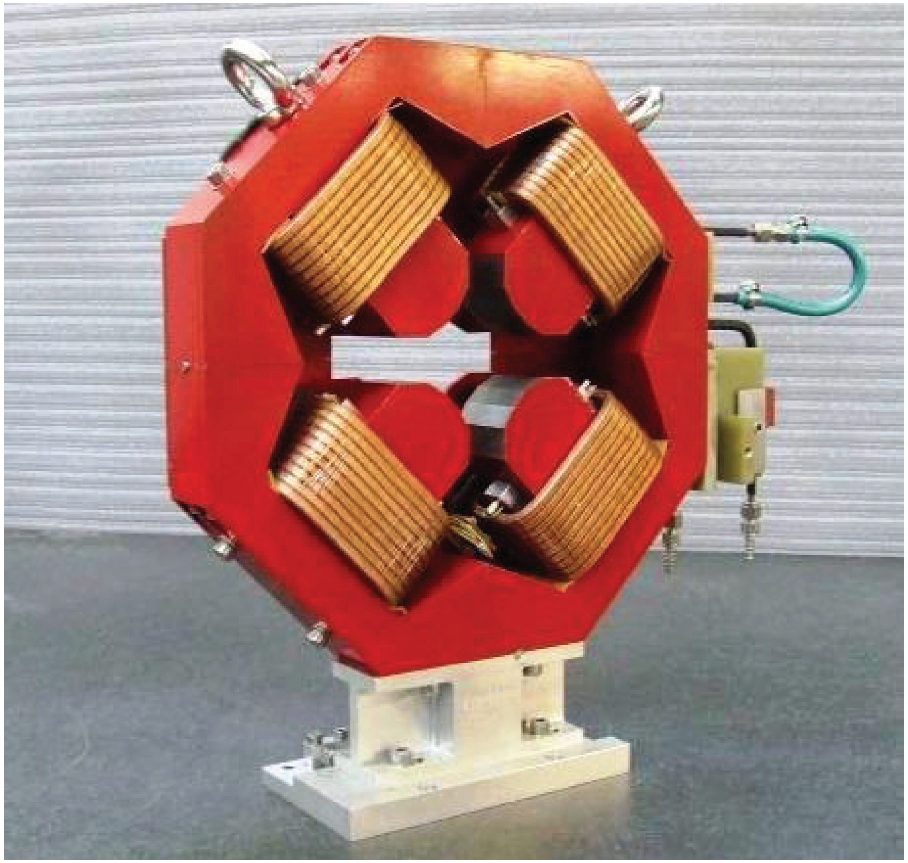
Optimal 3D Quadrupole Shapes

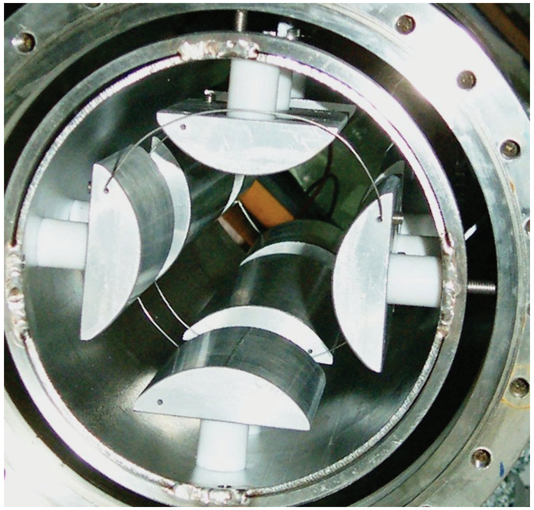
Rick Baartman, TRIUMF

Oct. 2, 2013

Introduction



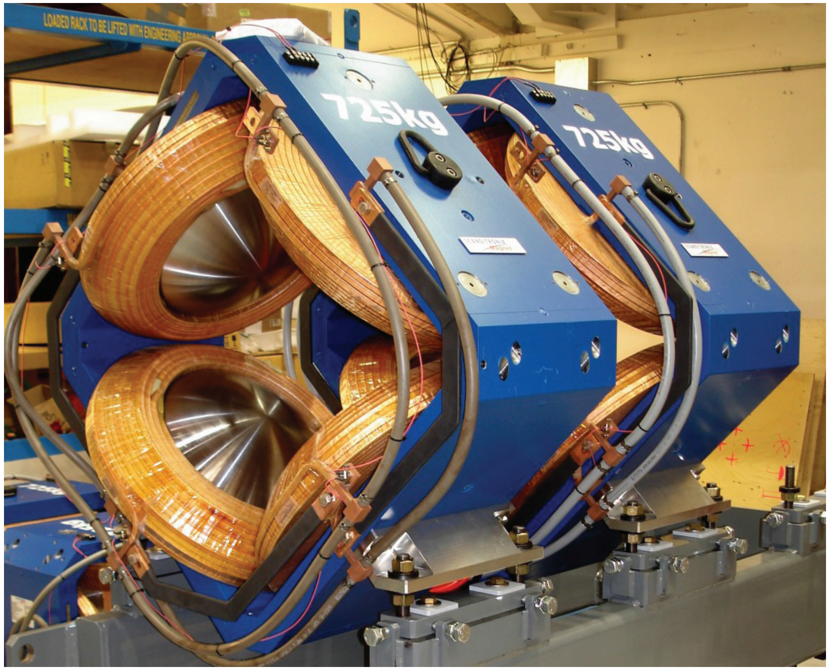
Square ends versus round: which is better?

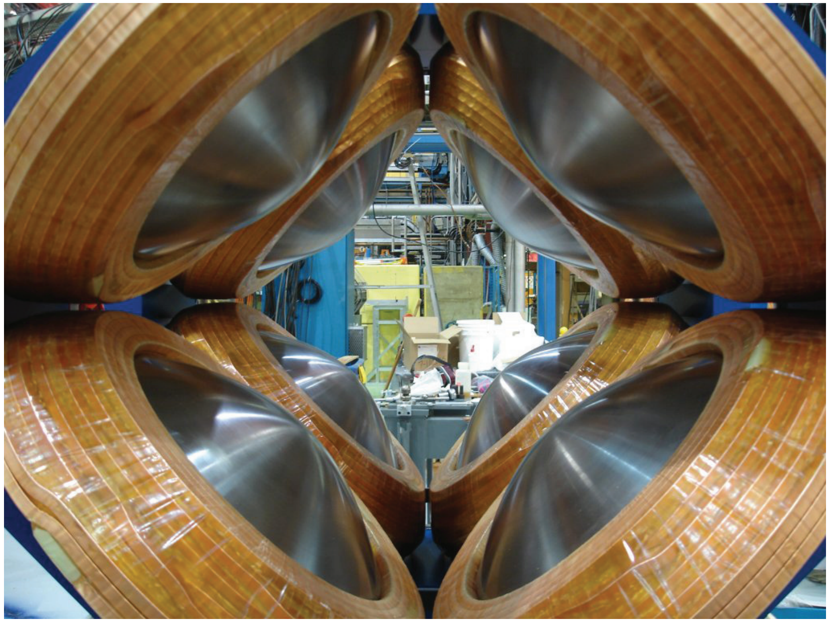


"Normal" quads
are those that are constant cross-section (cylinders), more-or-less truncated at each end. "Short" quads have length approximately same as aperture.

There is a pervasive myth that short quads, being "all fringe field", have higher aberrations than "normal" ones. This myth will be debunked. The analysis has implications for any "normal" quad design as well. Implications for the LHC high gradient quads are given as an example.

TRIUMF’s M20 Doublet





Little known Fact

The quadrupole was invented in 1952 by Kitagaki-san.

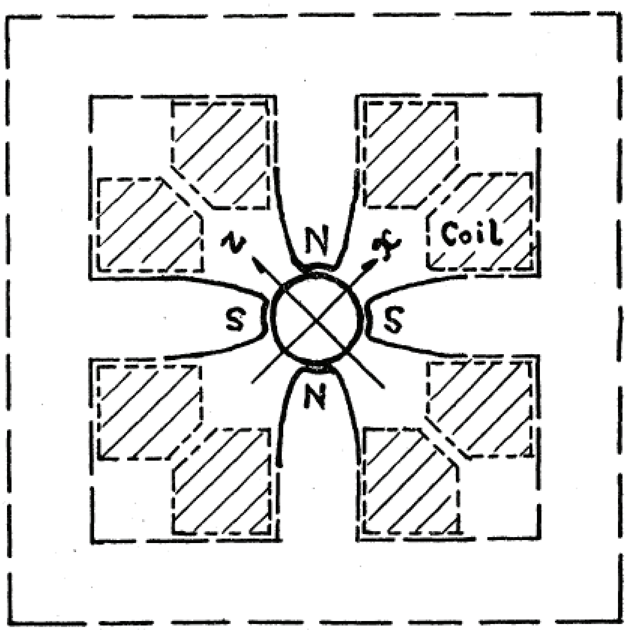


FIG. 2. Quadrupole focusing magnet.

