on an equilibrium orbit after N_o turns are

$$E = E_o\,(1 - \tfrac{1}{2}\phi^2)$$

$$r = r_o\,(1 - \tfrac{1}{4}\phi^2)$$

Particles with radial oscillations will in general have different radial positions depending on their oscillation phase and amplitude. Their positions are given by

$$r = r_o + \tau\Delta r - \frac{1}{4} r_o \phi^2 + A \sin \theta_f,$$

where the number of revolutions is given by $N = N_o + \tau$, the radial oscillation amplitude and final phase are given by A and θ_f respectively; Δr is the increase in radius per turn due to acceleration. We shall represent θ_f as follows:

$$\theta_f = \theta_i + \theta_c$$

where θ_i is the initial oscillation phase at the ion source and θ_c is the phase added during the acceleration. θ_c is found from the following expression:

$$\theta_c = 2\pi \int_i^f \frac{\nu - 1}{2V\cos[\phi_o(E) + \phi]} \mathrm{d}E$$

Here $\phi_o(E)$ is the phase of particles with minimum turn number. Using this last expression and assuming a slowly changing ν and a mean value of ϕ_o very close to zero we get:

$$r = r_o + \tau\Delta r - \frac{1}{4} r_o \ \phi^2 + A \sin \left[\theta_{co} + \pi k\phi^2 - \pi(\nu_r - 1)N\phi^2 + \pi(\nu_r - 1)2\tau + \theta_i\right]$$

where θ_{co} is the value for θ_c when $\phi = 0$, k equals the total number of radial oscillations with frequency $\nu - 1$, N is the turn number, ν_r is the value of ν at $R = r_o$. Now the radial position is represented in τ, ϕ and the initial radial phase plane co-ordinates A, θ_i. (As ν does not change much A remains practically constant.)

Let us now assume that r is defined by slits in the extraction system or by limitations in the beam handling system. Then the above expression yields a number of surfaces in the (A, θ_i, ϕ)-space, each surface belonging to a discrete value of τ:

$$S_\tau = S_\tau(\phi, A, \theta_i) = 0$$

The projection of S_τ on the radial phase plane gives a continuous area, which may have boundaries. Each point in the radial phase plane shows a discrete rf phase structure which follows from the turn number τ. At extraction the whole radial phase plane is contracted into the p_r-axis, since we defined the radial position: the surfaces S_τ transformed to the extraction region coincide with the (ϕ, p_r) plane. Then if no limitations at other places are used the extracted beam will not show any discontinuity in rf phase or geometrical extensions.

If beside radius energy is also defined by means of an analysing system we get the lines L_τ:

$$L_\tau = L_\tau(\phi_\tau, A, \theta_i)$$

each line having its ϕ_τ value according to the construction shown in Fig. 3. In Fig. 4 the lines are shown in the (A, θ_i, ϕ)-space. We will now get an external beam with a discontinuous rf phase structure but a continuous geometrical structure.

The extensions $\Delta\phi$ of the discontinuous regions on the ϕ-axis follow from

