A GENERAL PERTURBATION THEORY FOR CAVITY MODE FIELD PATTERNS

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Abstract

The electric and magnetic field patterns of all modes in a cavity each form a complete set of eigenfunctions with the square of the mode angular frequency serving as the corresponding eigenvalue. Slater's theorem provides a formula for predicting the first-order shift in the eigenvalue when the cavity surface is deformed slightly. A similar formula for predicting the shift in the eigenfunction (*i.e.*the field pattern) is derived from first principles. With this formula, it is possible to apply perturbation theory to find higher-order corrections to both the frequencies and the field patterns of a deformed cavity.

INTRODUCTION

Given a known cavity geometry, the electric and magnetic field pattern of the eigenmodes may be numerically solved by various computer codes. If the cavity geometry is perturbed slightly, Slater's theorem may be used to calculate the first-order shift $\delta\omega$ in the angular eigenfrequency ω:

$$\frac{\delta\omega}{\omega} = \frac{\int \left(\mu H^2 - \epsilon E^2\right) \mathrm{d}\tau}{4U} \quad (1)$$

where H and E are respectively the magnetic and electric fields of the unperturbed cavity mode; μ is the permeability; ϵ is the permittivity; and U is the total energy stored in the unperturbed cavity mode. The integration occurs only over the perturbed volume of the cavity space.

Eq. 1 allows us to calculate the new eigenfrequency of the perturbed cavity without needing to bother with the time and labor of numerically solving for the field pattern of the new eigenmode. It would be helpful to have similar formulas for the field patterns. Once the perturbed field patterns are known, the change in any cavity parameter or figure of merit may be calculated. Fortunately, such formulas can be written by using perturbation theory to solve for the new field patterns of the perturbed cavity modes. The method outlined here for doing this is a minor extension of Slater's derivation of his theorem[1].

DEFINITIONS

The cavity volume is a region contained by a closed surface which may be divided into regions having only one of two possible boundary conditions: electric walls which have $\mathbf{E}$ only normal to the surface EW while $\mathbf{H}$ has only tangential components; and magnetic walls which have $\mathbf{H}$ only normal to the surface MW while $\mathbf{E}$ has only tangential components.

The infinite set of electric field patterns for the cavity eigenmodes form a complete set by which any divergence-free vector field $\mathbf{F}$ may be uniquely described:

$$\mathbf{F} = \sum_i e_i \mathbf{E}_i. \quad (2)$$

The infinite set of magnetic field patterns for the cavity eignemodes form their own complete set by which we could expand the arbitrary vector field $\mathbf{F}$:

$$\mathbf{F} = \sum_i h_i \mathbf{H}_i. \quad (3)$$

The field patterns satisfy the following orthogonality relationships:

$$\frac{\epsilon}{2U_i} \int \mathbf{E}_i \cdot \mathbf{E}_j \mathrm{d}\tau = \delta_{ij} \quad (4)$$

$$\frac{\mu}{2U_i} \int \mathbf{H}_i \cdot \mathbf{H}_j \mathrm{d}\tau = \delta_{ij} \quad (5)$$

where the integration takes place over the entire cavity volume. The field patterns have zero divergence and we assume their curls are proportional to each other:

$$\nabla \times \mathbf{E}_i = k_i Z_0 \mathbf{H}_i \quad (6)$$

$$\nabla \times \mathbf{H}_i = \frac{k_i}{Z_0} \mathbf{H}_i \quad (7)$$

$$k_i = \frac{\omega_i}{c} \quad (8)$$

$$c = \frac{1}{\sqrt{\epsilon\mu}} \quad (9)$$

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}}. \quad (10)$$

MODELING THE CAVITY

Maxwell's equations allow us to predict, given known boundary conditions, the distribution of $\mathbf{E}$ and $\mathbf{H}$ in space and their evolution with respect to time. When $\mathbf{E}$ and $\mathbf{H}$ are the fields contained in a cavity, Eq. 2 and Eq. 3 are particularly useful:

$$\mathbf{E} = \sum_i e_i \mathbf{E}_i \quad (11)$$

$$e_i = \frac{\epsilon}{2U_i} \int \mathbf{E}_i \cdot \mathbf{E} \mathrm{d}\tau \quad (12)$$

$$\mathbf{H} = \sum_i h_i \mathbf{H}_i \quad (13)$$

$$h_i = \frac{\mu}{2U_i} \int \mathbf{H}_i \cdot \mathbf{H} \mathrm{d}\tau. \quad (14)$$

