Computer Codes for Cyclotron Orbit Calculations

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I should say that our philosophy on accelerator calculation has been pretty well prejudiced by certain circumstances which are probably quite different from those of most of you.

As you know, we have for some time been hoping to build a large cyclotron at Oak Ridge, where by large I mean a relativistic cyclotron, over 800 Mev. We don't have support for it, and at present Oak Ridge's resources are almost completely committed to building a 75-Mev proton cyclotron. However, because of the start that we have made in the direction of large cyclotrons, our methods depend to a relatively small extent on analytical procedures.

Our philosophy is in some ways similar to the MURA philosophy described by Dr. Symon. We use analytic formulas for orientation, to make sure that we don't get very far off, for understanding results from computer runs, and so on. In the type of situation which we encounter in the large cyclotron, the analytical formulas, if they are simple enough to be easily used, are likely to be rather grossly in error. We have not ourselves derived formulas which give good answers. We have the feeling that if these formulas were sufficiently complicated to give good results, they would be rather too complex to use easily. Also we have a computer available, which makes a difference. We have, therefore, relied to a very large extent on the development of computer codes, rather more specialized than the MURA design, for the calculation of cyclotron orbit properties. First, we used the Oracle which unfortunately is a custom-built machine and, therefore, unavailable except at ORNL. More recently, we have started to use the IBM 704, so that our results can be used by anybody who has access to a 704.

ω_0 = unit of frequency

b = unit of magnetic field

a = unit of length

$$b = \frac{m_0 c \omega_0}{e}$$

$$a = \frac{c}{\omega_0} = \frac{m_0 c^2}{e\,b}$$

b = central field in units of 10^4 gauss.

$1/\tau_0 = 2\pi/\omega_0 = 15.246b$ mc/sec

$a = c/\omega_0 = 123.2/b$ inches $= 313/b$ cm.

$r/a \longrightarrow r$

$\omega_0 t \longrightarrow t$

$B/b \longrightarrow B$

Fig. 31.

Figure 31 just sets up the general notation which we use. I think the units are very similar to those Dr. Parzen mentioned. If one takes ω_o, which is the r-f frequency, as the unit of frequency or the reciprocal unit of time, b, as the unit of magnetic field, a, as the unit of length, then b is that magnetic field which would give orbital frequency ω_o for the type of particle under consideration. The length a is simply the reciprocal wave number, for the r-f field. If one takes b to be the central field in units of 10 gauss, then the given numerical relations hold for protons. Dimensional quantities will not be used from this point on; r/a becomes r, $\omega_o t$ becomes t, B/b becomes B.

$$B(r,\theta) = B_z(r,\theta,0)$$

$$\mathbf{B} = \nabla\psi \; ; \; \nabla^2\psi = 0$$

$$\psi = \sum_n \frac{(-1)^n}{(2n+1)!}(L^n B)z^{2n+1}$$

$$L \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

$$\psi \cong z\,B(r,\theta)$$

Fig. 32.

The value of B for r = 0 will be unity.

Figure 32 shows the representation of the magnetic field used, which is general in principle. Actually we have specialized rather severely for most of the calculations performed. The median plane magnetic field is $B(r,\theta)$ and B at any point can be written as the gradient of the magnetic potential, which has zero Laplacian. This magnetic potential can be written as a simple series expansion in odd powers of z.

L is the two-dimensional Laplace operator. Actually, we have specialized in almost all calculations by taking the first term only which corresponds to just the usual rule of writing B_r and $B\theta$ in terms of derivatives of the median plane field. We have never found a place where this introduces a serious error, but I expect that some day we will.

$$s' = \frac{ds}{d\theta} = (r^2 + r'^2 + z'^2)^{1/2}$$

$$t' = \frac{1}{\beta}s'$$

$$p_r = p\frac{r'}{s'} \; ; \quad p_z = p\frac{z'}{s'} \; ; \quad p = \frac{\beta}{\sqrt{1-\beta^2}} \; ; \quad E = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{p^2+1}$$

$$q = (p^2 - p_r^2 - p_z^2)^{1/2} = p\frac{r}{s'}$$

Fig. 33.

Figure 33 contains some more definitions. The quantity s' is the derivative of arc length with respect to θ, (which is used as the independent variable), and t' is the derivative of orbit time with respect to θ, p_r is the radial momentum. It is just the total momentum in units of m c, and the z-component of momentum is similarly defined. The momentum, p, is related to β as shown, and E is the total relativistic energy. Finally the quantity Q comes in, with definition given.