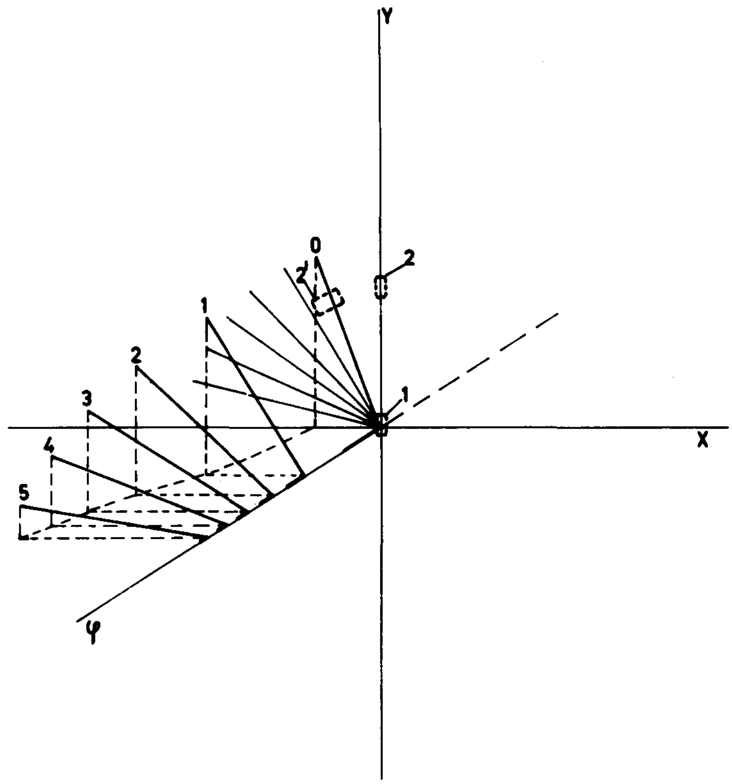

Fig. 4. The lines L_τ for $\tau = 0, 1 \ldots 5$ in the (X_i, Y_i, ϕ)-space. The areas (1) and (2) represent 'point sources' of equilibrium particles and particles off the equilibrium axis respectively. The positions of these areas are the positions in the (X_f, Y_f, ϕ)-space at extraction. Transforming these areas into the initial (X_i, Y_i, ϕ)-space means rotation around the origin. The area (2') is the projection in the initial radial phase space of the volume V_i given in the text. In fact (1) and (2) are the projections of volumes V_f. Extensions of the volumes V_i are indicated along the ϕ-axis

Fig. 3 and are due to effects mentioned in Section 3, which smear out the discreteness. If the ion source delivers a homogeneous particle distribution throughout the whole (A, θ_i, ϕ)-space and if we are looking at near-equilibrium particles these extensions are a good measure of the intensity in them as those particles start from a 'point' source in the (A, θ_i, ϕ)-space (see Fig. 4).

If now we look at particles which lie off the equilibrium axis and which belong to an area in radial phase space at extraction which has the same size as the above mentioned point source, we will find that these particles occupy the same volume in the (A, θ_i, ϕ)-space as the particles around the equilibrium orbit. Or, stated differently: if one wants a high beam quality with high energy resolution it does not matter from which part of the initial radial phase space it comes, assuming that the ion source gives a homogeneous particle distribution. To prove this we must show that a volume in the (A, θ_f, ϕ)-space is conserved after transformation to the (A, θ_i, ϕ)-space. Particles which lie off the equilibrium axis can be represented by a volume V_f:

$$V_f = A \times \delta\theta_f \times \delta A \times \delta\phi$$

Here $A \times \delta\theta_f \times \delta A$ is the radial phase space area representing a 'point source' and $\delta\phi$ a small interval in the ϕ-axis (see Fig. 4). This 'point source' rotates around the point corresponding to the equilibrium orbit after transformation into the (A, θ_i, ϕ)-space. The angle of rotation depends on the value of θ_c—given at the beginning of this section—and thus also on the rf phase ϕ. After this transformation the radial area becomes $A\,[2\pi k\phi - 2\pi(\nu_r - 1)N\phi]\,\delta\phi\,.\,\delta A$. The volume is now found by multiplying this area with the thickness of the transformed volume in the ϕ-direction, given by $\delta\theta_f/[2\pi k\phi - 2\pi(\nu_r - 1)N\phi]$, which yields:

$$V_i = A \times \delta\theta_f \times \delta A \times \delta\phi$$

Therefore $V_i = V_f$. We have to remark, however, that the radial phase space area after transformation may be much larger than the area before transformation.