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The $\mathbf{E}_i$ and $\mathbf{H}_i$ are stationary; i.e. they do not change with time. If the values of all e_i and h_i are known for all times t, then we know $\mathbf{E}(t)$ and $\mathbf{H}(t)$ via Eq. 11 and Eq. 13. If we know the values of $\mathbf{E}$ and $\mathbf{H}$ at all points in the cavity at $t = 0$, then the initial conditions may be calculated by performing the integrations in Eq. 12 and Eq. 14. The time evolution of the expansion coefficients is then produced by solving Maxwell's equations. The time dependence of $\mathbf{E}$ and $\mathbf{H}$ is completely contained in the time dependence of the expansion coefficients e_i and h_i.

If the cavity is unperturbed, then the time dependence of the expansion coefficients is trivial:

$$e_i = e_i(0)\, \mathrm{e}^{\mathrm{i}\,\omega_i t} \tag{15}$$

$$h_i = h_i(0)\, \mathrm{e}^{\mathrm{i}\,\omega_i t}\,. \tag{16}$$

If the cavity shape is slightly perturbed, then the $\mathbf{E}_i$ and $\mathbf{H}_i$ are no longer eigenmodes, and Eq. 15 and Eq. 16 are no longer solutions. Eq. 11 and Eq. 13 are still valid- the $\mathbf{E}_i$ and $\mathbf{H}_i$ are still a complete set-but the solution is complicated by the fact that the differential equations determining the dynamics of the expansion coefficients are all linearly coupled to each other. Only special linear combinations of the expansion coefficients will have the simple time dependence of Eq. 15 and Eq. 16:

$$\widetilde{e}_i = e_i + \sum_{j\neq i} a_j e_j \tag{17}$$

$$\widetilde{h}_i = h_i + \sum_{j\neq i} b_j h_j. \tag{18}$$

It follows that the stationary field patterns which undergo simple harmonic oscillations are

$$\widetilde{\mathbf{E}}_i = \mathbf{E}_i + \sum_{j\neq i} a_j \mathbf{E}_j \tag{19}$$

$$\widetilde{\mathbf{H}}_i = \mathbf{H}_i + \sum_{j\neq i} b_j \mathbf{H}_j. \tag{20}$$

These are the eigenmodes of the perturbed cavity. The new angular eigenfrequencies, to first order, are determined by Eq. 1:

$$\widetilde{\omega_i} = \omega_i + \delta\omega_i \tag{21}$$

By deriving the general form of Maxwell's equations in terms of the expansion coefficients, not only will we be able to calculate the values of the c_j, but also the higher order corrections to Eq. 21.

MAXWELL'S EQUATIONS

To find the general differential equation which determines the time evolution of the expansion coefficients, we must reformulate Maxwell's equations in terms of them. Beginning with Faraday's law

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{22}$$

we take the dot product of both sides with $\mathbf{H}_i$ and integrate over the cavity volume:

$$\int \mathbf{H}_i \cdot (\nabla \times \mathbf{E})\, \mathrm{d}\tau = -\mu \int \mathbf{H}_i \cdot \frac{\partial \mathbf{H}}{\partial t} \mathrm{d}\tau. \tag{23}$$

Substituting Eq. 13 and Eq. 64 into equation 23 gives Faraday's law in terms of the expansion coefficients:

$$\omega_i e_i + \frac{1}{2U_i} \oint (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\mathbf{a} = -\frac{\mathrm{d}h_i}{\mathrm{d}t}. \tag{24}$$