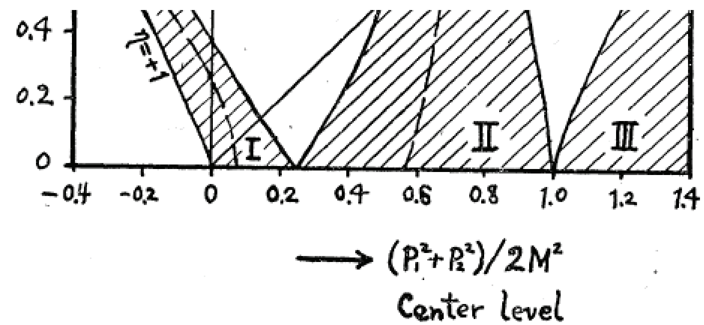


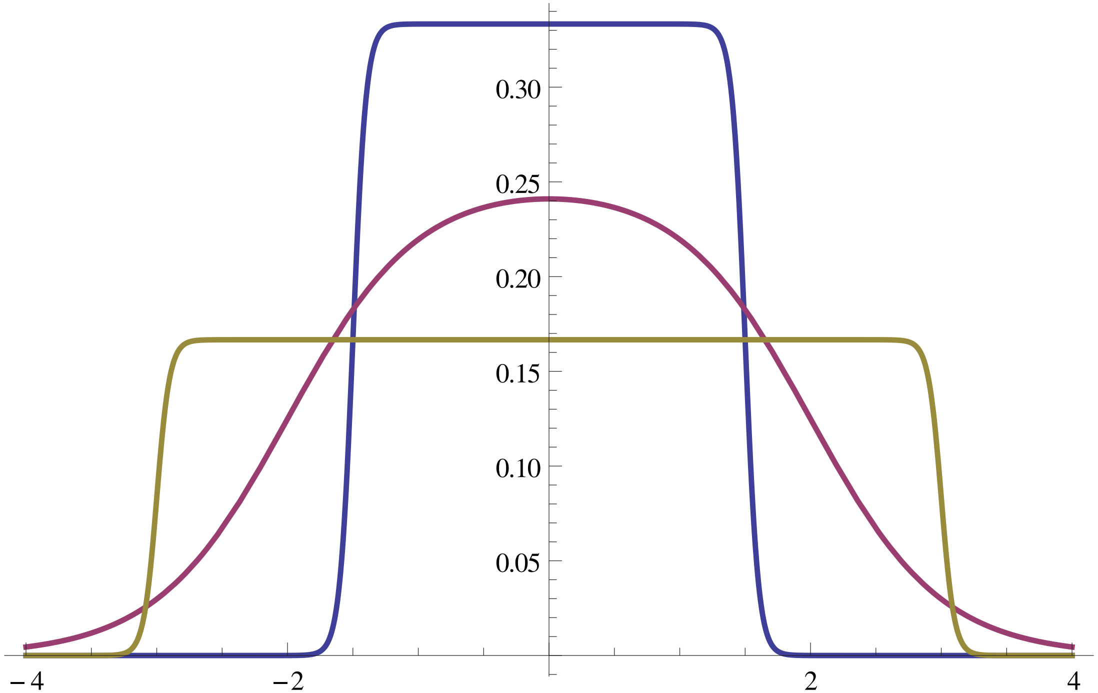
FIG. 3. (a) A general periodic focusing field. (b) Stable regions are shaded.

nantly in the earlier stages of acceleration. For the low energy period, from injection until the synchrotron phase starts, a conventional type of machine with solenoids on the donut may be enough.

These focusing principles are now being tested. The author wishes to express his thanks to Professor M. Kimura for his valuable suggestions and encouragement.

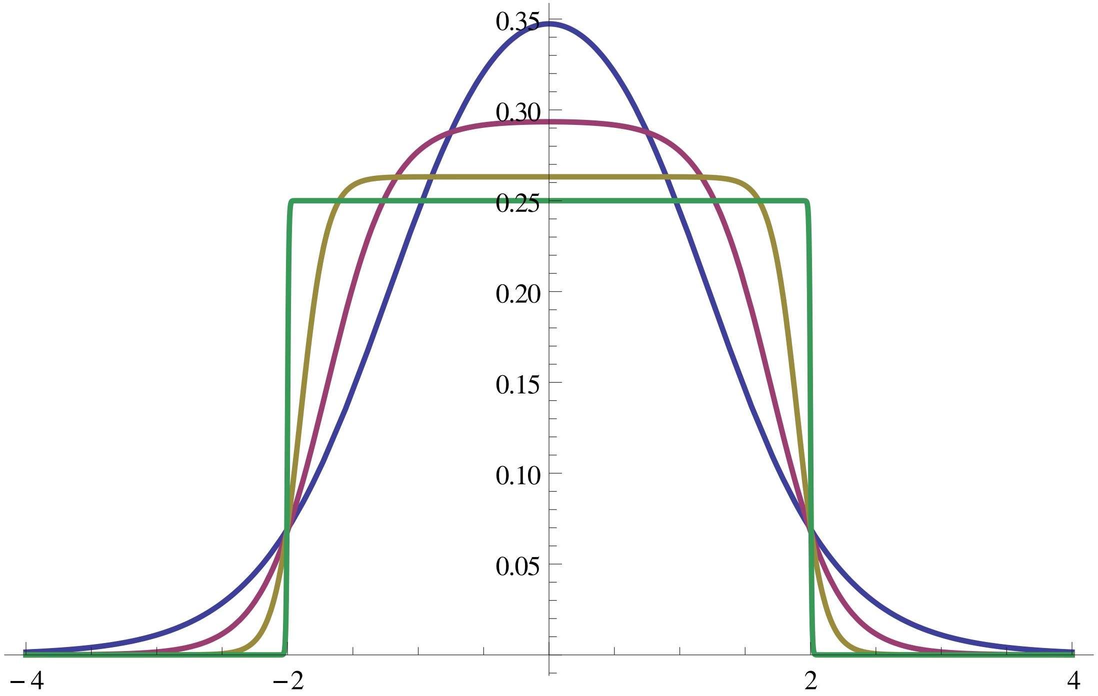
[1] Courant, Livingston, and Snyder, Phys. Rev. **88**, 1190 (1952).
[2] D. C. DePackh, Phys. Rev. **86**, 433 (1952).

Basics – first order



All these strength functions $k(z)$ have same first order effect (same focal length), since they have the same $\int k\,dz$.

Basics – higher order



All these strength functions $k(z)$ have same first order effect and same third order effect, since they have the same $\int k\,dz$ and $\int k^2\,dz$.

But guess which has higher 5^{th} and higher order?

.

Ans: In the green (hard-edged) case, 5^{th} and higher order diverge.

Quad Fields

Can be derived from a potential function that satisfies the Laplace equation. In 2 dimensions, solutions are terms n:

$$V(r,\theta) \sim r^n \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} n\theta$$

We want (only) $n = 2$, since that gives quadratic potential and LINEAR fields.

So $V \sim r^2 \cos 2\theta = x^2 - y^2$ (or $2xy$), equipotentials are hyperbolae, and they give us the linear fields we want, but only in the limit of infinitely long quadrupoles. For finite quads, we would like

$$V(x,y,z) = \frac{k(z)}{2}(x^2 - y^2),$$

but this violates Laplace equation:

$$\nabla^2 V = \frac{k''}{2}(x^2 - y^2).$$

We can fix by correcting for this “leftover”:

$$V(x, y, z) = \frac{k}{2}(x^2 - y^2) - \frac{k''}{24}(x^4 - y^4),$$

and again get a leftover k'''', so correct again, and so on:

$$V(x, y, z) = \frac{k}{2}(x^2 - y^2) - \frac{k''}{24}(x^4 - y^4) + \frac{k''''}{720}(x^6 - y^6) - ...$$

These new terms are not of the form $r^n \cos n\theta$ so it is incorrect to call them octupole, 12-pole, etc. They are usually referred to as “pseudo-multipoles”.

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QUADRUPOLE MAGNET FIELD MEASURING EQUIPMENT AT TRIUMF

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Received 17 March 1976

A Hall plate system with a CAMAC interfaced mini-computer for measuring the magnetic field harmonics of quadrupole magnets is presented. Errors and their minimization are discussed in detail.