In Section 2 it was shown that the injected beam sometimes may be represented by a surface $S_i(A, \theta_i, \phi) = 0$. (It must be remarked that there may exist more surfaces S_i if one slit is used. This is caused by the same effect which results in the surfaces S_T. The different surfaces lie generally far away from each other due to the large radial increase per turn at the cyclotron centre. We have neglected these extra surfaces.) This condition together with $L_T = 0$ yields a beam discontinuous both in rf phase and geometrical dimensions and may favour sometimes a special rf phase ϕ_T, when L_T lies in or nearly in S_i. In this last—very special—case we can conclude that when a slit is placed in the cyclotron centre the external beam may deliver mainly particles of one phase ϕ_T but having a continuous region of p_r values, when the energy and radius are defined. This then has the same behaviour as single turn acceleration. For multiturn extraction these effects are a second reason for not using any slit in the cyclotron centre (see Section 3).

Two slits in the cyclotron centre yield a line $L_i = 0$. If energy and radius are again determined, then for getting particles through the whole system this line has to intersect one of the lines L_T. In fact this means that the images of all slits in the cyclotron and the beam handling system have a common region. Small misalignments however have a large influence on the beam intensity. If we take $\delta E \approx .1/10\,\Delta E$ in a cyclotron with 400 turns the dee voltage must be stabilised much better than 1:4000 or large intensity fluctuations will result. Further large fluctuations can be caused by a change in the magnetic field or rf frequency of $2{:}10^5$ (giving a phase shift of $1{\cdot}5°$, see also Fig. 3) or by a change in radial position of the beam of 0·1 mm at extraction which gives a mismatching of the cyclotron emittance and the energy analysing system acceptance (e.g. maladjusted first harmonic components $\simeq 10^{-5}\,B_o$ can do this).

5. SOME INFLUENCES OF CENTRAL REGION SLITS ON THE PARTICLE ORBITS

In the preceding sections the selecting influence of slits on the radial particle orbits has been shown. We will mention shortly a few other properties of slits. When collimating the beam axially at one of the first turns,[4, 9] a phase selection may occur. Then the pulse width of the extracted beam is cut down to 3° (e.g. ∿0·5 ns). Experimental evidence for this was found some years ago on the Philips prototype machine.

When axial focusing is very small in the cyclotron centre the beam height increases such that the dee aperture may act as a phase selecting slit. A decrease in axial focusing can be acquired by tuning the innermost circular correction coils. This has been the explanation of the very narrow beam pulses found in the V.U. cyclotron at Amsterdam.[10]

Slits placed near the accelerating gap can have a special influence on the particle orbits. A radial defining slit may change the electric field shape, from being homogeneous along the dee gap into one homogeneous in the axial direction. This means that instead of electrical axial focusing an extra radially focusing term appears. The result will be a decrease in the influence of the electric fields on the axial motion, especially with regard to the rf phase of the particles. The axial focusing may be entirely due to magnetic forces in this case.

Sometimes grids are used in the cyclotron centre to define the electric field in a desired region. However, we must keep in mind the lens effect of the grid wires. Grids in the central region of the cyclotron may cause deterioration of the beam quality that exists even at the start of acceleration.

6. SOME REMARKS ON AXIAL FOCUSING IN THE CYCLOTRON CENTRE

For a description of the axial motion in the centre of an A.V.F. cyclotron we shall use our general orbit theory.[11] Coupling between the radial and axial oscillations will be neglected. Also for the moment we will not take into account space charge effects.

The Hamiltonian for the axial motion is given by:

$$H = \frac{p_s^2}{2m} + \frac{p_z^2}{2m} + eV(s,\ z)\cos(\phi + \omega t)$$

Here s is the co-ordinate along the central orbit, which is assumed to be a circle of radius r in the cyclotron centre, z is the axial co-ordinate, p_s and p_z are the corresponding kinetic momenta, V is the electric potential in the accelerating gap, ω is the angular frequency of the rf system, ϕ is an arbitrary phase.