Figure 34 gives the equations of motion; a prime means $d/d\theta$, and the equations are general. Approximations enter in representing the three field components. At this point there is no acceleration in the equations. That is introduced in a trivial way.

Figure 35 gives the actual approximations used, which amount to neglecting terms quadratic in z, of which there are a great many, and the equation is then specialized as indicated. The field components simplify, and the radial equations of motion become a little simpler.

We have devised a long series of codes at Oak Ridge for handling the basic cyclotron design problem, which is that of finding the equilibrium orbit properties and the near equilibrium orbit properties in an experimental magnetic field. All of these codes have a great deal in common as they have evolved in a steady progression. The process for finding the equilibrium orbit is shown in Figure 36. R and P_r are the radius and radial momentum at $\theta = 0$ an arbitrary azimuth), and they satisfy the

$$p_r' = q - rB_z + \frac{r}{q} p_z B_\theta$$

$$r' = \frac{r}{q} p_r$$

$$p_z' = rB_r - \frac{r}{q} p_r B_\theta$$

$$z' = \frac{r}{q} p_z$$

$$t' = E \frac{r}{q}$$

Fig. 34.

Neglect z^2, zp_z, p_z^2, etc.

$$q \to Q = (p^2 - p_r^2)^{1/2}$$

$$B_z = B$$

$$B_r = z \frac{\partial B}{\partial r}$$

$$B_\theta = z \frac{\partial B}{r \partial \theta}$$

$$p_r' = Q - rB$$

$$r' = \frac{r}{Q} p_r$$

Fig. 35.

definition in the first two lines, namely, that R and P_r must repeat in one sector. Actually, if one starts with an arbitrary r and $\theta = 0$, different values will be obtained at the end of the sector.

Let's suppose that the desired R is equal to the assumed r plus a correction, x, and similarly for the radial momentum. Then, what is desired is that after passage through one sector r + x should be equal to the initial r + x and similarly for the radial momentum.

We approximate x and p_r at the end of one sector by an expression linear in the x and p_r at the beginning of the sector, as shown in Figure 37. This is an approximation only, but one which becomes perfect as x and p_r become small, which means that as we approach the equilibrium orbit in a series of tries, the approximation becomes more accurate. Under these conditions the matching equation between the beginning and the end of the sector becomes just the pair of linear equations, appearing at the bottom, which tell us what the $x(0)$ and $p_x(0)$ should be, on all of these assumptions, in order to find the correct equilibrium orbit. In other words, we run an orbit with some r and p_r initially. Then if we know the indicated derivatives, we can immediately find what $x^{(0)}$ and $p_x^{(0)}$ would have to be added to these assumed $r^{(0)}$

$$R(2\pi/N) = R(0)$$

$$P_r(2\pi/N) = P_r(0)$$

$$r(0) \longrightarrow r(2\pi/N)$$

$$p_r(0) \longrightarrow p_r(2\pi/N)$$

$$R(0) = r(0) + x(0)$$

$$P_r(0) = p_r(0) + p_x(0)$$

$$r(0) + x(0) = r(2\pi/N) + x(2\pi/n)$$

$$p_r(0) + p_x(0) = p_r(2\pi/N) + p_x(2\pi/N)$$

Fig. 36.

$$x(2\pi/N) = \frac{\partial x(2\pi/N)}{\partial x(0)} x(0) + \frac{\partial x(2\pi/N)}{\partial p_x(0)} p_x(0)$$

$$p_x(2\pi/N) = \frac{\partial p_x(2\pi/N)}{\partial x(0)} x(0) + \frac{\partial p_x(2\pi/N)}{\partial p_x(0)} p_x(0)$$

$$\left[\frac{\partial x(2\pi/N)}{\partial x(0)} - 1\right] x(0) + \frac{\partial x(2\pi/N)}{\partial p_x(0)} p_x(0) = r(0) - r(2\pi/N)$$

$$\frac{\partial p_x(2\pi/N)}{\partial x(0)} x(0) + \frac{\partial p_x(2\pi/N)}{\partial p_x(0)} p_x(0) = p_r(0) - p_r(2\pi/N)$$

Fig. 37.

$$p_x' = -\frac{p_r}{Q} p_x - \frac{\partial}{\partial r}(rB)\, x$$

$$x' = \frac{p_r}{Q} x + \frac{p^2 r}{Q^3} p_x$$

$$x_1(0) = \delta x \qquad x_2(0) = 0$$

$$p_{x1}(0) = 0 \qquad p_{x2}(0) = \delta p_x$$

$$x_1(2\pi/N) = \frac{\partial x(2\pi/N)}{\partial x(0)} \delta x \qquad x_2(2\pi/N) = \frac{\partial x(2\pi/N)}{\partial p_x(0)} \delta p_x$$

$$p_{x1}(2\pi/N) = \frac{\partial p_x(2\pi/N)}{\partial x(0)} \delta x \qquad p_{x2}(2\pi/N) = \frac{\partial p_x(2\pi/N)}{\partial p_x(0)} \delta p_x$$

Fig. 38.

and $p_r^{(0)}$ to get the true equilibrium orbit. The method used here is just exactly Newton's method for a two-dimensional problem; the simultaneous equations are solved easily.