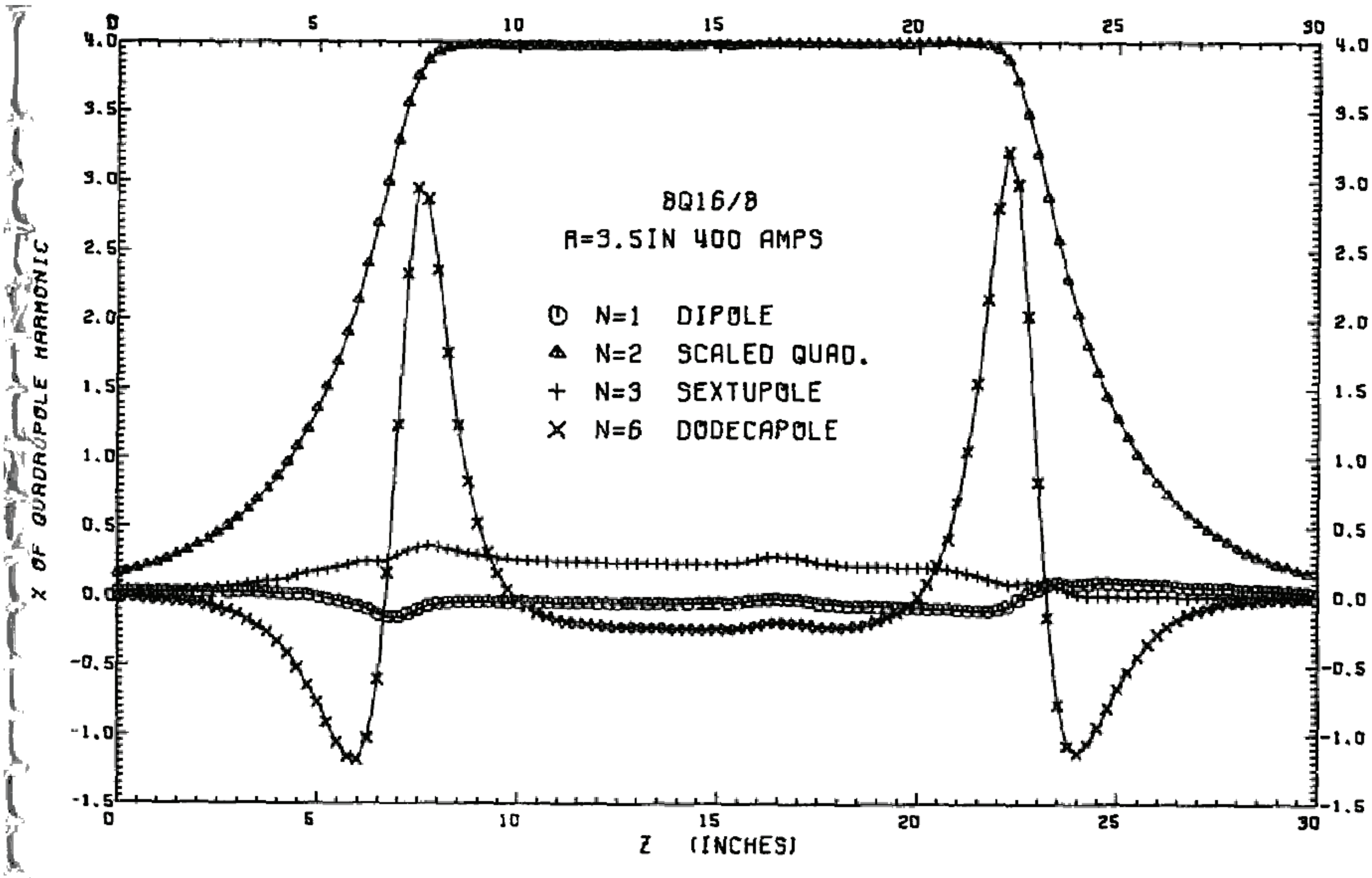
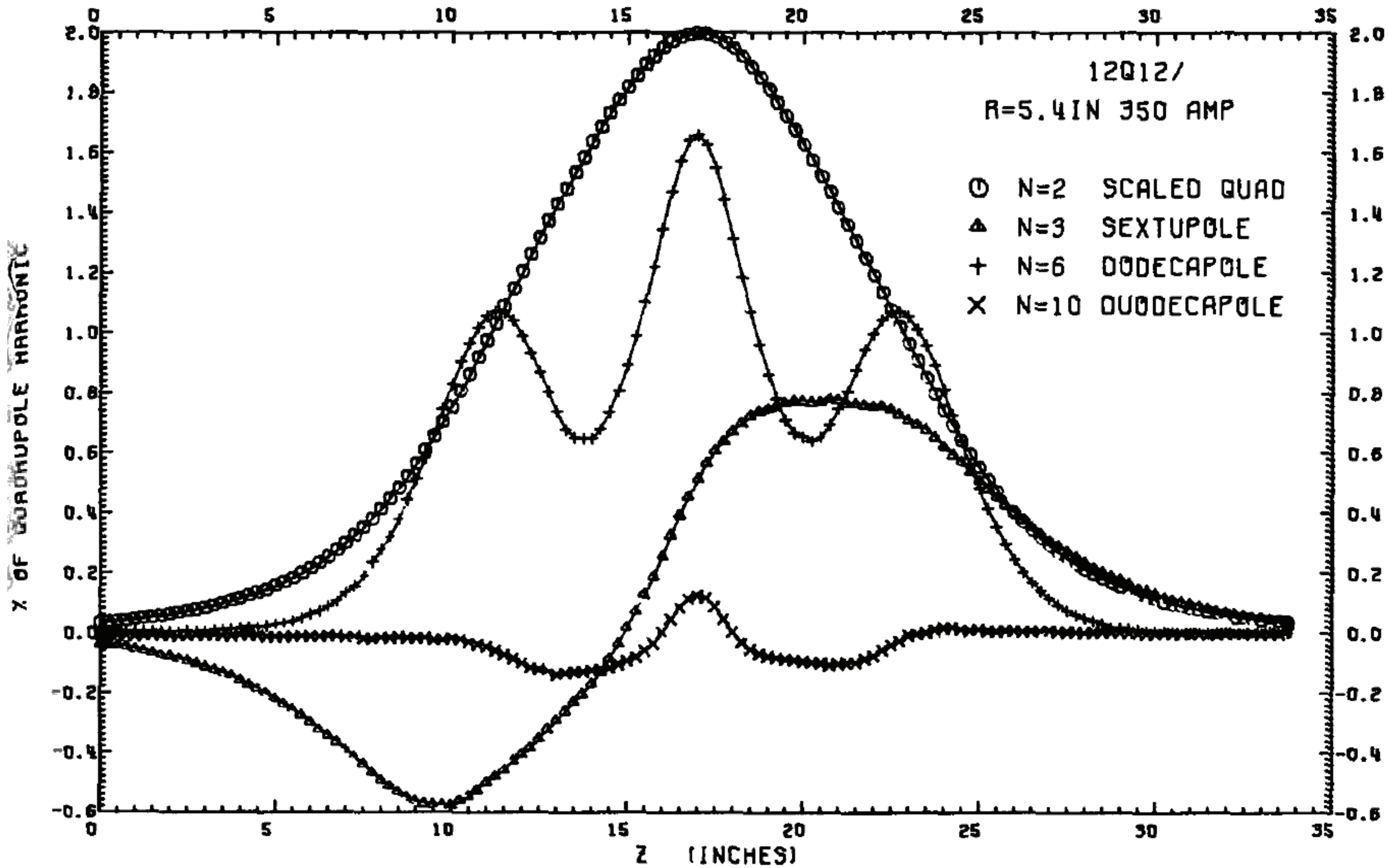
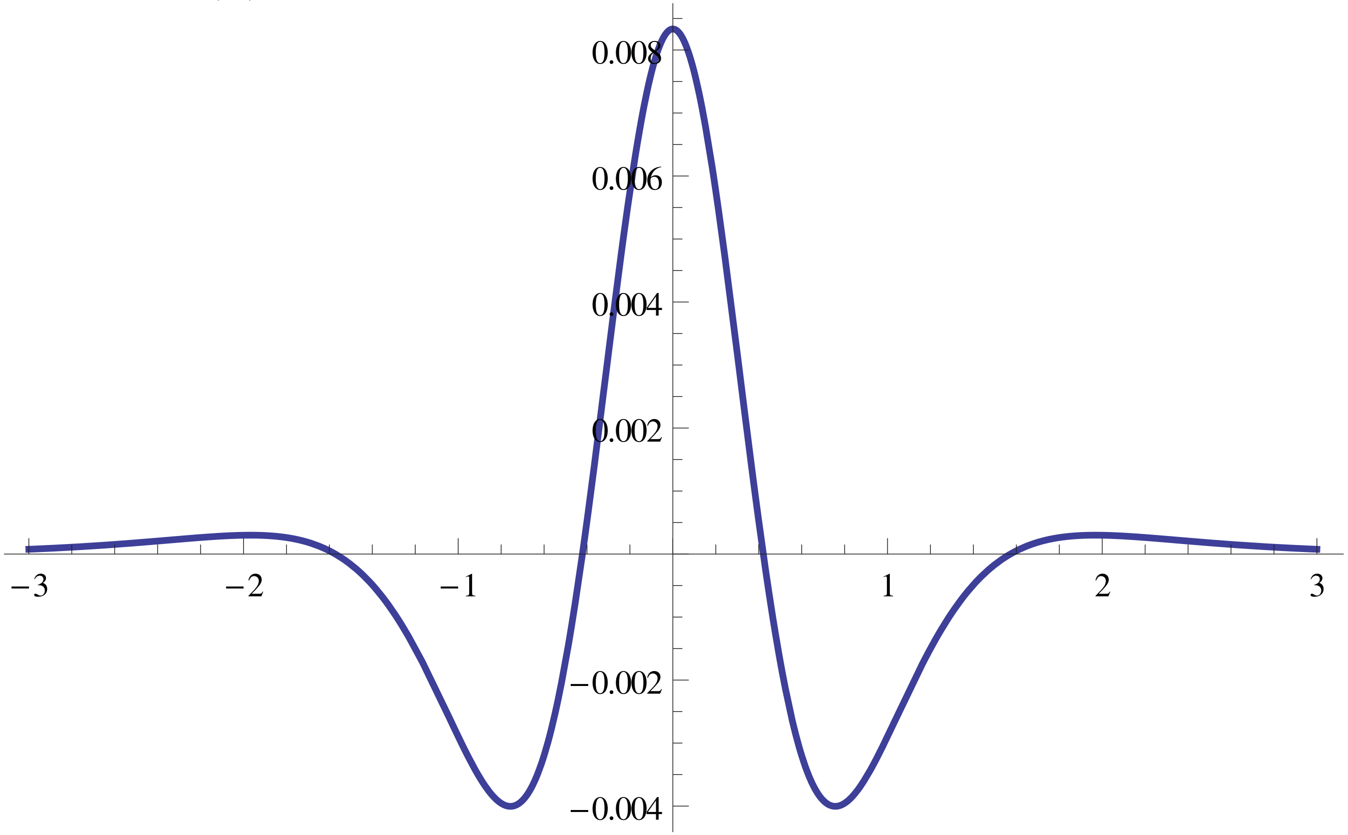


Fig. 9. Field harmonics of 8Q16/8 quadrupole magnet.


12Q12/
R=5.4IN 350 AMP
N=2 SCALED QUAD
N=3 SEXTUPOLE
N=6 DODECAPOLE
N=10 DUODECAPOLE
% OF QUADRUPOLE HARMONIC
Z (INCHES)

$k''''(z)$ This is how the $n = 6$ (12-pole) should look:



Integrated percentage harmonics at 90% of aperture radius[a].

Field harmonic number (n)	Magnet type			
	8Q16/8	8QN16M/7	12Q12/5	4Q19/8
2	100	100	100	100
3	0.24	0.18	0.3	0.2
4	0.15	0.15	0.1	<0.3
5	0.06	0.02	<0.2	<0.2
6	0.11	0.04	1.3	0.4
7–9	<0.05	<0.07	<0.2	<0.4
10	0.16	0.05	0.06	0.6

[a] Note: errors are typically smaller than the noise harmonics ($n = 7$–9).

SCANDITRONIX Magnet	ROTATING COIL MEASUREMENTS	

Scanditronix Magnet Order No:	Description:	Serial No:
C1032	Qpole magnet	1032-10617-0003

Evaluation

- harmonic content in units (1/10000 of the main component)

current [A]	0	40,048	80,052	120,032	160,020	200,015	160,004	120,060	80,028	39,996	
Cn [1E-4]											
n= 1	1772,930	39,531	26,203	21,396	20,613	19,359	19,583	20,338	23,688	33,947	142
2	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000	
3	233,217	5,717	5,025	5,231	4,826	5,401	7,716	10,483	3,924	5,186	46
4	117,839	4,200	0,775	2,563	1,352	1,580	1,957	1,759	2,047	6,303	12
5	199,300	3,899	1,504	1,313	1,935	0,551	0,387	0,937	2,136	3,715	7
6	64,955	31,817	31,457	32,039	32,156	32,312	32,437	30,748	31,443	31,400	4
7	316,258	0,983	1,274	1,337	1,455	0,540	0,098	1,079	0,287	1,960	6
8	190,250	5,676	1,508	1,513	0,839	0,589	1,155	0,725	0,441	2,044	20
9	68,952	4,342	2,218	1,460	0,884	0,572	0,873	0,351	0,176	1,768	17
10	129,244	2,322	0,883	0,940	0,615	0,209	0,642	0,727	0,793	1,321	3
11	121,298	2,709	0,981	0,077	0,459	0,698	0,757	1,156	1,183	1,986	11
12	235,751	4,036	3,531	0,389	1,399	0,396	0,473	1,491	0,627	2,624	11
13	28,373	1,866	1,325	0,283	0,929	0,305	1,434	0,758	1,051	1,925	19
14	56,079	2,385	1,316	1,339	0,783	0,307	0,371	0,882	1,053	4,475	5
15	172,510	3,477	1,783	1,692	0,777	0,491	0,138	0,702	0,590	0,909	11
16	158,155	4,617	1,554	0,879	0,259	0,191	0,188	0,610	1,075	2,714	4
17	273,819	0,934	2,179	0,521	0,888	0,372	0,277	0,416	0,196	1,144	3
18	245,098	5,039	1,088	0,739	0,946	0,883	0,441	0,881	3,997	4,199	5
19	210,403	3,608	1,446	0,697	1,282	0,522	0,754	1,303	2,786	1,460	4
20	139,483	3,353	1,514	0,889	0,158	0,025	0,218	0,317	1,521	2,544	5

The results (with phase angle) for a magnet which has a relatively short length (12″) but wide aperture (12″) are shown in fig. 11. This magnet is unusual in that the poles have circular symmetry about an axis normal to the yoke. The magnetic field as a function of axial distance is therefore peaked instead of being flat-topped; i.e., it is all fringe field. The $n=6$ harmonic is very large for this design (table 2).