The magnetic field in the cyclotron centre is assumed to be constant. The potential V is represented by

$$V(s,z) = V_O(s) - \frac{1}{2}z^2 V''(s) + \ldots\ldots$$

We now introduce a new Hamiltonian $K = -P_s$, where P_s is the canonical momentum given by $P_s = p_s + eA_s$:

$$K = -\sqrt{2mH}\left[1 - \frac{p_z^2}{2mH} - \frac{eV}{H}\cos(\phi + \omega t) + \frac{1}{2}z^2\,\frac{eV''}{H}\cos(\phi + \omega t)\right]^{1/2}$$

It is convenient to use relative momentum co-ordinates $\pi_z = p_z/\sqrt{2mH}$. Further we introduce $\Phi = eV/H$. The quantity Φ is of order $\frac{1}{2}N$, where N is the number of revolutions. The change in particle energy H is assumed to be adiabatic, so that we solve our problem correspondingly. K becomes

$$K = -\left[1 - \pi_z^2 - \Phi\cos(\phi + \omega t) + \frac{1}{2}z^2\,\Phi''\cos(\phi + \omega t)\right]^{1/2}$$

We now introduce a new time co-ordinate t_o given by:

$$t = t_o + s/v,$$

where v is the adiabatically changing particle velocity. In this way t_o then represents a starting time and is a constant (instead of s/v a better approximation would be $\int ds/v$). The transformation $t \rightarrow t_o$ is canonical. Substituting the new time co-ordinate and expanding the square root we get:

$$K = \frac{1}{2}\pi_z^2 \left[1 + \frac{1}{2}\Phi(\cos\phi_o \cos\frac{s}{r} - \sin\phi_o \sin\frac{s}{r}) + \frac{3}{8}\Phi^2 \cos^2\phi_o \cos^2\frac{s}{r}\right]$$
$$+ \frac{1}{2}z^2 \left[-\frac{1}{2}\Phi''(\cos\phi_o \cos\frac{s}{r} - \sin\phi_o \sin\frac{s}{r}) - \frac{1}{4}\Phi\Phi''(\cos\phi_o \cos\frac{s}{r} - \sin\phi_o \sin\frac{s}{r})^2\right]$$

where $\phi_o = \phi + \omega t_o$, ϕ_o is the phase of the particles in the middle of the accelerating gap. If higher harmonic acceleration is studied s/r must be replaced by hs/r with h the harmonic number. The potential function Φ is expanded in a Fourier series:

$$\Phi = \Sigma\Phi_n \sin\frac{ns}{r}$$

where $s = 0$ lies in the middle of the accelerating gap. By assuming a pulse shaped electric field occurring twice each revolution we find

$$\Phi_n = \frac{2\Phi}{\pi n} \quad \text{with} \quad \Phi = V_D/E,$$

where V_D is the dee voltage.

The Hamiltonian K contains several rapidly oscillating terms. A general rule states that these terms can be removed to higher orders by a suitable canonical transformation: e.g. a first order oscillating term will result in a combination of a second order constant term and a second order oscillating term. In our approximation we shall keep only terms up to second order. Therefore in K, oscillating terms of first order and constant terms up to second order must be kept. First order quantities are Φ and $\sin\phi_o$.

The Hamiltonian K is abbreviated to

$$K = \frac{1}{2}\pi_z^2 f + \frac{1}{2}z^2 g,$$

where $f = 1$ + first and second order terms, g contains only first and second order terms.

To get rid of the extra terms in f, a transformation function $G(\pi_z, \bar{z})$ is used, given by[11]

$$G = -\pi_z \bar{z} f^{1/2} + \frac{1}{4} f^{-3/2} f' \bar{z}^2$$

The new Hamiltonian then becomes

$$\bar{K} = \frac{1}{2}\bar{\pi}_z^2 + \frac{1}{2}\bar{z}^2 \left(fg - \frac{3}{4}\frac{f'^2}{f^2} + \frac{1}{2}\frac{f''}{f}\right)$$

The relations between old and new co-ordinates and momenta are

$$\bar{\pi}_z = \pi_z f^{1/2} \quad \frac{1}{2} f^{-3/2} f' \bar{z}$$

$$z = \bar{z} f^{1/2}$$

Substitution of the real coefficients in $\bar{K}$ gives:

$$\bar{K} = \frac{1}{2}\bar{\pi}_z^2 + \frac{1}{2}\bar{z}^2 \left(\frac{7}{16}\Phi'^2 - \frac{1}{4}\Phi'' + \frac{1}{4}\Phi'' \sin\phi_o \sin\frac{s}{r}\right)$$

