First and second harmonic extraction

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ABSTRACT

The orbit separation desired for good beam extraction can be obtained in different ways. The separation is mainly determined by the harmonic field disturbances. A simple analytic representation of the motion of the orbit centre under the influence of first and second harmonic field components is presented. This description gives a clear and accurate insight into the extraction process. It is also used for numerical calculations with slowly varying field parameters at acceleration.

A large gradient of the second harmonic field component can drive the orbit centre in a fixed direction, determined by the first and second field harmonics. This effect resembles the operation of regenerative extraction. The method with two independently adjustable harmonics has the advantage of permitting flexible control of the increase of the coherent oscillation amplitude and its radial and azimuthal directions.

The numerical calculations are made for the field parameters of a Compact Isochronous Cyclotron.

1. INTRODUCTION

For good beam extraction in cyclotrons a relatively large orbit separation is advantageous. The orbit separation can be brought about in different ways, determined by harmonic magnetic and electric field disturbances and by the energy increase per turn.

The precessional[1] and regenerative[2] systems are well known. In the first the separation is obtained by the use of a very small first harmonic component. The second uses the radially decreasing peeler field and increasing regenerator field. These fields can be resolved into a number of harmonic field components, of which the second harmonic mainly drives the particle orbit centre in a desired direction. The first harmonic component normally has a minor influence, and moreover it has an absolute coupling with the second harmonic.

In this paper we will discuss the influence of first and second harmonic magnetic field components on the radial motion of the orbit centre, based upon a rather simple analytical description (Section 2). We will investigate the usefulness of separately adjustable harmonic components for flexible control of the radial

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and azimuthal motions of the orbit centre, which is an important feature in optimising an extraction system.

Some numerical calculations using the analytical formulae taking into account the change of the magnetic field parameters due to acceleration, will be given in Section 3. It should be emphasised that the analytical formulation presented here is a first order approximation, which is quite suitable before the point $\nu_r = 1$. Far out in the fringing field however, it gives only qualitative information.

We will not go into the details of the nonlinear effects and the induced vertical particle motion due to too large an oscillation amplitude, but to determine the maximum permissible amplitude these effects should certainly be taken into account.[1] Another phenomenon, the influence of the oscillation amplitude on energy compression and retardation, is discussed elsewhere.[3,4]

2. ANALYTICAL CALCULATIONS

Hagedoorn and Verster[1] pointed out that a time-independent Hamiltonian function $H(\varphi, I)$ for the action (I) and angle (φ) variables can be found, representing in a first approximation the motion of the actual orbit centre (Eqn 8.8, ref. 1). With this Hamiltonian the region of stable motion can be determined for a magnetic field configuration without disturbances.

An extension of this Hamiltonian for a magnetic field with perturbations in the average field and first and second harmonic components leads to (Eqn 11.2, ref. 1):

$$H(\varphi, I) = \tfrac{1}{2}(A_1 \cos\varphi + B_1 \sin\varphi)(2I)^{\frac{1}{2}} + \left[(\nu_r - 1) + \tfrac{1}{2}A_0' + (\tfrac{1}{2}A_2 + \tfrac{1}{4}A_2')\cos 2\varphi\right] I, \tag{1}$$

A_1, B_1 and A_2 are the amplitudes of the relative harmonic field components; A_0' and A_2' represent the radial gradients of the average field disturbance and the second harmonic component given by

$$A_{0,2}' = r\,\mathrm{d}A_{0,2}(r)/\mathrm{d}r;$$

ν_r is the relative radial oscillation frequency.

The angle φ is the angular co-ordinate of the orbit centre in the cyclotron, with its zero value along the direction of the second harmonic component. The angle is positive in the direction of rotation of the particle.

The radial co-ordinate ρ of the orbit centre is given by

$$\rho = r_0 \left(\frac{2I}{\nu_r}\right)^{\frac{1}{2}} \approx r_0 (2I)^{\frac{1}{2}} \text{ for } \nu_r \approx 1 \tag{2}$$

where r_0 is the radius of the particle orbit.