32 parts per 10000 at 94mm/156mm or 1.3% at 90% aperture... Is this a problem?

If we are to use the full aperture, it's 2% in both cases. Measurable effect on focusing, and easily avoidable.

Integrated harmonics (Fallacy #1)

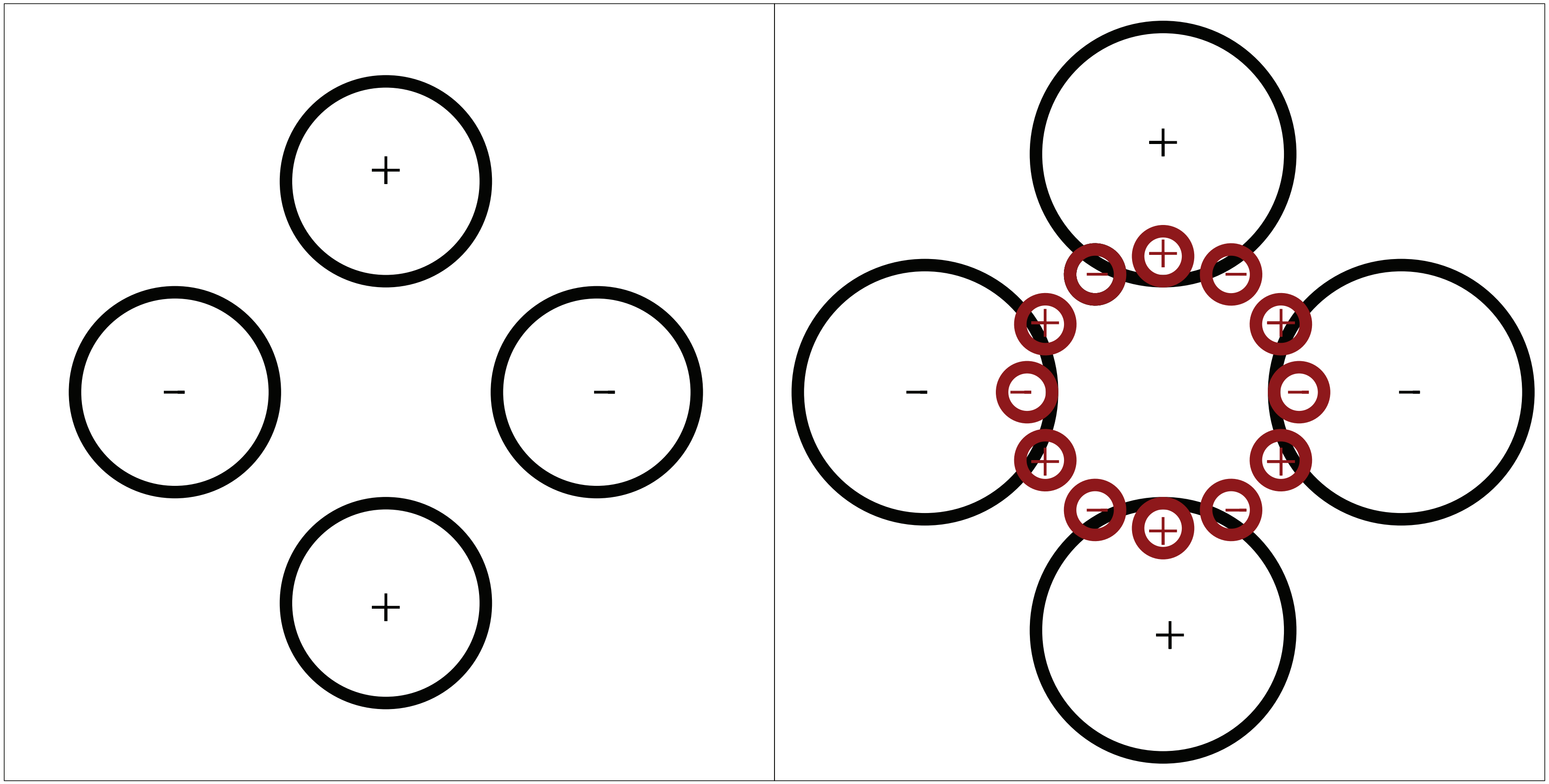
The rotating probe is used in two modes: a short loop to measure local fields, and a long one that extends completely through the quad to measure the integrated harmonic. According to the expansion above, the integrated quad should be exactly linear:

$$\int_{-\infty}^{\infty} V(x,y,z)dz = \int_{-\infty}^{\infty} k(z)dz \left(\frac{x^2-y^2}{2}\right)$$

because all the other terms integrate to zero.

Thus there is no reason to expect a short quad to have larger integrated 12-pole.

Quad with too-narrow poles = = Perfect quad + Duodecapole



Linearity (Fallacy #2)

For long quadrupoles, the extra nonlinear terms are zero where k is constant well away from the entrance or exit. This leads one to believe that long quads with sharp edges are more linear than short quads where k is nowhere constant.

One would be WRONG.

The reason is that aberrations are sensitive to two effects: the size of the derivatives of k, and their extent in the z direction. Aberrations scale with a higher power of the former than the latter.

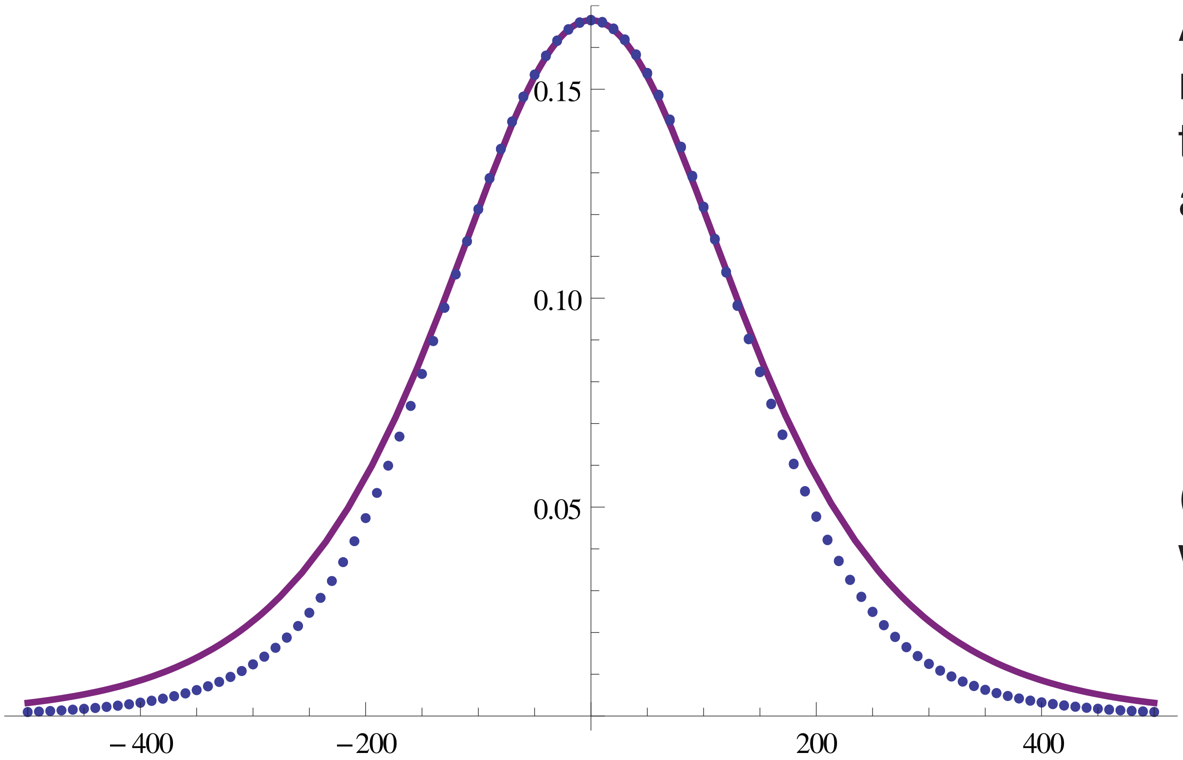
The perfect Quad

So if the hyperbolic equipotentials are not optimum, what is?

It turns out to be possible to find the equipotentials for a “perfect” quad for any choice of $k(z)$. The technique is “analytic continuation”. Explicit formula due to Derevjankin (1971):

$$V(x,y,z) = -\Re\left\{\int_{z+iy}^{z+ix} dt \int_0^t k(\zeta)d\zeta\right\}$$

It satisfies Laplace, and is $k(z)(x^2-y^2)/2$ in the limit of small x and y.