(In a cylindrically symmetric electric field we get only (3/16) Φ'^2 as coefficient for the second term.) Though there are still time dependent coefficients in this Hamiltonian it has a clear meaning. This can be seen if the function Φ'' is drawn. It represents two first order lenses of equal strength but opposite sign, separated from each other by a distance of the order of the accelerating gap. The term (7/16) Φ'^2 and the constant part in the third term between the brackets give a second order lens in the middle of the accelerating gap.

Only the second term between the brackets thus gives an oscillating part of first order, which can be removed by a transformation function $\bar{G}(\bar{\pi}_z, \bar{z})$ given in ref. 11:

$$\bar{G}(\bar{\pi}_z, \bar{z}) = -\bar{\pi}_z \bar{\bar{z}} - a_1(s)\bar{\bar{z}}^2 + a_2(s)\bar{\pi}_z\bar{\bar{z}} - a_3(s)\bar{\pi}_z^2$$

This yields the final Hamiltonian $\bar{\bar{K}}$

$$\bar{\bar{K}} = \frac{1}{2}\bar{\bar{\pi}}_z^2 + \frac{1}{2}\bar{\bar{z}}^2 \left(\frac{8}{16} < \Phi'^2 > \frac{\Phi}{4\pi r^2} \sin\phi_o\right)$$

where $<>$ means the average value. The magnetic focusing is of second order—i.e. same order as the electric focusing. It is therefore sufficient to add in this last Hamiltonian the magnetic focusing term:

$$\bar{\bar{K}} = \frac{1}{2}\bar{\bar{\pi}}_z^2 + \frac{1}{2}\bar{\bar{z}}^2 \left(\frac{1}{2} < \Phi'^2 > + \frac{\Phi}{4\pi r^2} \sin\phi_o + \frac{n}{r^2}\right)$$

with $n/r^2 = -\frac{1}{p/r}\frac{dB}{dr}$. Now magnetic focusing and electric focusing can be compared to each other. Especially the gain or loss acquired with an off-resonant magnetic field can easily be studied. Such an off-resonant field may be a magnetic bump or cone in the centre. The axial oscillating frequency is given by

$$\nu_z^2 = \frac{1}{2} r^2 < \Phi'^2 > + \frac{\Phi}{4\pi} \sin\phi_o + n$$

Let us now assume that the central field B is given by

$$B = B_o\left(1 + \frac{\Delta B}{B_o} - \alpha r^2\right),$$

where B_o is the resonant field. If we substitute the turn number N we get:

$$B = B_o(1 + \frac{\Delta B}{B_o} - \beta N).$$

For this field a change in ν_z^2 is found equal to

$$\Delta\nu_z^2 = -\frac{1}{4}\frac{\Delta B}{B} + \frac{17}{8}\beta N$$

Now the turn number can be found at which the increase in magnetic focusing becomes more important than the decrease in electric phase focusing. One finds in this case $N \geqslant 3$ for not too sharply changing field bumps for which $\beta \approx 1/20\ \Delta B/B$. It must, however, be realised that the total focusing in the first revolutions is decreased. The above treatment is not suited for synchrocyclotrons as there $\sin\phi_o$ generally has large values compared to those in A.V.F. cyclotrons. An article on the synchrocyclotron centre has been given by Holm.[16]

The effect of phase focusing disappears if too many radial limitations in the electric fields are present during the first revolutions. The theory can be extended to multi-dee systems by substituting the corresponding Fourier expansions in the equations.