The equations of motion of the orbit centre can be found from Eqn (1) by

$$\frac{\mathrm{d}I}{\mathrm{d}\theta} = \frac{\partial H}{\partial \varphi} = \tfrac{1}{2}(2I)^{\frac{1}{2}}(-A_1 \sin\varphi + B_1 \cos\varphi) - I(A_2 + \tfrac{1}{2}A_2')\sin 2\varphi \tag{3a}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\theta} = -\frac{\partial H}{\partial I} = -\tfrac{1}{2}(2I)^{-\frac{1}{2}}(A_1 \cos\varphi + B_1 \sin\varphi) - (\nu_r - 1) - \tfrac{1}{2}A_0' - (\tfrac{1}{2}A_2 + \tfrac{1}{4}A_2')\cos 2\varphi \tag{3b}$$

where θ is the time co-ordinate, expressed in the azimuthal position of the particle.

The Hamiltonian is a constant of motion and it defines the lines along which the orbit centre moves as a function of time.

When

$$\begin{aligned} A_1 &= C_1 \cos\psi \\ B_1 &= C_1 \sin\psi \\ \nu_r - 1 + \tfrac{1}{2}A_0' &= C_0 \\ \tfrac{1}{2}A_2 + \tfrac{1}{4}A_2' &= C_2 \end{aligned}$$

these lines are given by

$$\left(\frac{\rho}{r_0}\right)^2 \left(C_0 + C_2 \cos 2\varphi\right) + \frac{\rho}{r_0} C_1 \cos(\varphi - \psi) = C, \tag{4}$$

putting the value of the Hamiltonian H equal to $C/2$.

2.1. *First harmonic component only*

The orbit centre moves under the influence of a first harmonic field component along lines which are given by

$$\left(\frac{\rho}{r_0}\right)^2 C_0 + \frac{\rho}{r_0} C_1 \cos(\varphi - \psi) = C. \tag{5}$$

These lines form concentric circles with radius and centre co-ordinates:

$$\left.\begin{aligned} r_a &= r_0 \left[\left(\frac{C_1}{2C_0}\right)^2 + \frac{C}{C_0}\right]^{\frac{1}{2}} \\ x_a &= -r_0 \frac{C_1}{2C_0} \cos\psi \\ y_a &= -r_0 \frac{C_1}{2C_0} \sin\psi \end{aligned}\right\} \tag{6}$$

This means that the centre of the circle is located on the line through the cyclotron centre pointing in the direction of the maximum of the first harmonic component. For $\nu_r < 1$ the centre is attracted by this maximum and for $\nu_r > 1$ it is pushed away.

The equation of angular motion is given by

$$\dot{\varphi} = -\tfrac{1}{2}\frac{r_0}{\rho}C_1 \cos(\varphi - \psi) - C_0 \tag{7}$$

This equation shows that the sense of rotation depends on the sign of $\nu_r - 1$. The angular velocity with which the orbit centre moves along the circle of Eqn (6) is

$$\frac{d\alpha}{d\theta} = -C_0$$

from which the arc length $\Delta S = r_a C_0 2\pi$ corresponding to one particle revolution can be found. For $C = 0$ this results in the well-known equation for the maximum displacement due to a first harmonic:

$$\Delta S = \pi r_0 C_1$$

At extraction the particle orbit passes the region where $\nu_r - 1$ changes from positive to negative values. Here the orbit centre moves along a circle with gradually increasing radius and centre position. At $\nu_r = 1$ the circle has an infinite radius. Beyond this point the circle has again a decreasing radius but its centre lies on the opposite side of the cyclotron centre and the sense of rotation is reversed.

Quantitative information of this motion as a function of time can only be obtained by solving the equations of motion (3) with the gradually changing coefficients (as a function of θ or r_0) given by the magnetic field configuration. (See Section 3.)

2.2. *Second harmonic component only*

A similar treatment can be given for the motion of the orbit centre under the influence of a second harmonic perturbation only. In this case the orbit centre follows the lines given by

$$\left(\frac{\rho}{r_0}\right)^2 = \frac{C}{C_0 + C_2 \cos 2\varphi} \tag{8}$$

When $|C_0| > |C_2|$ (or $|\nu_r - 1| > |\tfrac{1}{2}A_2 + \tfrac{1}{4}A_2'|$) these curves form concentric ellipses, whose centres coincide with the cyclotron centre.

The direction of motion of the orbit centre along the ellipse as a function of time is opposite in the two cases $\nu_r > 1$ and $\nu_r < 1$.