A particularly nice (because the integrals are easy) and quite realistic k is

$$k(z) = \frac{K}{2}\,\mathrm{sech}^2 z$$

(left, compared with 12Q12):

$$V(x,y,z) = \frac{K}{2}\,\Re\left\{-\log[\cos(x-iz)] + \log[\cos(y-iz)]\right\}$$

The properties of V are perhaps more readily apparent if it is written in real form:

$$V(x,y,z) = \frac{K}{4}\log\frac{\cos^2 y\cosh^2 z + \sin^2 y\sinh^2 z}{\cos^2 x\cosh^2 z + \sin^2 x\sinh^2 z}$$

Let $\vec{F} = \nabla V$ denote the magnetic field or electric field as appropriate for magnetic or electrostatic quadrupoles. This has components:

$$\begin{aligned}
F_x &= \frac{K}{2}\frac{\sin 2x}{\cos 2x + \cosh 2z} = \frac{K}{2}\frac{\sin x\cos x}{\cos^2 x + \sinh^2 z} \\
F_y &= -\frac{K}{2}\frac{\sin 2y}{\cos 2y + \cosh 2z} = -\frac{K}{2}\frac{\sin y\cos y}{\cos^2 y + \sinh^2 z} \\
F_z &= \frac{K}{2}\left(-\frac{\sinh 2z}{\cos 2x + \cosh 2z} + \frac{\sinh 2z}{\cos 2y + \cosh 2z}\right)
\end{aligned}$$

These make apparent the fact that the transverse fields are nonlinear; at $z = 0$,

$$F_x \propto \tan x, \ F_y \propto \tan y.$$

Not linear at all! Nevertheless, the integral of the transverse field is exactly linear:

$$\int_{-\infty}^{\infty} F_x dz = Kx,$$

and similarly, $\int F_y dz = -Ky$. The reason for this behaviour is that farther from axis, the transverse field is stronger at $z = 0$, but weaker at the tails (it's 'peakier'). This is clarified in Fig. 1.

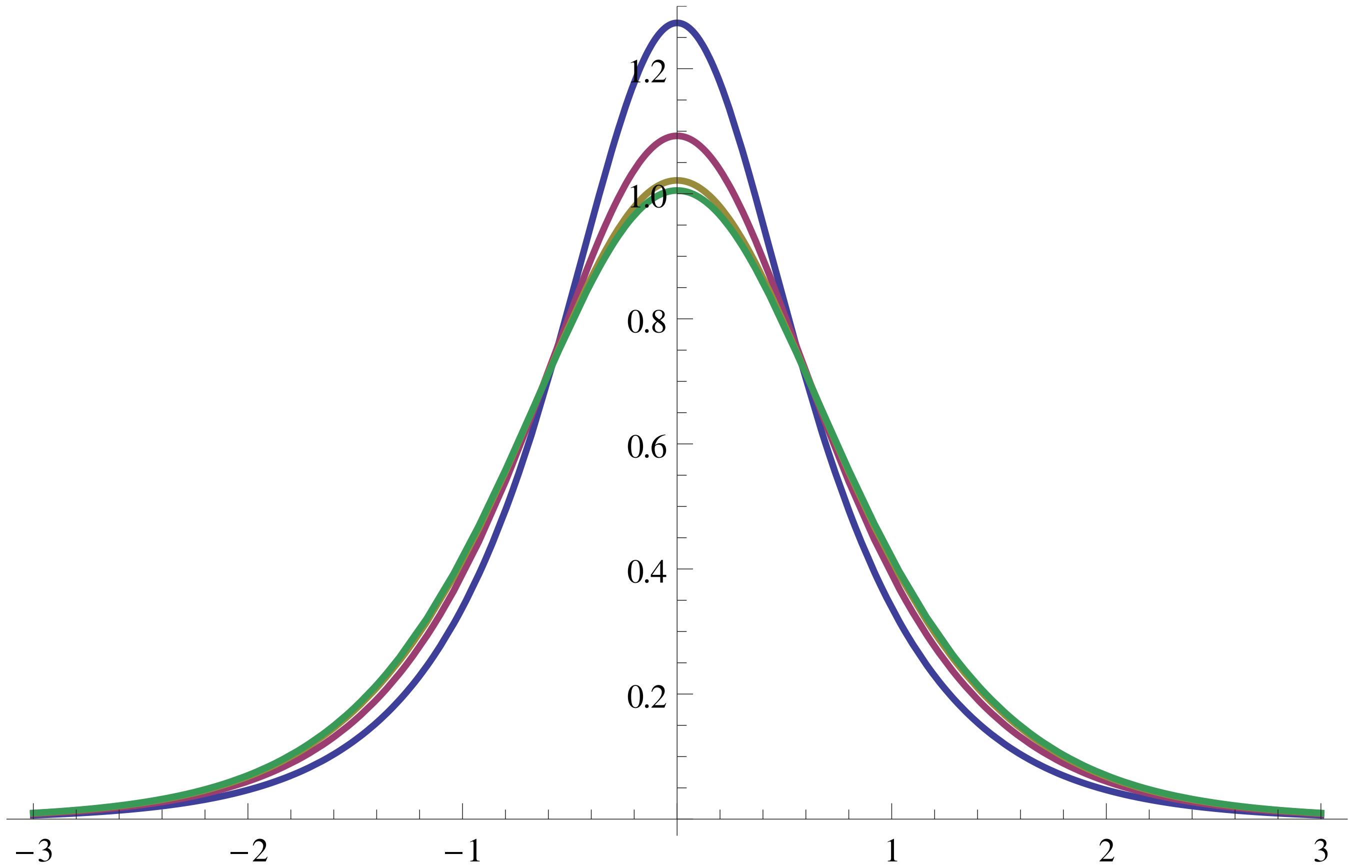
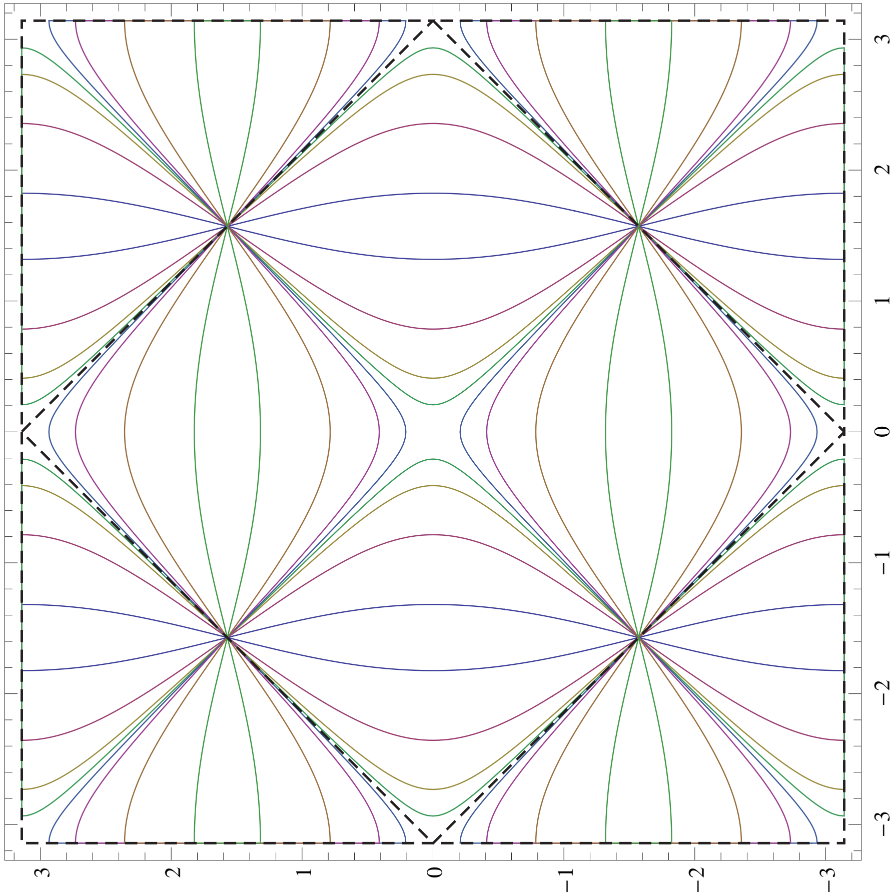
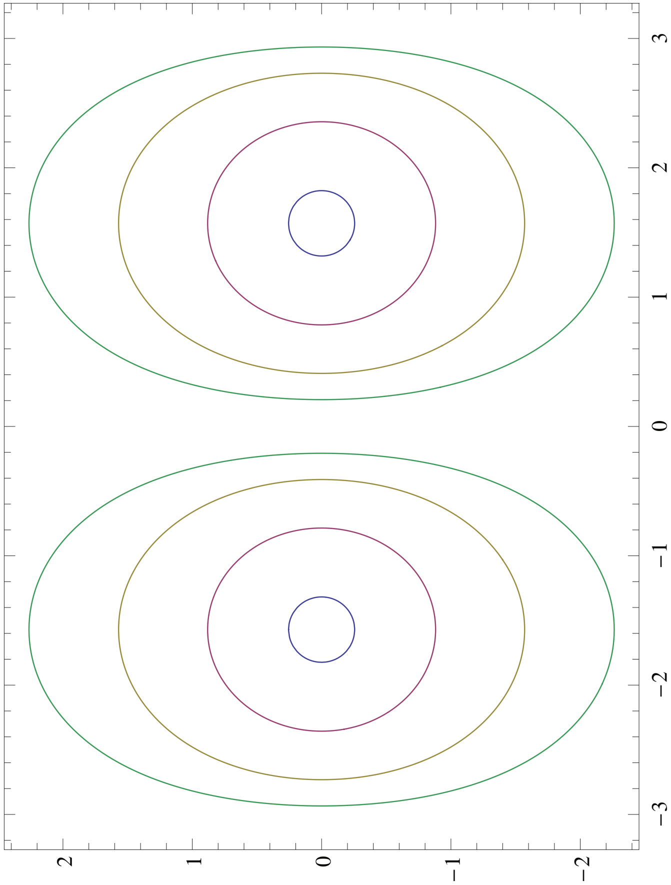
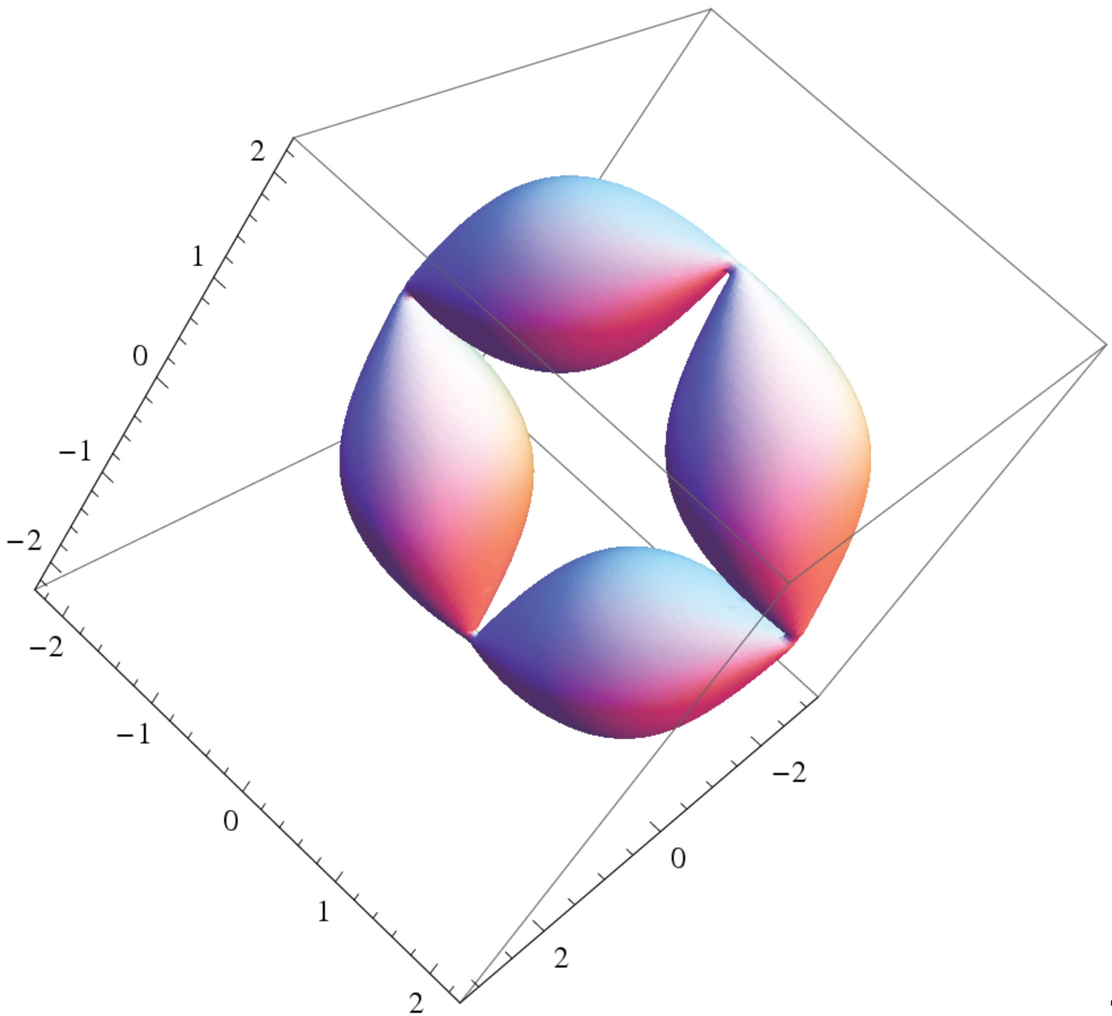