Next, we take the dot product of $\mathbf{E}_i$ with both sides of Ampere's law:

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \tag{25}$$

$$\int \mathbf{E}_i \cdot (\nabla \times \mathbf{H})\, \mathrm{d}\tau = \epsilon \int \mathbf{E}_i \cdot \frac{\partial \mathbf{E}}{\partial t} \mathrm{d}\tau. \tag{26}$$

Substituting Eq. 11 and Eq. 68 into Eq. 26 gives Ampere's law in terms of the expansion coefficients:

$$\omega_i h_i - \frac{1}{2U_i} \oint (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\mathbf{a} = \frac{\mathrm{d}e_i}{\mathrm{d}t}. \tag{27}$$

Eq. 24 and Eq. 27 are two coupled, first-order differential equations which are decoupled into two independent, second-order differential equations by solving for e_i or h_i in one expression and substituting into the other:

$$\begin{aligned}\frac{\mathrm{d}^2 e_i}{\mathrm{d}t^2} + \omega_i^2 e_i = & -\frac{\omega_i}{2U_i} \oint (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\mathbf{a} \\ & -\frac{1}{2U_i} \frac{\mathrm{d}}{\mathrm{d}t} \oint (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\mathbf{a}\end{aligned} \tag{28}$$

$$\begin{aligned}\frac{\mathrm{d}^2 h_i}{\mathrm{d}t^2} + \omega_i^2 h_i = & +\frac{\omega_i}{2U_i} \oint (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\mathbf{a} \\ & -\frac{1}{2U_i} \frac{\mathrm{d}}{\mathrm{d}t} \oint (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\mathbf{a}.\end{aligned} \tag{29}$$

PERTURBATION THEORY

Now, we would like to describe the fields inside a cavity which has exactly the same shape except for a small, local deformation on the boundary. If we knew the field patterns of the eigenmodes of the new, perturbed system, then any field within the new cavity boundary could be described in terms of this complete set, and Maxwell's equations would take the same form as Eq. 24 and Eq. 27:

$$\widetilde{\omega}_i \widetilde{e}_i + \frac{1}{2\widetilde{U}_i} \oint \left(\mathbf{E} \times \widetilde{\mathbf{H}}_i\right) \cdot \mathrm{d}\widetilde{\mathbf{a}} = -\frac{\mathrm{d}\widetilde{h}_i}{\mathrm{d}t}. \tag{30}$$

$$\widetilde{\omega}_i \widetilde{h}_i - \frac{1}{2\widetilde{U}_i} \oint \left(\widetilde{\mathbf{E}}_i \times \mathbf{H}\right) \cdot \mathrm{d}\widetilde{\mathbf{a}} = \frac{\mathrm{d}\widetilde{e}_i}{\mathrm{d}t} \tag{31}$$

where all surface integrations occur over the perturbed geometry. In this case, the surface integrals in Eq. 30 and

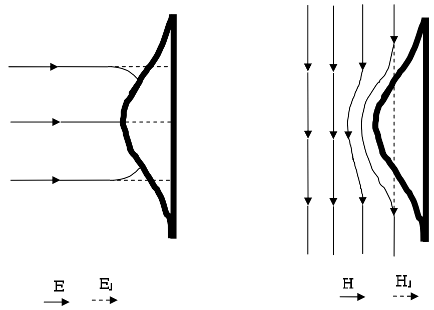

Figure 1: Along the perturbed electric wall, both the perturbed fields and the unperturbed fields are non-zero; however, the perturbed fields are zero along the unperturbed geometry.