When the particle orbit approaches the region where $\nu_r = 1$ it must have passed the point where $|C_0| = |C_2|$. In this case we get an unstable situation, since for $|C_0| \leqslant |C_2|$ Eqn (8) yields hyperbolas. The angle φ_a of the asymptotes is given by

$$\varphi_a = \arctan\left[\left|\frac{C_2 + C_0}{C_2 - C_0}\right|\right]^{1/2} (+\ 180^\circ) \tag{9}$$

The sign of $\nu_r - 1$ has no influence on the orientation of the motion of the orbit centre along the curves as a function of time.

At the asymptotes (where the equation of angular motion shows $\dot{\varphi} = 0$ since $\cos 2\varphi_a = -C_0/C_2$) the relative amplitude grows or diminishes according to Eqn (3) as

$$\frac{\rho(\theta)}{r_0} = \frac{\rho_0}{r_0} \exp \pm \left[\tfrac{1}{2} (C_2^2 - C_0^2)^{\frac{1}{2}} \theta \right] \tag{10}$$

The instability occurring at two of the four angles φ_a can be used for a controlled orbit separation in the extraction region. The orbit centre is pushed in a constant direction while its distance from the cyclotron centre increases exponentially, showing the normal regenerative action.

The point $(\rho/r_0) = 0$ is an unstable point. In a region around it the orbit centres can move in two directions. A fixed displacement made by a first harmonic will move the unstable point away and will drive the orbit centres into one desired direction.

2.3. *Combination of first and second harmonic components*

The curves determined by the constant of motion show a more complicated behaviour in this case. However instabilities again occur when $|C_2| \geqslant |C_0|$. Considering the function $\rho(\varphi)$ for $C = 0$ (the only curve going through the cyclotron centre) we see that ellipses and hyperbolas again are obtained when we make a co-ordinate transformation from $x = (\rho/r_0) \cos\varphi$, $y = (\rho/r_0) \sin\varphi$ to x', y' by

$$\left.\begin{aligned} x' &= x + \frac{C_1}{2(C_2 + C_0)} \cos\psi \\ y' &= y - \frac{C_1}{2(C_2 - C_0)} \sin\psi \end{aligned}\right\} \tag{11}$$

The angles of the asymptotes of the hyperbolas in the new co-ordinate system are again given by Eqn (9).

The semi-axes are given by

$$\begin{aligned} a_{x'} &= \frac{C_1}{2} \left[\left| \frac{C_2 \cos 2\psi - C_0}{(C_2^2 - C_0^2)(C_2 + C_0)} \right| \right]^{\frac{1}{2}}, \\ b_{y'} &= \frac{C_1}{2} \left[\left| \frac{C_2 \cos 2\psi - C_0}{(C_2^2 - C_0^2)(C_2 - C_0)} \right| \right]^{\frac{1}{2}}, \end{aligned} \tag{12}$$

which shows that in the case $|C_2| < |C_0|$ the ellipses are large for $\psi \approx 90°$. It is then possible that the orbit centre can make large steps per particle revolution along a nearly straight line.

Some examples of orbit centre paths are given in Fig. 1(a) for the case $|C_2| < |C_0|$ and in Fig. 1(b) when $|C_2| > |C_0|$ for different angles ψ of the first harmonic.