7. THE AXIAL ENVELOPE OF THE BEAM

Due to the periodicity of axial focusing the axial beam envelope also will show a periodical structure. If the beam occupies the area of an eigen ellipse in phase space we find the envelope by means of the co-ordinate transformation relations following from the last two canonical transformations in Section 6. Keeping terms up to first order one finds

$$z \approx z_o f^{1/2}\ [1 - a_2\ (s)],$$

where z_o is the envelope belonging to the eigen ellipse in the phase space after the last transformation and z is the envelope of the beam in real co-ordinates. If we substitute numerical quantities in this expression we find the envelope oscillating around a mean value (z_o) with amplitudes normally smaller than 10%, except for the first turn.

The envelope may be much more affected by space charge. The influence of space charge on the axial motion has been discussed by several authors.[12–15] McKenzie[12] and Lawson[13, 14] describe the motion of a particle on the outer edge of a wedge-shaped continuous current sheet, whereas Reiser[5] derives formulas taking into account the finite size of the bunched beam and the influence of neighbouring orbits.

The axial oscillation frequency is given by

$$\nu_z^2 = \nu_{zo}^2 - \frac{eE_z}{m\omega^2 z_o},$$

where E_z is the electric field strength acting on a particle moving at a height $z = z_o$ in the outer beam edge. The value of E_z depends on the selected model. Reiser gives:

$$E_z = \frac{I\,G_z}{4\epsilon_o v \psi z_o},$$

where I is the beam current, ψ the azimuthal extent of the wedge, ϵ_o the dielectric constant in vacuum, v the particle velocity, G_z a constant depending on the beam dimensions.

When ν_{zo}^2 is slightly changing we can consider the change in ν_z^2 to be adiabatic. Then the height of the beam envelope $\bar{z}_o$ is:

$$\frac{\bar{z}_o}{z_o} = \left(\frac{\nu_z}{\nu_{zo}}\right)^{-1/2}$$

where z_o is the beam envelope without space charge.

If the finite bunch length is ignored, the quality of the beam is not changed

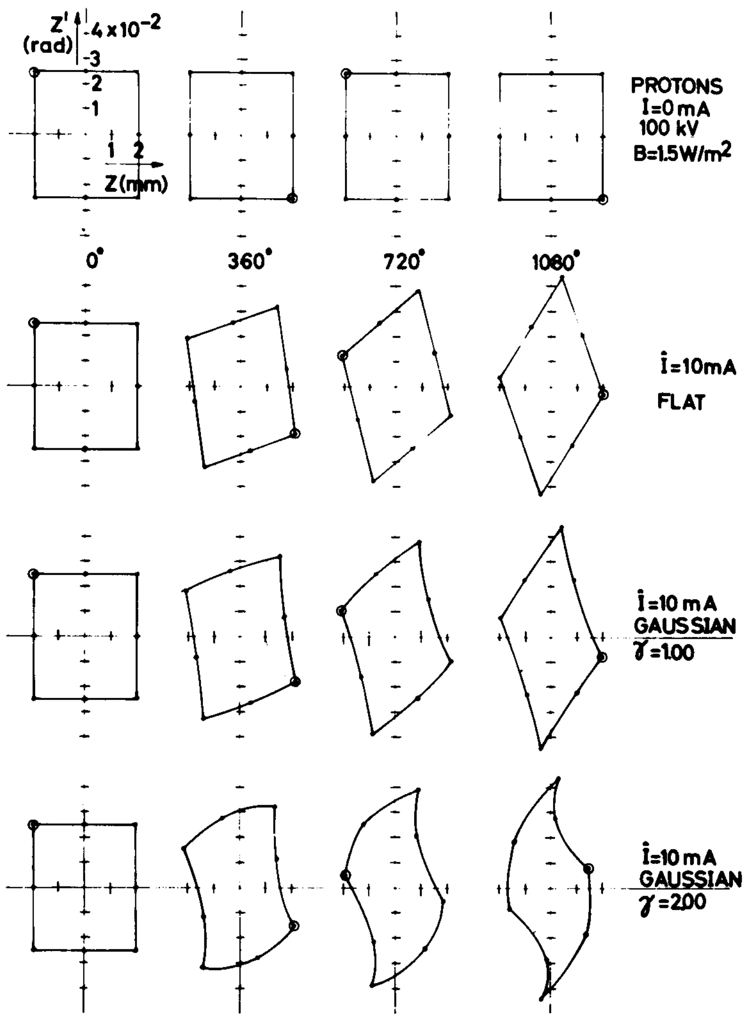

Fig. 5. The motion of axial phase space figures with and without space charge

by space charge effects when the current distribution is homogeneous, since then the particle motion is linear.