Eq. 31 are all zero and the perturbed expansion coefficients exhibit simple harmonic motion:

$$\widetilde{e}_i = \widetilde{e}_i(0)\, \mathrm{e}^{\,\mathrm{i}\,\widetilde{\omega}_i t} \tag{32}$$

$$\widetilde{h}_i = \widetilde{h}_i(0)\, \mathrm{e}^{\,\mathrm{i}\,\widetilde{\omega}_i t}\,. \tag{33}$$

While this solution is simple, it requires numerically solving for the perturbed field patterns, and we would like to avoid doing this extra work. We choose instead to analyze the fields contained within the perturbed boundary using the complete set of eigenmodes of the unperturbed cavity. Maxwell's Equations now take the form

$$\omega_i e_i + \frac{1}{2U_i} \oint (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\widetilde{\mathbf{a}} = -\frac{\mathrm{d}h_i}{\mathrm{d}t}, \tag{34}$$

$$\omega_i h_i - \frac{1}{2U_i} \oint (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\widetilde{\mathbf{a}} = \frac{\mathrm{d}e_i}{\mathrm{d}t} \tag{35}$$

and produce the same differential equations as Eq. 28 and Eq. 29 but with the surface integration performed over the perturbed geometry $\widetilde{a}$. When the cavity is unperturbed, the right sides of Eq. 28 and Eq. 29 are zero and each differential equation is independent of the rest. The i^{th} unperturbed cavity mode oscillates with angular frequency ω_i. When the cavity geometry is perturbed, the right sides of Eq. 28 and Eq. 29 are generally finite and the differential equations are now linearly coupled to each other. By evaluating the surface integrals over the perturbed geometry, the coupling coefficients are then known and the new eigenvectors and new eigenvalues may be found. The surface integrals can be simplified by noting that the total perturbed fields, $\mathbf{E}$ and $\mathbf{H}$, will be equal to that of the unperturbed fields for regions far from the perturbation; therefore, the contribution to the surface integrals is zero. The surface integrals only need to be evaluated at places where the perturbed fields are different from the unperturbed ones-i.e at the locations of the perturbation; see Fig. 1.

When a magnetic wall is perturbed, the second integral on the right side of Eq. 28 is zero since $\mathbf{n}\times\mathbf{H} = 0$ along the perturbed magnetic wall of $\widetilde{a}$. The remaining integration to perform is

$$\int_{\widetilde{\mathrm{MW}}} (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\widetilde{\mathbf{a}} = \int_{\delta\widetilde{\mathrm{MW}}} (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\widetilde{\mathbf{a}} \tag{36}$$

where $\delta\widetilde{\mathrm{MW}}$ is the area on the perturbed surface that deviates from the unperturbed magnetic wall. The integral can be more easily evaluated by including the unperturbed surface into the region of integration. This is permissible, since, at this location, $\mathbf{E} = 0$ along the unperturbed magnetic wall, the contribution to the total surface integral is zero:

$$\int_{\delta\widetilde{\mathrm{MW}}} (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\widetilde{\mathbf{a}} = -\oint_{\delta\widetilde{\mathrm{MW}}} (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\widetilde{\mathbf{a}}. \tag{37}$$

This allows us to use the divergence theorem to evaluate the left side of equation 37:

$$\oint_{\delta\widetilde{\mathrm{MW}}} (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\widetilde{\mathbf{a}} = \int_{\delta\widetilde{\mathrm{MW}}} \nabla \cdot (\mathbf{E} \times \mathbf{H}_i)\, \mathrm{d}\widetilde{\tau}. \tag{38}$$

Substituting equation 11 into the right side of equation 38 gives

$$\int_{\delta\widetilde{\mathrm{MW}}} \nabla \cdot (\mathbf{E} \times \mathbf{H}_i)\, \mathrm{d}\widetilde{\tau} = \sum_j e_j \int_{\delta\widetilde{\mathrm{MW}}} \nabla \cdot (\mathbf{E}_j \times \mathbf{H}_i)\, \mathrm{d}\widetilde{\tau}. \tag{39}$$

The right side of equation 39 can be further evaluated by noting that

$$\nabla \cdot (\mathbf{E}_j \times \mathbf{H}_i) = \mathbf{H}_i \cdot (\nabla \times \mathbf{E}_j) - \mathbf{E}_j \cdot (\nabla \times \mathbf{H}_i)\,. \tag{40}$$