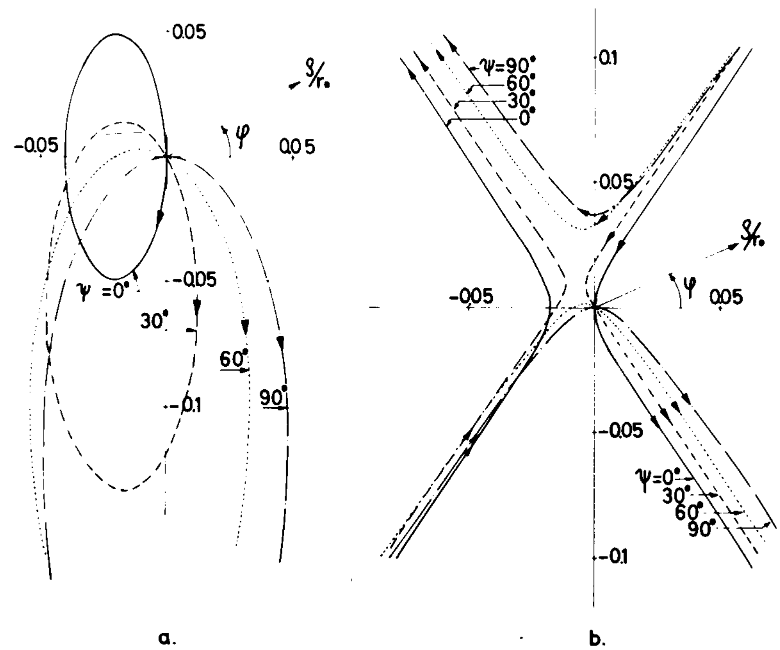

Fig. 1. Orbit centre paths under the influence of first and second harmonic field components. ψ is the angle of the first harmonic. $C_0 = 190 \times 10^{-4}$, $C_1 = 13 \times 10^{-4}$. *(a)* $C_2 = 135 \times 10^{-4}$, *(b)* $C_2 = 540 \times 10^{-4}$

It is seen here that the ellipses and hyperbolas are placed such that the orbit centre leaves the cyclotron centre in a direction perpendicular to the first harmonic maximum, which is in accordance with the equation of angular motion (3).

The rate of change of the oscillation amplitude as a function of time is given by

$$\frac{d\left(\frac{\rho}{r_0}\right)}{d\theta} = \tfrac{1}{2} C_1 \sin(\psi - \varphi) - \frac{\rho}{r_0} C_2 \sin 2\varphi \qquad (13)$$

From this it follows that the optimum contribution from the first harmonic is obtained when $\psi = \varphi + 90°$. In order to maintain this optimum contribution during a part of the extraction process, it is desirable that φ remains constant. This is obtained in the case $|C_2| > |C_0|$ for an angle $\varphi = \varphi_i$ given by

$$\left.\begin{aligned} \cos(\varphi_i - \psi) &= 0 \\ C_0 + C_2 \cos 2\varphi_i &= 0 \end{aligned}\right\} \qquad (14)$$

This means that when the strength of the second harmonic is given, the angle ψ of the first harmonic should be set such that

$$\psi = \pm \tfrac{1}{2} \arccos\left(-\frac{C_0}{C_2}\right) \pm 90^\circ \tag{15}$$

In this case the orbit centre, starting in the cyclotron centre, is pushed in a constant direction and follows the asymptote of the hyperbola. The strength of the first harmonic gives a separate independent parameter to control the amplitude increase in the first instance. The strength of the second harmonic also has its influence at larger amplitudes.

The total effect is maximised when C_2 is chosen such that φ_i given by Eqn (14) is equal to -45° In that case we get

$$\frac{d(\rho/r_0)}{d\theta} = \tfrac{1}{2} C_1 + \frac{\rho}{r_0} C_2$$

Although the phenomenon of growing amplitude is similar to that occurring in regenerative action with fixed field bumps, the Eqns (13) and (14) show the advantage of separate adjustable first and second harmonic field components. With such a system great flexibility is obtained both in direction and in rate of change of the amplitude.

In the case where the coefficients in the Hamiltonian equation change adiabatically during acceleration, the motion of the orbit centre is much more complicated. In the region where ν_r changes rapidly (near $\nu_r = 1$) an optimum setting of the angle of the first harmonic is difficult.

In the next section some numerical results of orbit centre displacements for accelerated particles are discussed.

3. *Numerical calculations*

For the numerical calculations it is useful to present the equations of motion in X and Y co-ordinates:

$$\begin{aligned} X &= r_0 \sqrt{2I} \cos \varphi \\ Y &= r_0 \sqrt{2I} \sin \varphi \end{aligned} \tag{16}$$

This is a canonical transformation where X is the canonical co-ordinate and Y the canonical momentum. The new Hamiltonian function becomes:

$$\begin{aligned} K(x,y) = \tfrac{1}{2} A_1 r_0 x + \tfrac{1}{2} B_1 r_0 y + \tfrac{1}{2} (\nu_r - 1 + \tfrac{1}{2} A_0') (x^2 + y^2) \\ + \tfrac{1}{2} (\tfrac{1}{2} A_2 + \tfrac{1}{4} A_2') (x^2 - y^2) \end{aligned} \tag{17}$$

