NUMERICAL STUDY OF BEAM TRAPPING IN STABLE ISLANDS FOR SIMPLE 2D MODELS OF BETATRONIC MOTION

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Abstract

An essential ingredient for the proposed Multi-Turn Extraction (MTE) at the CERN PS is the beam trapping in stable islands. The control of the trapping process is essential for the quality of the final beam in terms of intensity sharing and emittance. In this paper, the splitting process is studied quantitatively by means of numerical simulations performed on 2D model representing the horizontal non-linear betatronic motion. The results are reviewed and discussed in details.

INTRODUCTION

The multi-turn extraction (MTE) is based on the trapping of a fraction of particle beam in stable islands created in the horizontal phase space by means of sextupoles and octupoles [1, 2]. The beam is split by crossing a resonance using a variation of the tune. Different resonances have been successfully tested at the CERN PS [3] and the actual implementation uses the fourth order resonance. The resulting beam is made of two different structures: the core and four beamlets corresponding to the particles trapped inside the stable islands. By inducing a closed orbit bump around the extraction septum it is then possible to extract the beam in five turns. A crucial point of this method is to obtain an equal intensity and emittance sharing among the core and the four islands. Such a sharing depends on the strengths of the non-linearities, the tune variation, and the initial beam distribution. The effect of these parameters has been studied by means of a simple numerical model. The motion in the accelerator is represented by a linear transformation except for a sextupole and an octupole, located at the same place, and represented by a single, non-linear kick [2]. In this paper we discuss the results of numerical simulations based on a 2D generalised Hénon mapping given by

\[
\begin{align*}
\hat{X}_{n+1} = & R(\omega_n) \left( \hat{X} + \hat{X}^2 + \kappa \hat{X}^3 \right) \\
\hat{X}'_{n+1} = & 2 K_3 \frac{1}{3} K_2^2 \beta_x
\end{align*}
\]

where \( \omega_n = 2\pi \nu(n) \) is time-varying and \( \kappa \) represents the ratio between the strengths of the non-linear elements, weighted by the value of the beta function at that location. These coordinates are dimensionless and are related to the usual Courant-Snyder coordinates by the scaling factor \( \frac{1}{2} K_3 \beta_x^{3/2} \). The trapping of a particle inside the island occurs when it crosses the separatrix, which is moving due to the tune variation. At that point the particle has a certain probability to be trapped in the island. That probability increases if the adiabaticity of the crossing is increased, i.e., if the motion of the separatrix is slow compared to the motion of the particle in the phase space. Fig. 1 represents the evolution of the beam distribution during the tune variation.

\[ T(\epsilon, N, \kappa) = A(\kappa, N) \left[ 1 - e^{-B(\kappa, N)\epsilon} + C(\kappa, N) \right] \]  

where the results show that the following simple expressions can be assumed for the functions \( A, B \) and \( C \):

\[
\begin{align*}
A(\kappa, N) &= (c_1 \kappa + c_2)(d_1 \sqrt{N} + d_2) \\
B(\kappa, N) &= (c_3 \kappa + c_4)(d_3 N + d_4) \\
C(\kappa, N) &= (c_5 \kappa + c_6)(d_5 N + d_6).
\end{align*}
\]

Simulations performed for a large range of parameters set\(^1\) allowed to obtain the numerical values of the coefficients listed in Table 1 together with the error associated with the fit. The increase of the trapping with a slower tune vari-

Figure 1: Beam distribution at the beginning (\( n = 0 \)) and end (\( n = 20000 \)) of the trapping process. In all simulations \( 10^6 \) particles have been used.

TRAPPING MODEL

The trapping fraction \( T \), defined as the percentage of particles trapped in one island, is crucial to assess the performance of the splitting process as the goal is to reach \( T = 20\% \), which corresponds to beamlets and core equally populated. Tracking simulations using the model (1) were performed to obtain \( T \) as a function of the total number of turns \( N \) over which the tune variation is performed, the strength \( \kappa \), and the initial emittance \( \epsilon_i \). The resonance is crossed from above with a tune varying linearly from 0.252 to 0.245.

Fig. 2 displays \( T \) for an initial Gaussian beam distribution of \( 10^6 \) initial conditions as a function of \( \epsilon_i \) for a given set of \( N \) and \( \kappa \). The trapping increases with \( \epsilon_i \) in a similar way for the range of variation of \( N \) and \( \kappa \) that was used. A model for \( T \) is proposed with the following form

\[ T(\epsilon, N, \kappa) = A(\kappa, N) \left[ 1 - e^{-B(\kappa, N)\epsilon} + C(\kappa, N) \right] \]  

\(^1\)Beam distributions with \( 2 \times 10^{-4} \leq \epsilon_i \leq 80 \times 10^{-4}, -1.9 \leq \kappa \leq -1.1 \) and \( 5 \times 10^3 \leq N \leq 25 \times 10^3 \).
Figure 2: Trapping fraction as a function of $\epsilon_i$ for different sets of parameters $N$ and $\kappa$. The markers represent the simulation results, while the lines the fitted curves.