Upon substituting Eq. 6 and Eq. 7 into the right side of Eq. 40 we have

$$\int_{\delta\widetilde{\mathrm{MW}}} \nabla \cdot (\mathbf{E} \times \mathbf{H}_i)\, \mathrm{d}\widetilde{\tau} = \sum_j e_j \int_{\delta\widetilde{\mathrm{MW}}} (\omega_j \mu \mathbf{H}_i \cdot \mathbf{H}_j - \omega_i \epsilon \mathbf{E}_j \cdot \mathbf{E}_i)\, \mathrm{d}\widetilde{\tau}. \tag{41}$$

We define a new matrix

$$M_{ij} \equiv \int_{\delta\widetilde{\mathrm{MW}}} (\omega_j \mu \mathbf{H}_i \cdot \mathbf{H}_j - \omega_i \epsilon \mathbf{E}_j \cdot \mathbf{E}_i)\, \mathrm{d}\widetilde{\tau} \tag{42}$$

and have finally

$$\int_{\delta\widetilde{\mathrm{MW}}} (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\widetilde{\mathbf{a}} = -M_{ij} e_j. \tag{43}$$

For perturbations along electric walls, it is similarly convenient to define a new matrix

$$E_{ij} \equiv \int_{\delta\widetilde{\mathrm{EW}}} (\omega_i \mu \mathbf{H}_j \cdot \mathbf{H}_i - \omega_j \epsilon \mathbf{E}_i \cdot \mathbf{E}_j)\, \mathrm{d}\widetilde{\tau} \tag{44}$$

because it will be found that the second integral on the right side of Eq. 29 is zero since $\mathbf{n} \times \mathbf{E} = 0$ along the perturbed electric wall of $\widetilde{a}$. It is similarly found that the remaining integration may be written as

$$\int_{\delta\widetilde{\mathrm{EW}}} (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\widetilde{\mathbf{a}} = -E_{ij} h_j. \tag{45}$$

Magnetic Walls

For the case that perturbations occur only along electric walls, the differential equations which determines the time evolution of the expansion coefficients are

$$\frac{\mathrm{d}^2 e_i}{\mathrm{d}t^2} + \omega_i^2 e_i = \sum_j \left(\frac{\omega_i}{2U_i}\right) M_{ij} e_j \tag{46}$$

$$\frac{\mathrm{d}^2 h_i}{\mathrm{d}t^2} + \omega_i^2 h_i = \sum_j \left(\frac{\omega_j}{2U_i}\right) M_{ij} h_j . \tag{47}$$

Electric Walls

For the case that perturbations occur only along magnetic walls, the differential equations which determines the time evolution of the expansion coefficients are

$$\frac{\mathrm{d}^2 e_i}{\mathrm{d}t^2} + \omega_i^2 e_i = -\sum_j \left(\frac{\omega_j}{2U_i}\right) E_{ij} e_j \tag{48}$$

$$\frac{\mathrm{d}^2 h_i}{\mathrm{d}t^2} + \omega_i^2 h_i = -\sum_j \left(\frac{\omega_i}{2U_i}\right) E_{ij} h_j . \tag{49}$$

Electric and Magnetic Walls

For the case that perturbations occur along both electric and magnetic walls, the differential equations which determines the time evolution of the expansion coefficients are

$$\frac{\mathrm{d}^2 e_i}{\mathrm{d}t^2} + \omega_i^2 e_i = \sum_j \Bigg[\left(\frac{\omega_i}{2U_i}\right) M_{ij} - \left(\frac{\omega_j}{2U_i}\right) E_{ij} + \sum_k \left(\frac{E_{ik} M_{kj}}{4U_i U_k}\right)\Bigg] e_j . \tag{50}$$

$$\frac{\mathrm{d}^2 h_i}{\mathrm{d}t^2} + \omega_i^2 h_i = \sum_j \Bigg[\left(\frac{\omega_j}{2U_i}\right) M_{ij} - \left(\frac{\omega_i}{2U_i}\right) E_{ij} + \sum_k \left(\frac{M_{ik} E_{kj}}{4U_i U_k}\right)\Bigg] h_j . \tag{51}$$