The equations of motion are then given by

$$\frac{dx}{d\theta} = \tfrac{1}{2} B_1 r_0 + y \left[\nu_r - 1 + \tfrac{1}{2} A_0' - (\tfrac{1}{2} A_2 + \tfrac{1}{4} A_2')\right] \tag{18}$$

$$\frac{dY}{d\theta} = -\tfrac{1}{2}A_1 r_0 - x\left[\nu_r - 1 + \tfrac{1}{2}A_0' + (\tfrac{1}{2}A_2 + \tfrac{1}{4}A_2')\right] \tag{19}$$

In order to introduce the acceleration we transform the particle time θ into the average radius r_0 with

$$\frac{dx}{dr_0} = 2\pi \frac{dx}{d\theta}\frac{dn}{dr_0}$$

$$\frac{dy}{dr_0} = 2\pi \frac{dy}{d\theta}\frac{dn}{dr_0}$$

where n is the number of revolutions given by

$$n = \frac{1}{2}\frac{e^2B^2}{m\Delta E_{\text{turn}}} r_0^2 = \alpha r_0^2$$

Then the equations of motion become:

$$\frac{dx}{dr_0} = 2\pi\alpha r_0 \left\{r_0 C_1(r_0)\sin\psi + 2y\left[C_0(r_0) - C_2(r_0)\right]\right\} \tag{20}$$

$$\frac{dy}{dr_0} = 2\pi\alpha r_0 \left\{- r_0 C_1(r_0)\cos\psi - 2y\left[C_0(r_0) + C_2(r_0)\right]\right\} \tag{21}$$

These simultaneous differential equations can be numerically solved for given $C_1(r_0)$, $C_2(r_0)$, $\nu_r(r_0)$ and ψ.

Since a large amplitude of the coherent oscillation causes the particles to cross different values of ν_r, A_1, A_2 and A_2' during one revolution, the coefficients in the Eqns (20)–(21) are taken as averages along the particle path in the numerical calculations. At large oscillation amplitudes the effect associated with the crossing of $\nu_r = 1$ occurs at smaller average radius r_0 than would be expected.

Some results of the calculations are shown in Figs 2 and 3 for the magnetic field parameters of our compact isochronous cyclotron.[5] In Fig. 2 the orbit displacement is given for a first harmonic field perturbation only. The value of $\nu_r - 1$ ranges in the region of the calculation from 110×10^{-4} to its maximum value of 400×10^{-4}. After its maximum it drops very rapidly to negative values. Three curves are given for different values of the first harmonic; in all cases $\psi = 0$. The first harmonic is a gradient field disturbance with zero value at the starting point of the calculations, increasing to relative values of $0{\cdot}75 \times 10^{-3}$, $1{\cdot}5 \times 10^{-3}$, 3×10^{-3} respectively for the curves 1, 2 and 3.

The dots on the curves represent the number of revolutions. Fig. 3 shows the behaviour of the orbit centre under the influence of a first and a second harmonic component, for different angles ψ of the first harmonic. Both harmonics are gradient fields, A_1 ranging between 0 and 6×10^{-3} and A_2 between 0 and 12×10^{-3}.

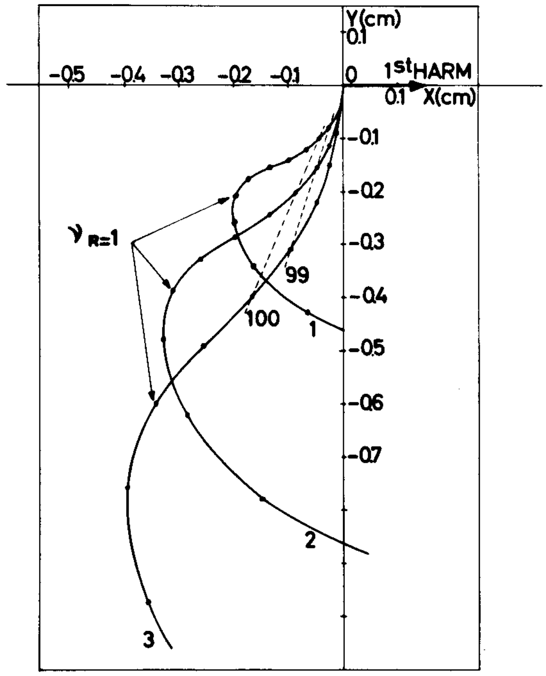

Fig. 2. Motion of the orbit centre for a first harmonic field component only. The dots represent the turn-number. Curve 1: A_1 *from 0 to 0·75 × 10^{-3}, curve 2:* A_1 *from 0 to 1·5 × 10^{-3}, curve 3:* A_1 *from 0 to 3 × 10^{-3}*