The problem is to find the derivatives which are required to tell how the final r and p move about as the initial r and p_r are changed by arbitrary amounts. We formerly did this by making definite movements in the two directions, finding how the final point would move and then taking the differences. This is a little clumsy because of the fact that if the displacement is large nonlinear effects come in, while if the displacement is small, round-off trouble comes in, and some loss of significance is inevitable.

Figure 38 shows the procedure presently used in the 704 code. The equations for the small quantities p_x and x linearized and solved exactly, that is, to the same approximation to which all our differential equations are integrated. We introduce two sets of quantities, an x_1 and p_x, which always start out with an x displacement only, and x_2 and p_x which start out with a p_x displacement only. From the values at the end of the sector the required derivatives can immediately be read off without any loss of significance. All the numbers gotten are accurate to the total number of figures that are carried in the register of the machine.

Figures 39 and 40 show the steps in the process. First, a guess must be made for r and p_r. There is a definite method for making the guess which I cannot describe for lack of time. The code is set up to integrate simultaneously $r_1 p_r$ with these conditions, x_1, p_x^1, x_2, and p_x^2.

The virtue of solving simultaneously all these equations, which require a common set of field values, derivatives, etc., is that once the

1. Guess (r, p_r) for equilibrium orbit.
2. Integrate $r, p_r, x_1, p_{x_1}, x_2, p_{x_2}$ equations through one sector.
3. Obtained improved (r, p_r) and check $\epsilon = |r(2\pi/N) - r(0)| + |p_r(2\pi/N) - p_r(0)|$.
4. If ϵ is too large, repeat. If ϵ is small enough, integrate $r, p_r, x_1, p_{x_1}, x_2, p_{x_2}, z_1, p_{z_1}, z_2, p_{z_2}$, t equations through one sector.
5. Calculate and print:

   T, ν_r, ν_z, $\langle r \rangle$, $r(0)$, $p_r(0)$, etc.

Fig. 39.

machine goes to the trouble of getting the field values it should use them as many times as it can before it drops them to get something else. This is a simple matter of efficiency. Then from these results, improved R,p values are calculated by solving the linear equation. The discrepancy between the beginning and the end of the sector is then checked, as shown. If ε is too large, the process is then repeated. If it is small enough, the integration scheme is changed, and the equations for R, P_r, x_1, p_x^1, x_2, p_x^2, z_1, p_z^1, z_2, p_z^2 are integrated through one sector, again because of economy. Finally the indicated quantities are calculated and printed out. We have a very flexible printout with a variety of quantities available. Eventually everything that one might want to know about the equilibrium orbits and small axial and radial deviations can be printed out. We find that on the 704 for one energy value we can calculate all of these properties and get very good accuracy in a matter of about six seconds. On the 704 with a realistic charge, this means a cost of about 50 cents per energy value. Unfortunately, everybody does not have a 704. All quantities are given to at least four significant figures, and for some quantities much more. This is hard to beat by analytical procedures.

I would like to make it clear that this work was done by a number of people. In particular I would like to give credit for the detailed planning of the 704 code to Dr. Gordon, of the University of Florida, who spent some 15 months with us rather recently. A number of his ideas are incorporated in the code which I have just described, and they are largely responsible for its fine performance.

POWELL: I am interested in knowing how small that epsilon is to do it in six seconds.