Figure 1: F_x/x vs. z for $x = \pi/4$ (blue), 0.5 (purple), 0.25 (brown), 0.125 (green).

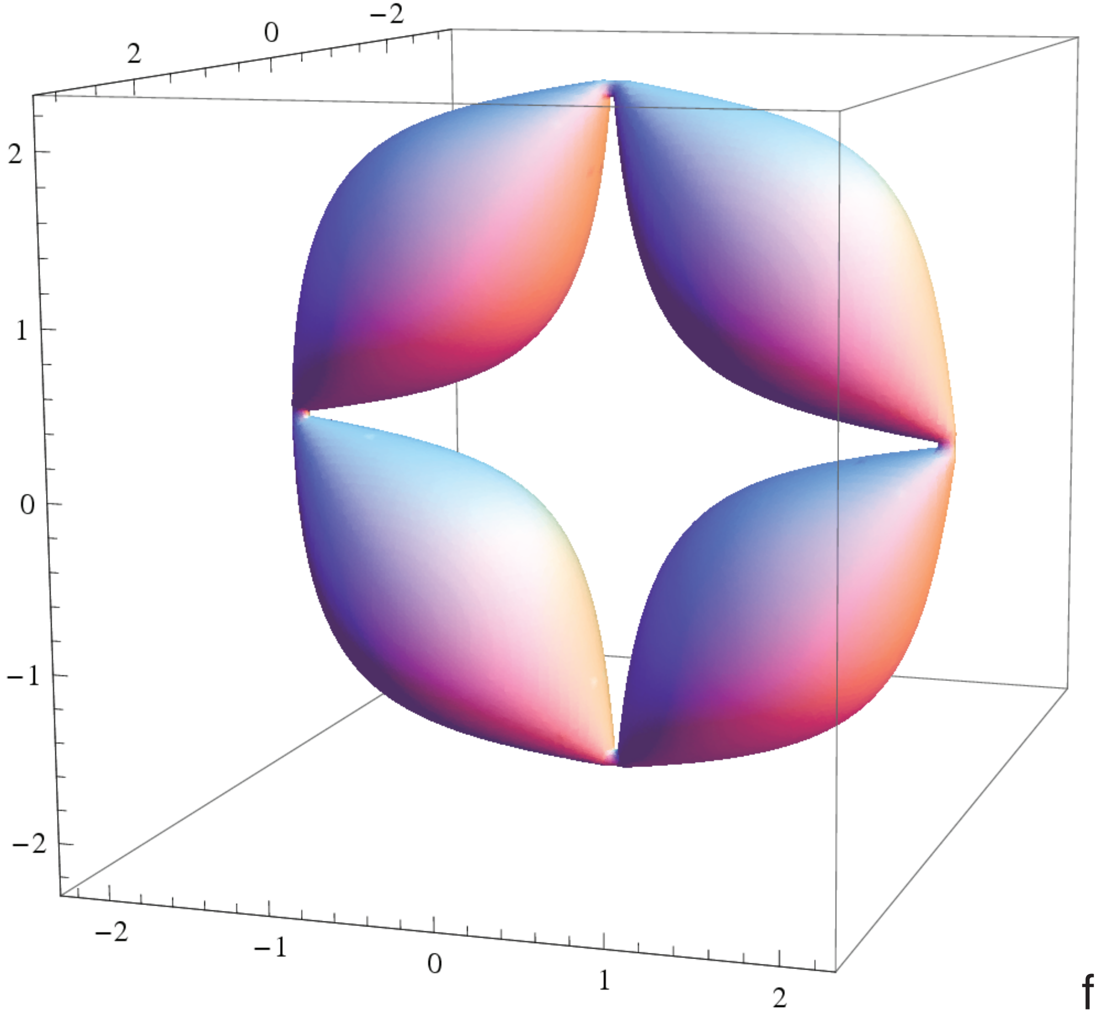




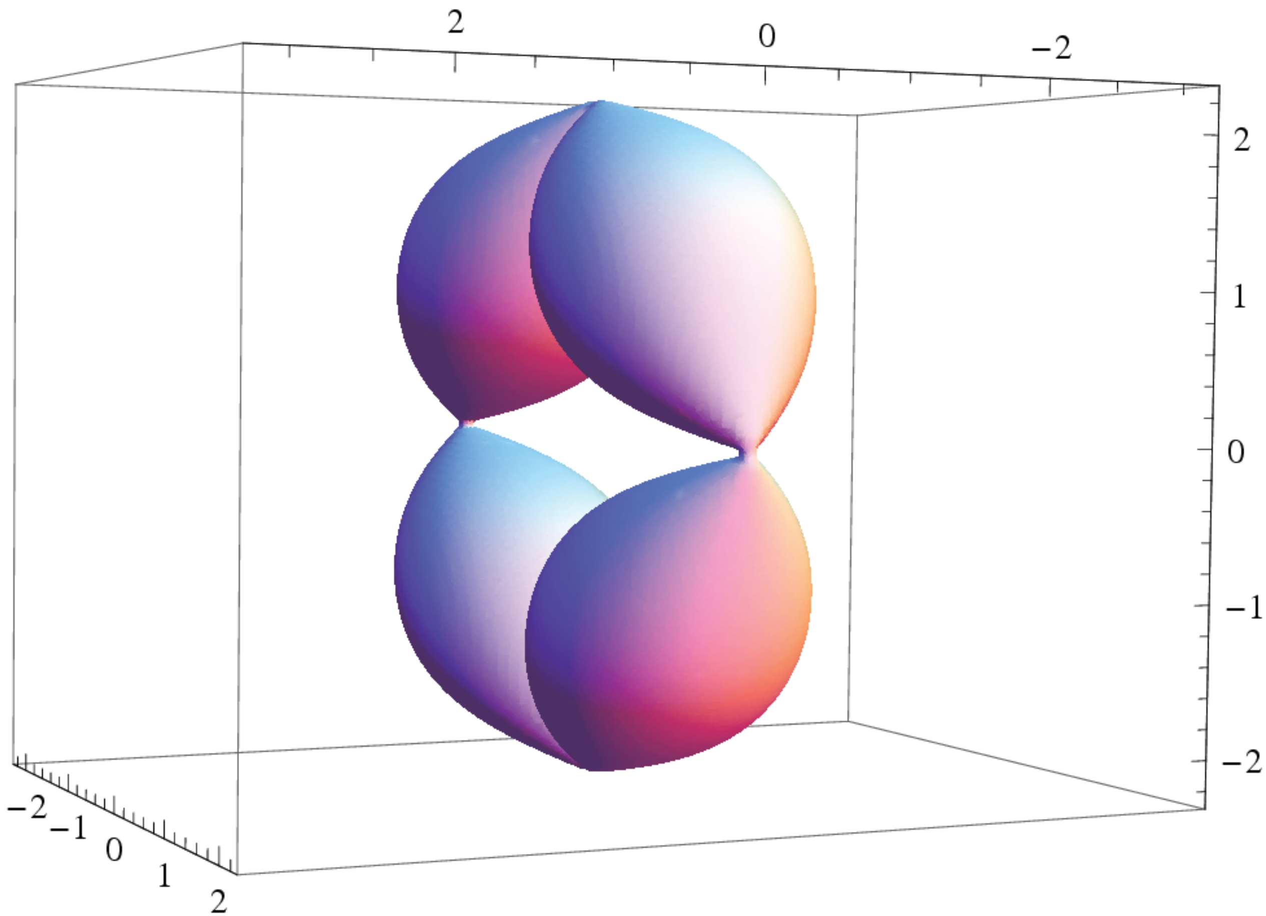
Equipotentials. Left: xy-plane. Right: xz-plane