In practice the source will emit a beam with an approximately Gaussian distribution. Then non-linearities occur, as is illustrated in Fig. 5, where numerical results are shown.[15] A 200 mm mrad, 100 keV proton beam, initially 4 mm wide in both r and z directions, is circulating in a 15 kG magnetic field. Axial electric focusing occurs twice per revolution. The dee gaps are approximated by thin lenses. The corresponding vertical focusing frequency has been chosen as 0·5. The upper row of figures shows the vertical motion at zero current. In the other figures a peak current of 10 mA is taken. The second row has a flat charge distribution, in which case the motion remains linear. We observe a relative change in ν_z of about 7%, which agrees well with the value of 9% calculated from Reiser's work. In the last two rows the charge distribution is Gaussian:

$$\rho \approx \exp\left[-\gamma^2\left(\frac{r^2}{r_m^2} + \frac{z^2}{z_m^2}\right)\right]$$

where r_m and z_m are the semi axes of the ellipse representing the beam.

8. CONCLUSION

At present much loss of beam intensity occurs at the cyclotron centre. A careful study of axial focusing and space charge influence must be made for each cyclotron. The effect of a field bump in the central region will approximately disappear if axial electric focusing is not disturbed by grids or diaphragms. Space charge effects do not necessarily lead to an appreciable decrease in beam quality. Its influence can be calculated in detail.

A speculation about the images in the initial radial phase space of various slits used in the cyclotron centre and in the high energy resolution analysing system indicates a way of improving the beam properties behind the analysing system with regard to intensity and duty-cycle. In this respect the present survey may also be used as a scheme for numerical calculations.

REFERENCES

1. Gordon, M. M., *I.E.E.E. Trans. N.S.* **NS-13, 4,** 48, (1966).
2. Mallory, M. L., Phase space density studies on cyclotron ion sources, MSUCP-23, Michigan State University.
3. Blosser, H. G., private communication (1969).
4. Kramer, P., Hagedoorn, H. L. and Verster, N. F., CERN 63-19, 214, Geneva 1963.
5. Reiser, M., *I.E.E.E. Trans. N.S.* **NS-13, 4,** 171, (1966).
6. Blosser, H. G., Lectures at the Informal Meeting on electron- and gamma ray spectroscopy at heavy ion beams, Stockholm 1969, and at this Conference.
7. Hagedoorn, H. L. and Kramer, P., *I.E.E.E. Trans. N.S.* **NS-13, 4,** 64, (1966).
8. Welton, t. A., Sectored Focused Cyclotrons (Proc. Sea Ireland Conference), Nat. Acad. Science–Nat. Res. Counc. publ. 656, Washington D.C., 192, (1959).
9. Blosser, H. G., Gordon, M. M. and Reiser, M., CERN 63-19, 193, Geneva 1963.
10. Rethmeier, J., Nijenhuis, J. G. and Langkau, R., *Nucl. Instr. Meth.*, **68,** 135, (1969).
11. Hagedoorn, H. L. and Verster, N. F., *Nucl. Instr. Meth.* **18, 19,** 201, (1962).
12. McKenzie, K. R., *Nucl. Instr. Meth.* **31,** 139, (1964).
13. Lawson, J. D., *Nucl. Instr. Meth.* **34,** 173, (1965).
14. Lawson; J. D., *Nucl. Instr. Meth.* **64,** 214, (1968).
15. Hazewindus, N., Proceedings of this Conference, p. 625.
16. Holm, S., *Nucl. Instr. Meth.* **64,** 317 (1968).