First-Order Perturbations

For an unperturbed cavity, the differential equation for the i^{th} electric expansion coefficient is

$$\frac{\mathrm{d}^2 e_i}{\mathrm{d}t^2} = -\sum_j \omega_i^2 e_j \delta_{ij} . \tag{52}$$

When the cavity is perturbed, these differential equations are linearly coupled since the $\mathbf{E}_i$ are no longer eigenmodes for the new perturbed geometry. In this case, Eq. 52 becomes

$$\frac{\mathrm{d}^2 e_i}{\mathrm{d}t^2} = \sum_j A_{ij} e_j . \tag{53}$$

The matrix A_{ij} may be factored as

$$A_{ij} = -\omega_i^2 \delta_{ij} + \alpha_{ij} \tag{54}$$

where the components of α_{ij} are found by inspecting the right sides of Eq. 46, Eq. 48, or Eq. 50 for the case that the perturbation is along a magnetic wall, an electric wall, or both electric and magnetic walls respectively. When solving for the new eigenvalues and eigenvectors, we assume that the perturbation is small enough that only terms to first-order in the components of α_{ij} need to be retained. According to this first-order approximation, the new i^{th} eigenvalue is

$$\widetilde{\omega}_i^2 = \omega_i^2 + \alpha_{ii} . \tag{55}$$

This first-order correction to the angular frequency results in the same formula as in Slater's theorem. The new eigenmode electric field pattern for the perturbed cavity is primarily that of the unperturbed cavity with an additional, small correction from all of the cavity higher-order and lower-order modes as expressed in Eq. 19. To first-order, the formula for expansion coefficient a_j is

$$a_j = \frac{\alpha_{ij}}{(\omega_i^2 - \omega_j^2)} . \tag{56}$$

Similarly, the differential equation for the i^{th} cavity mode magnetic field may be written as

$$\frac{\mathrm{d}^2 h_i}{\mathrm{d}t^2} = \sum_j B_{ij} h_j \tag{57}$$

where the matrix B_{ij} may be factored as

$$B_{ij} = -\omega_i^2 \delta_{ij} + \beta_{ij} \tag{58}$$

and the components of β_{ij} are found by inspecting the right sides of Eq. 47, Eq. 49, or Eq. 51 for the case that the perturbation is along a magnetic wall, an electric wall, or both electric and magnetic walls respectively. The expression for the first order correction to the eigenvalue

$$\widetilde{\omega}_i^2 = \omega_i^2 + \beta_{ii} \tag{59}$$

is identical to Eq. 55 and the first-order formula for the expansion coefficients in Eq. 20 is

$$b_j = \frac{\beta_{ij}}{(\omega_i^2 - \omega_j^2)} . \tag{60}$$

DISCUSSION

Eq. 56 and Eq. 60 provide a formula for the first-order correction to the electric and magnetic field patterns; however, higher-order corrections are possible by retaining higher-order terms of the α_{ij} and β_{ij} elements in the algebraic solution for a_j and b_j. Additionally, higher-order corrections to the angular frequency which exceed the accuracy of Slater's theorem may be possible. These higher-order corrections would involve the additional computational complexity of the non-orthogonality of $\mathbf{E}_i$ and $\mathbf{H}_i$ in the region of the perturbed cavity. Eq. 4 and Eq. 5 are no longer true when the volume integration is performed over the perturbed geometry. There is an additional first-order correction to the orthogonality relation whose elements will appear in the final results for orders second or higher.

The first-order derivation involved expansions over an infinite set of cavity modes; however, a real numerical solver will have an upper limit to the higher-order modes which it can faithfully reproduce. This upper limit on the higher-order modes determines the lower limit on the size of the cavity wall deformation, since a perturbation of size λ will require the higher-order mode expansions in Eq. 19 and Eq. 20 to be carried out to at least the frequency $\nu \approx c/\lambda$, where c is the speed of light, for accurate convergence to the perturbed field.