4 footballs

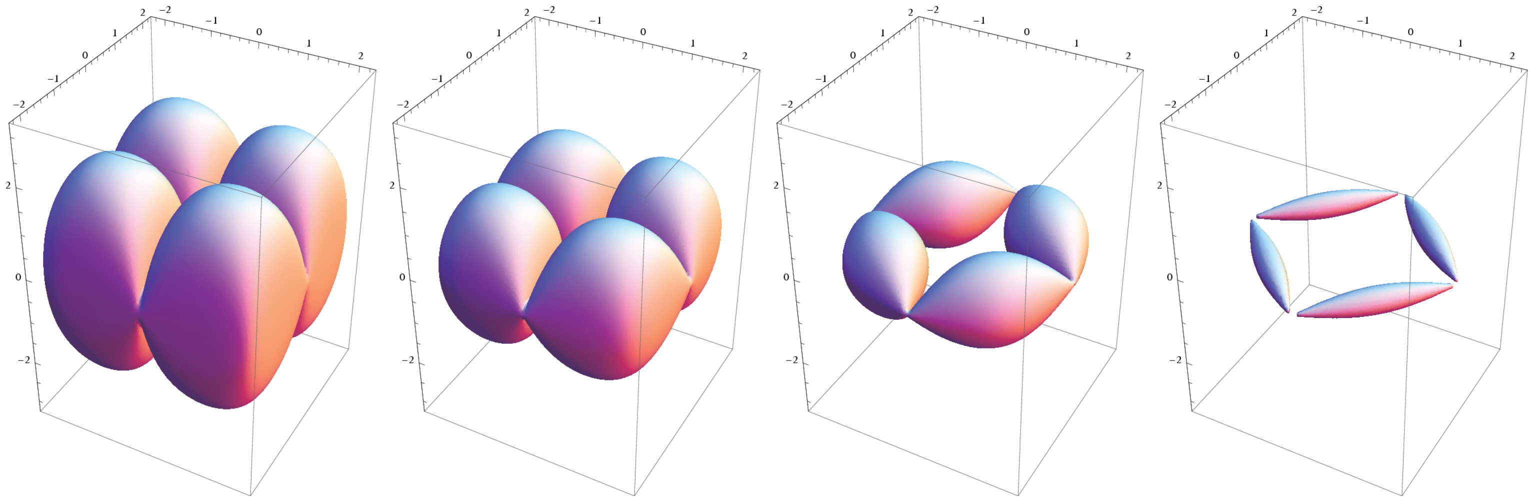


frontal view



side (or top) view

some other equipotentials

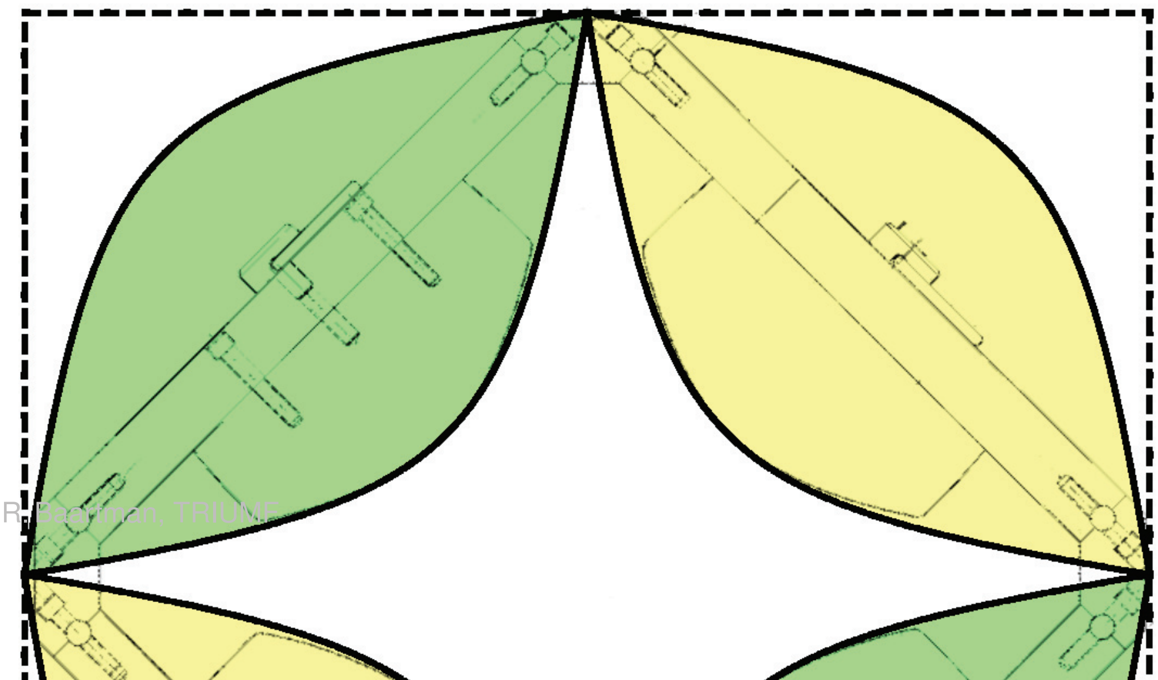


(Axis is vertical here.)

Practical Quad

Since we really only care about the fields around the beam axis, the fact that they are singular at the vertices is of no concern; we simply truncate the footballs.

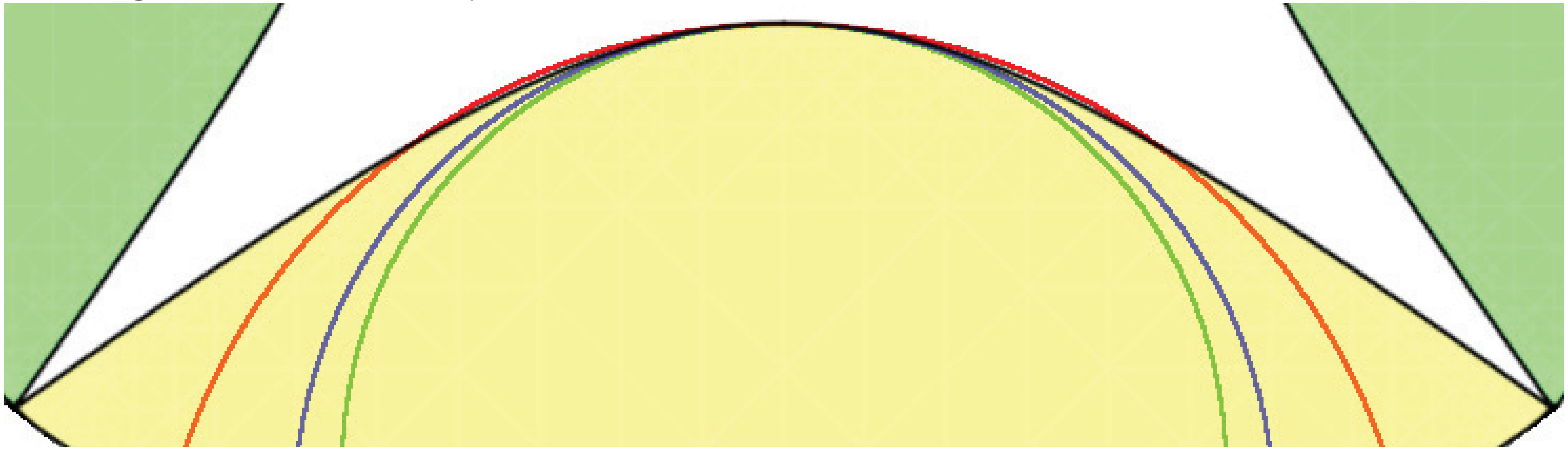
If we make the poles spherical, what is the curvature that matches the football quad? Ans: Radius of sphere = $4/\pi$ times aperture radius. (Reminder: for the 2D quad, if poles are circular, the radius to zero the 12-pole is 1.15 times the aperture radius.)



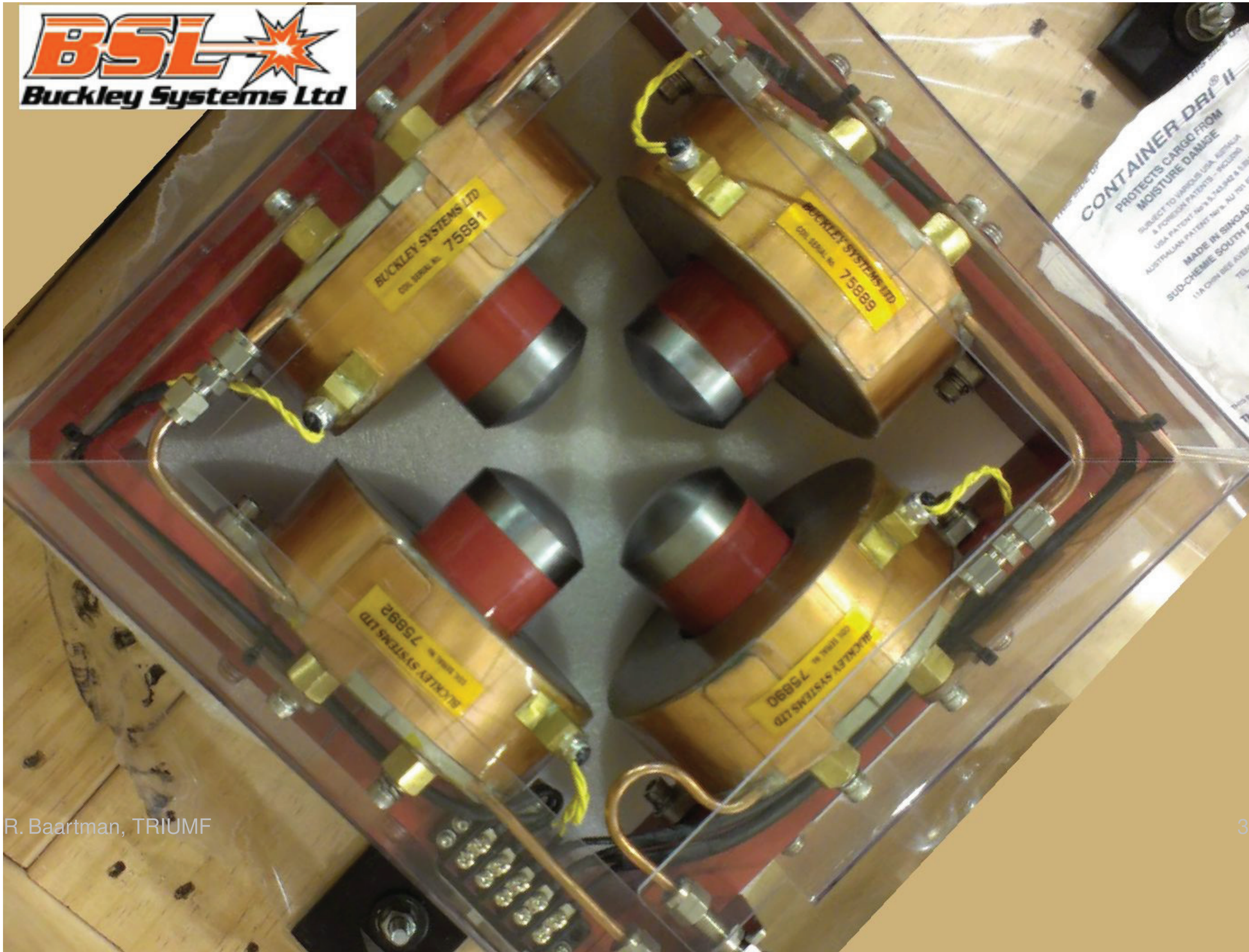
Here we compare the football quad with the 12Q12, cut through the $z = 0$ plane. Notice the slightly too sharp curvature and the missing corners.