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>$0.288 \pm 0.009$</th>
<th>$c_2$</th>
<th>$3.588 \pm 0.007$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_3$</td>
<td>$1.742 \pm 0.056$</td>
<td>$c_4$</td>
<td>$-1.322 \pm 0.074$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$-0.111 \pm 0.007$</td>
<td>$c_6$</td>
<td>$-0.383 \pm 0.011$</td>
</tr>
<tr>
<td>$d_1$</td>
<td>$0.44 \pm 0.01$</td>
<td>$d_2$</td>
<td>$5.400 \pm 0.007$</td>
</tr>
<tr>
<td>$d_3$</td>
<td>$-0.054 \pm 0.002$</td>
<td>$d_4$</td>
<td>$-0.125 \pm 0.001$</td>
</tr>
<tr>
<td>$d_5$</td>
<td>$0.341 \pm 0.006$</td>
<td>$d_6$</td>
<td>$-1.217 \pm 0.004$</td>
</tr>
</tbody>
</table>

Table 1: Coefficients of the fit defined in Eqs. (2) and (3)

The plot shows the trapping fraction as a function of $\epsilon_i$ for different sets of parameters $N$ and $\kappa$. The markers represent the simulation results, while the lines the fitted curves.

Asymptote for $k = -1.1$

Asymptote for $k = -1.5$

Asymptote for $k = -1.7$

Figure 3: $R_{\min}$ as a function of $N$, comparison of different values of $\kappa$.

The value of $T$ has also been evaluated using a time dependent variation of $\kappa$. The results show that the effect of $\kappa$ on the trapping is mainly given by its value at the beginning of the trapping, i.e., when the islands are growing close to the origin. In addition, using a non-linear tune variation where the slope of the tune curve is zero at the resonance crossing allowed to confirm our argument of adiabaticity, as it is clearly observed that the trapping fraction is increased in such a case. Such a phenomenon was already reported in Ref. [4]

**EMITTANCE EVOLUTION**

After the splitting, the emittances of the core $\epsilon_{\text{core}}$ and of the beamlets $\epsilon_{\text{beamlets}}$ are different and are reduced compared to $\epsilon_i$. The value of the resulting $\epsilon_{\text{core}}$ depends on two counteracting effects: the reduction due to the trapping of an important part of the initial intensity in the islands and the growth due to particles crossing the separatrix without being trapped. The $\epsilon_{\text{beamlets}}$ depends on the topology of the phase space, the density of the beam located in the islands can be varied if the islands are squeezed and dilated by the non-linearities.

The values of $\epsilon_{\text{beamlets}}$ and $\epsilon_{\text{core}}$ have been computed from the simulations and the ratios to $\epsilon_i$ have been computed. Fits like $A(N,\kappa)\epsilon_i^{-B(N,\kappa)} + C(N,\kappa)$ for the core emittance ratio and of the form $A(N,\kappa)\epsilon_i^{-B(N,\kappa)} + C(N,\kappa)$ for the beamlets emittance ratio.
For the island emittance ratio have shown good agreement with the results at least for $\epsilon_i$ not too small. The functional form for $\epsilon_{\text{beamlets}}$ features $\hat{B} > 0$. In the case of small $\epsilon_i$ the trapping will be very small, due to the presence of the no-trapping zone of size $R_{\text{min}}$, and large emittance growth is observed. Fig. 4 (upper) shows one case. Results for different sets of $\kappa$ and $N$ shows that the core emittance ratio reaches less than 20% for large enough $\epsilon_i$. For large values of $|\kappa|$ the island emittance ratio also reaches a value close to 20% but for small $|\kappa|$ the resulting emittance ratio can be close to 50% even for large initial emittances. The

For large $N$ the trapping will be very small, due to the presence of the no-trapping zone of size $R_{\text{min}}$, and large emittance growth is observed. Fig. 4 (upper) shows one case. Results for different sets of $\kappa$ and $N$ shows that the core emittance ratio reaches less than 20% for large enough $\epsilon_i$. For large values of $|\kappa|$ the island emittance ratio also reaches a value close to 20% but for small $|\kappa|$ the resulting emittance ratio can be close to 50% even for large initial emittances. The

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Impact of spherical nonlinear tune variation on the emittance ratios has also been analysed. Fig 4 (lower) shows the results where the slope of the tune curve is zero at the resonance crossing. This case of the core seems to feature a smaller final emittance independently on $\epsilon_i$. On the other hand, the beamlets feature a smaller $\epsilon_{\text{beamlets}}$ only when $\epsilon_i$ is not too large.

Simulations where a time variation of $\kappa$ is applied led to interesting conclusions. The results show that it allows to improve both the trapping fraction (which is found to be better for smaller $|\kappa|$) as well as the emittance ratios. The time variation of $\kappa$ was linear from $-1.1$ to $-1.9$. Results for one case are shown in Fig. 5. The trapping fraction reaches a value close to the static case $\kappa = -1.1$ while the emittance ratios are lowered to values corresponding to cases in-between $\kappa = -1.1$ and $-1.9$.

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\section*{References}

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