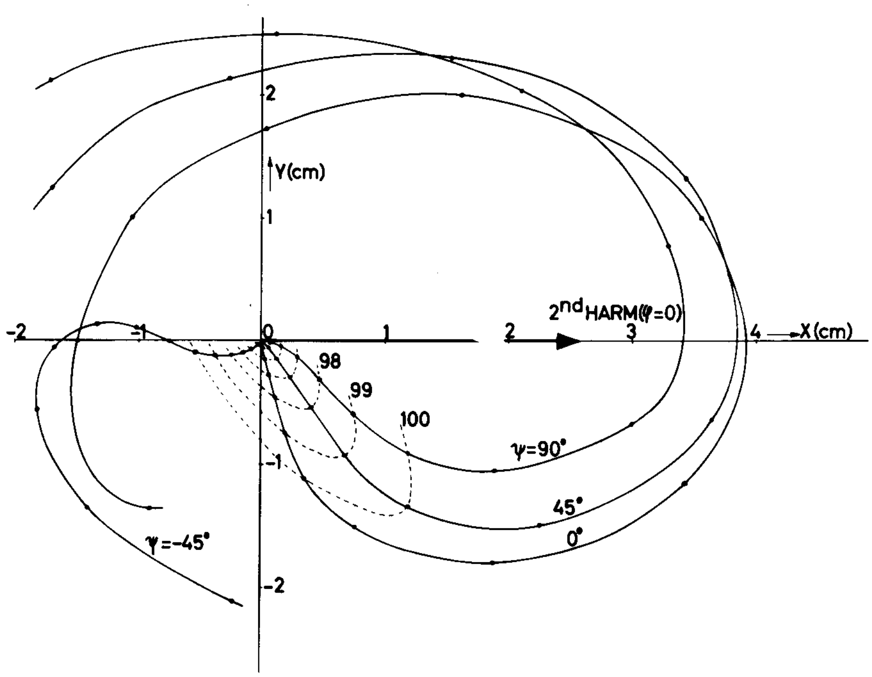

Fig. 3. Motion of the orbit centre for the first and second harmonic components. ψ is the angle of the first harmonic. A_1 *from 0 to 6 × 10^{-3},* A_2 *from 0 to 12 × 10^{-3}*

The figures indicate the strong dependence on the angle ψ of the first harmonic. According to Eqn (15) a nearly straight line should be obtained with this value of A_2, for $\psi \approx 35°$. It is seen that in a small range of angles around this value the orbit centre initially follows a nearly straight line. Close to the point $\nu_r = 1$ all curves turn around and a precessional motion with a very large amplitude is obtained.

4. *Concluding remarks*

The analytical description of the orbit centre displacement provides a quick and clear insight into the extraction process and the order of magnitude of the desired harmonic field components for many different cases. This is of great help before starting the more complicated orbit integration. The quantitative information is fairly accurate up to the point $\nu_r = 1$. Some of the calculations have been checked by numerical orbit integration methods and show good agreement.

It is clear that a separate setting of both harmonics has many advantages over a fixed coupling but in the case of regenerative extraction it is difficult to obtain the optimum situation, and there is no flexible tuning.

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DISCUSSION

Speaker addressed: J. M. van Nieuwland (Philips)

Questions by H. Liesem (A.E.G.): (1) What is the bump strength in a practical case? (2) Did you calculate the corresponding axial motion?
Answers: (1) In the examples given the harmonics are formed by gradient coils, giving for example, a second harmonic of 0 to 100 gauss. This is of course very large and is used to show the effect clearly. It yields an orbit separation of about 1 cm and this is not necessary in the machine.
(2) Of course the axial motion should be calculated in order to determine the maximum permissible oscillation amplitude. This depends very strongly on the field parameters.

Question by P. Wucherer (A.E.G.): Did you use only the trim coils for producing first and second harmonics?
Answer: Yes, this gives the flexibility to the system.

Question by P. Wucherer (A.E.G.): Did you choose the four-sector concept in compact cyclotrons because first and second harmonics are easily produced by the trim coils.
Answer: No, but it helps.

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