Spherical poles' optimum radius

We can compensate the truncation be flattening the poles. The pole radius that achieves this is 1.55 times the aperture radius. (Thanks to Buckley for refining our calculation.)



green: 1.15 (the 2D value)
blue: $4/\pi = 1.27$ (the football curvature)
red: 1.55 (compensates the 12-pole, but not the 20-pole...)


BSL
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CONTAINER DRI II
PROTECTS CARGO FROM MOISTURE DAMAGE
MADE IN SINGAPORE

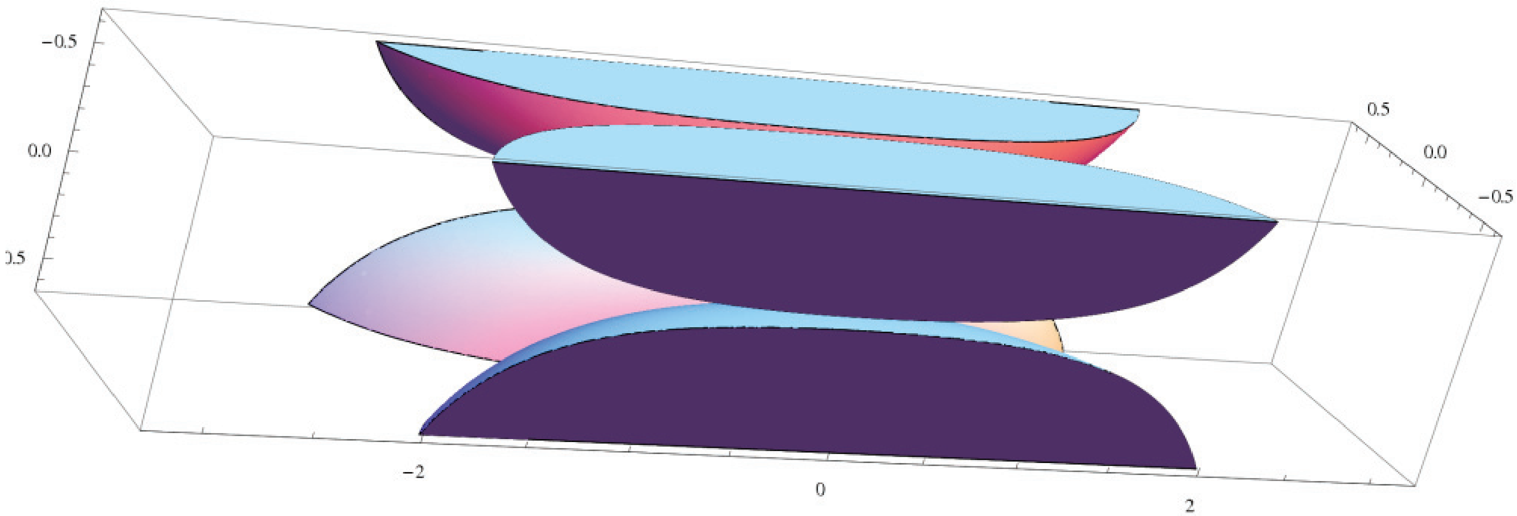
Rotating probe integrated harmonic measurement of new Buckley spherical-pole quad (typical).

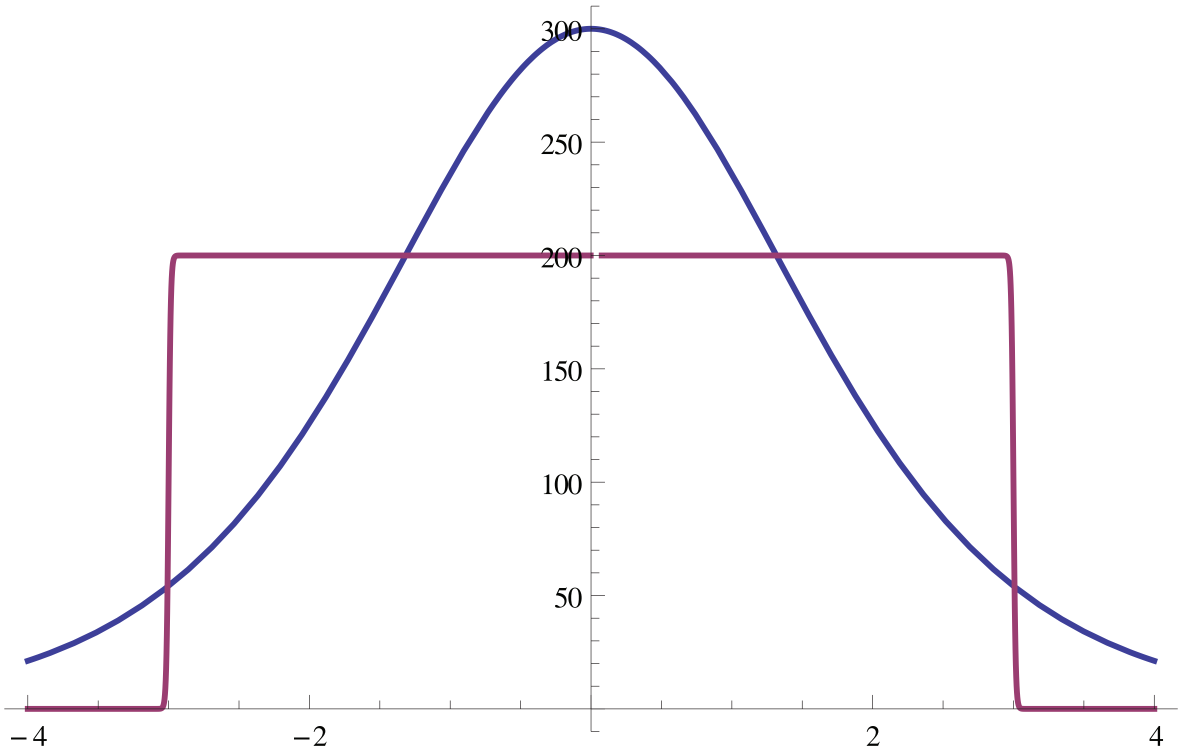
n	Strength$\times 10^4$
1	20.5
2	10000.0
3	9.8
4	1.0
5	1.7
6	0.8
7	1.4
8	0.6
9	0.3
10	27.3

Aberrations

What if we use this $k(z) \propto \mathrm{sech}^2 az$ quad instead of a conventional quad? Let us try the CERN LHC high gradient quad (LHCHGQ). This is a demanding high gradient quad, through which the proton beams pass as much as a cm off-axis. Length is 6 metres, aperture radius is 35 mm.

Choosing the largish aspect ratio equipotentials (green ones in slide 24):





Blue is $k(z) = 300\,\frac{\mathrm{T}}{\mathrm{m}}\,\mathrm{sech}^2\frac{z}{2\,\mathrm{m}}$, magenta is LHCHGQ.
Same length, same integrated strength.

We use COSY-∞ to calculate the aberrations...

Aberration coefficients of the LHC “high gradient quadrupole”, compared with quadrupole of same length and strength but sech^2 strength function. m and n are the exponents of x and y; for example, the fourth line is the coefficients for x^2y. Units are metres and radians.

LHCHGQ quad		sech^2 quad		
$\Delta x'$	$\Delta y'$	$\Delta x'$	$\Delta y'$	mn
-0.4877909E-01	0	-0.4879330E-01	0	10
0	0.5406070E-01	0	0.5407527E-01	01
-0.1154336E-03	0	-0.1134717E-03	0	30
0	-0.3724576E-03	0	-0.4038506E-03	21
-0.4881636E-03	0	-0.4714397E-03	0	12
0	-0.1822114E-03	0	-0.1867242E-03	03
0.1017362E-02	0	-0.2979146E-05	0	50
0	-0.7305652E-02	0	-0.1326917E-04	41
0.4749991E-02	0	-0.2637724E-04	0	32
0	-0.7580379E-02	0	-0.3096770E-04	23
0.2348281E-02	0	-0.1566155E-04	0	14
0	-0.8101367E-03	0	-0.3084560E-05	05
-0.1109434	0	-0.1215269E-06	0	70
0	0.9689789	0	-0.6573820E-06	61
-0.7151169	0	-0.1949172E-05	0	52
0	1.312496	0	-0.2858019E-05	43
-0.2433022	0	-0.2866705E-05	0	34
0	0.5811675E-02	0	-0.2532962E-05	25
0.6676453	0	-0.8507323E-06	0	16
0	-0.1404954	0	-0.1333598E-06	07

Third order are the same for the two quads (expected), the fifth order aberrations are about 200 times smaller for the sech^2 quadrupole, and the largest seventh order coefficients are roughly 100,000 times smaller for the sech^2 quadrupole. For ninth order, the ratio is 10^8.

This can be understood as follows. The aberrations depend on the derivatives of $k(z)$. For truncated quads, this is 1/aperture to appropriate power. For smoothly varying $k(z)$, it is 1/length.

Conclusions

We have derived an analytic potential for quadrupoles, both long and short. The strength function, instead of stepping up abruptly at the entrance and down at the exit, varies smoothly throughout the quadrupole.

We have shown that such quadrupoles, compared with conventional 2-dimensional ones:

1. needn't have any larger multipole component;

2. have much smaller 5^{th} and higher order intrinsic aberrations.

Acknowledgements

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