The case of degenerate modes may be handled using the conventional approach of degenerate perturbation theory: any degeneracies or near-degeneracies are identified and diagonalized first, so that the corresponding elements of α_{ij} and βij are zero.

For cavities which are azimuthally symmetric, the mode patterns and frequencies may be numerically calculated with a 2D code. The perturbative formulas for the field patterns have the advantage of being applicable for perturbations which are not azimuthally symmetric while requiring field values from only the azimuthally symmetric cavity. In this way, the use of computationally intensive 3D codes may be avoided.

APPENDIX

Some care is required in finding an expression for the expansion coefficient of the curl of either $\mathbf{E}$ or $\mathbf{H}$. We are often interested in cases where both $\mathbf{E}$ and $\mathbf{H}$ have tangential components along some portion of the cavity boundary, allowing power transfer; however, either $\mathbf{E}_i$ or $\mathbf{H}_i$ will have zero tangential component along the boundary. For this reason, the rate of convergence of the infinite series in Eq. 11 or Eq. 3 will be much lower at the boundaries. For such cases of non-uniform convergence, the limiting process of the infinite expansion does not generally commute with the limits implicit in the derivatives of $\nabla \times \mathbf{E}$ or $\nabla \times \mathbf{H}$: *i.e.* the curl of the infinite expansion will not equal the infinite expansion of the curl. We start by noting that

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}_i) = \mathbf{H}_i \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}_i)\,, \tag{61}$$

$$\int \nabla \cdot (\mathbf{E} \times \mathbf{H}_i)\, \mathrm{d}\tau = \oint (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\mathbf{a}, \tag{62}$$

$$\oint (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\mathbf{a} = \int \mathbf{H}_i \cdot (\nabla \times \mathbf{E})\, \mathrm{d}\tau - \int \mathbf{E} \cdot (\nabla \times \mathbf{H}_i)\, \mathrm{d}\tau. \tag{63}$$

After substituting Eq. 5 and Eq. 7 into Eq. 63, we arrive at an expression for the i^{th} expansion coefficient for the curl of $\mathbf{E}$:

$$\int \mathbf{H}_i \cdot (\nabla \times \mathbf{E})\, \mathrm{d}\tau = \oint (\mathbf{E} \times \mathbf{H}_i) \cdot \mathrm{d}\mathbf{a} + \left(\frac{2U_i k_i}{\epsilon Z_0}\right) e_i. \tag{64}$$

There is an extra term on the right side of this equation which does not appear if one simply takes the curl of both sides of equation 11. By including the extra term on the right of equation 64, we can now use these field patterns to describe more general cases.

A similar term is added to expansion for the curl of $\mathbf{H}$:

$$\nabla \cdot (\mathbf{E}_i \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}_i) - \mathbf{E}_i \cdot (\nabla \times \mathbf{H}) \tag{65}$$

$$\int \nabla \cdot (\mathbf{E}_i \times \mathbf{H})\, \mathrm{d}\tau = \oint (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\mathbf{a} \tag{66}$$

$$\oint (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\mathbf{a} = \int \mathbf{H} \cdot (\nabla \times \mathbf{E}_i)\, \mathrm{d}\tau - \int \mathbf{E}_i \cdot (\nabla \times \mathbf{H})\, \mathrm{d}\tau. \tag{67}$$

After substituting Eq. 4 and Eq. 6 into Eq. 67, we arrive at an expression for the i^{th} expansion coefficient for the curl of $\mathbf{H}$:

$$\int \mathbf{E}_i \cdot (\nabla \times \mathbf{H})\, \mathrm{d}\tau = \left(\frac{2U_i k_i Z_0}{\mu}\right) h_i - \oint (\mathbf{E}_i \times \mathbf{H}) \cdot \mathrm{d}\mathbf{a}. \tag{68}$$

There is an extra term on the right side of this equation which does not appear if one simply takes the curl of both sides of equation 13.

REFERENCES

[1] J.C. Slater, *Microwave Electronics*, D. Van Nostrand Company, Inc., 1950, p. 80