WELTON: The epsilon is so set that the equilibrium orbit conditions are matched all the way across the machine register, or in other words, approximately 10 significant

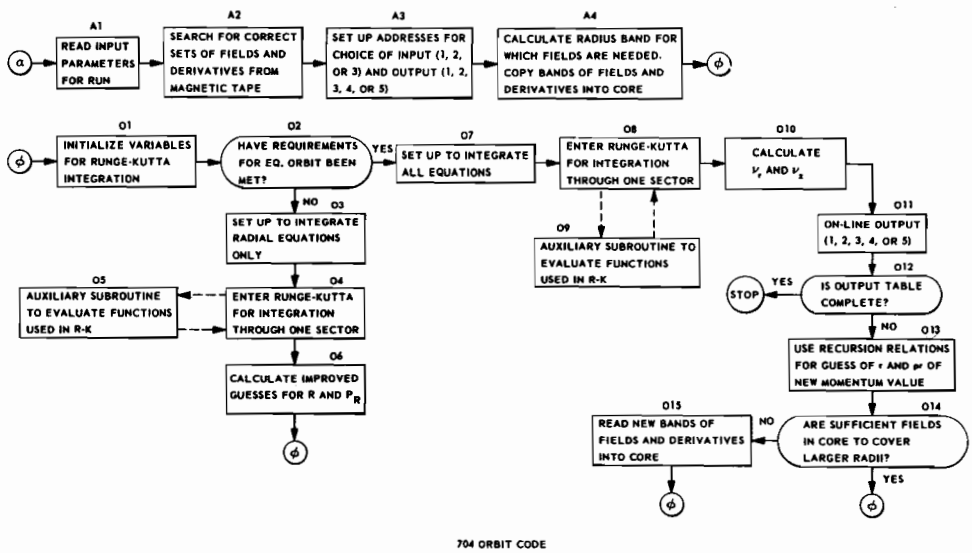

Fig. 40.

figures. This is the beauty of Newton's method. If you are not familiar with numerical methods the name may not mean anything; but if you have ever used it, you realize that the convergence is magical. Essentially what happens is that if on the first try, you can get two significant figures, the next one will give you four, the next one eight, and the next one sixteen. It happens that Dr. Gordon was able to get a guessing scheme for the equilibrium orbit so that the first time it gives him about four or five figures and the next time through it gives him everything. Figure 41 shows the accuracy obtained in successive stages.

SMITH: I would like to offer a testimonial on Welton's sales talk [laughter]. I think this Oak Ridge system works so beautifully that it is clearly the thing to use if you have a 704 available. We certainly will use the Oak Ridge code to get these various quantities and relegate our approximation formulas to the role of trying to figure out what to change in case something turns out in a way we don't like. So I heartily endorse it.

WELTON: I will make a further advertisement. There is a detailed writeup of the philosophy involved here which Dr. Gordon has recently finished, and it will be issued in the next few weeks as Oak Ridge report 2765.

One other point is that a code which Dr. Smith described in some detail, for running general orbits, making phase plots, etc., in the cyclotron, which we now have available for the Oracle, is shortly going to be transcribed with at first a few minor improvements for the 704. This will be available later.

LIVINGOOD: May I ask a simple-minded question? Are all of these calculations done on the basis of using any field you know you can obtain?

WELTON: A long time ago we were committed to the philosophy that it would be a good idea to have a code with which one simply took experimental values and put them directly into the machine without processing. This family of codes, therefore, always requires experimental information or analytical information in the form of stored numerical values in a polar mesh in the median plane. A considerable part of the speed of the code arises from this field representation.

We have recently found that this philosophy has a weak point. If the magnetic field measurements are not very accurate, phase plots may look bad and it will be necessary to smooth the fields in some way. If the measurements are very good, much time can be saved.

	$r(0)$	$r\left(\frac{2\pi}{N}\right)$	$p_r(0)$	$p_r\left(\frac{2\pi}{N}\right)$
$\beta = .24$	0.24000000	0.25172663	0.00000000	0.00361747
	0.24720228	0.24708448	−0.00002974	−0.00000626
	0.24715142	0.24715141	−0.00000012	−0.00000012
$\beta = .26$	0.26774736	0.26780753	−0.00000013	+0.00001696
	0.26778323	0.26778323	−0.00000013	−0.00000013
$\beta = .28$	0.28842057	0.28840913		
	0.28841384	0.28841383		

Fig. 41.

TENG: How do you explain the discrepancy between the two computer code results?

SMITH: As I said, we have not had enough time to check. The Ill-Tempered Five requires first analysis of the experimental data and getting the amplitudes for the harmonic analyses.

WELTON: It is extremely confusing because the harmonic analysis

is the same as Smith used in calculating his analytical results.

SMITH: For judging the approximation formulas it might be better to compare the IT-V with the approximate ones. There are complications. As I mentioned, I am not sure that we have unearthed all possible magnetic field measurement errors and these would come out differently perhaps in their effect. Also the integration system, and so on, have not been checked as they should. But I don't know if we could do much about that, because the agreement among all three is certainly magnificent as far as any practical outcome is concerned.