An Overview of Lie Methods for Accelerator Physics

Acknowledgements

Ancestors Who Have Shown the Way:

Galileo, Newton, Leibniz, Euler, Lagrange, Hamilton, Poisson, Jacobi, Poincare, … Sophus Lie…

Students and Post Docs:
John Finn, David Douglas, Etienne Forest, Liam Healy, Robert Ryne, Govindan Rangarajan, Dan Abell, Marco Venturini, Chad Mitchell, Filippo Neri, Johannes van Zeijts, Fernando Casas, Sergio Blanes

Support

- Institut des Hautes Etudes Scientifiques: Louis Michel
- Los Alamos National Laboratory: Richard Cooper
- U. S. Department of Energy: Melvin Month, David Sutter, Jerry Peters, Bruce Strauss
- University of Maryland

Finally, in the Tradition of Galileo, Newton, Leibniz, Euler, Lagrange, … and in Gratitude for the Universe, thanks to its Maker:

If the doors of perception were cleansed, everything would appear to man as it is, infinite. William Blake

What is a Lie algebra? Algebra involves addition and multiplication. A Lie algebra is

- A linear vector space (generalizes concept of addition)
- Equipped with a multiplication (Lie product) rule, denoted by [*,*], having the two properties
  1. [f,g]=-[g,f]
  2. [f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0

     (Jacobi identity)

Hamiltonian Mechanics Has a Lie Algebraic Structure

- Functions f on phase space form a linear vector space (they can be added).
- Define the Lie product of any two such vectors (functions) f and g by the Poisson bracket rule

$$[f,g] = \sum_i \{(\partial f / \partial q_i)(\partial g / \partial p_i)$$

$$-(\partial f / \partial p_i)(\partial g / \partial q_i)\}$$

Definition of Lie Operator :f: that acts on functions g

$$:f: := \sum_i \{(\partial f / \partial q_i)(\partial / \partial p_i)$$

$$-(\partial f / \partial p_i)(\partial / \partial q_i)\}$$

$$:f:^0 g = g$$

$$:f: g = [f, g]$$

$$:f:^2 g = [f,[f, g]]$$

Lie operators do not commute, but their commutator is again a Lie operator. Introduce the notation

$$\{:f:,:g:\}=:f::g:-:g::f:$$

Then, from the Jacobi identity, there is the result

$$\{:f:,:g:\}=:[f,g]:$$

Definition of Lie Transformation

$$\exp(:f:) = \sum_{n=0}^{\infty} :f:^n / n!$$

$$\exp(:f:)g = g + [f,g] + [f,[f,g]/2! + \ldots$$

Given phase-space variables $z=(q,p)$, a Hamiltonian $H(z,t)$, and initial conditions z^{in}, there is a transfer map M such that the final conditions z^{fin} are given by the relation

$$z^{fin} = M z^{in}.$$

M obeys the equation of motion

$$dM/dt = M{:}-H{:}. \quad (1)$$

Factorization Theorem:
Given any transfer map M arising from a Hamiltonian, it can be written uniquely as a product of Lie transformations in the form

$$M=\exp(:f_1:)\exp(:f_2:)\exp(:f_3:)\ldots \quad (2)$$

where the f_m are homogeneous polynomials of degree m.

In the context of Accelerator Physics the polynomials f_m have the following significance:

- f_1 describes misalignment, misplacement, and mis-powering errors
- f_2 describes linear transformations $R=\exp(:f_2:)$ such as produced, in the linear approximation, by drifts, quadrupoles, and dipoles
- f_3 describes sextupole effects
- f_4 describes octupole effects, etc.

Lie Algebraic Calculus for Manipulating (Inverting and Multiplying) Maps: If

$$M=\exp(:f:)$$

then

$$M^{-1}=\exp(-:f:)$$

Also,

$$\exp(:f:)\exp(:g:)=\exp(:h:)$$

with

$$h=f+g+(1/2)[f,g]$$

$$+(1/12)([f,[f,g]]+[g,[g,f]])+\ldots$$

Applications

- Concatenation: Given the maps for individual beam-line elements, they can be multiplied together to find the map for a full beam line for a linac, or the one-turn map for a ring.
- Normal Form: Suppose M is the one-turn map for a ring, and the tunes are not resonant. Then there exist maps A and N such that

$$N = AMA^{-1}$$

and N has a simple (normal) form.

The Normal Form Procedure is the Nonlinear Generalization of Matrix Diagonalization

- The normal form map N contains all information about tunes, anharmonicities, and chromaticities.
- The transforming/normalizing map A contains all information about closed-orbit distortions and linear and nonlinear lattice functions.
- From/with A one can also manufacture matched beams (including nonlinear effects) and nonlinear generalizations of the Courant-Snyder invariants, and examine tracking data for KAM tori.

Computation of Maps

Suppose the Hamiltonian for a beam-line element has the expansion

$$H=H_2+H_3+H_4+\ldots$$

Then, from the equation of motion (1) for M and for the factorized representation (2) for M, it follows that there are equations of motion for $R=\exp(:f_2)$, the linear part of M, and for the nonlinear generators f_3, f_4, etc. These equations involve the H_m, and integrating these equations yields M.

Accurate Computation of M for Realistic Beam-Line Elements

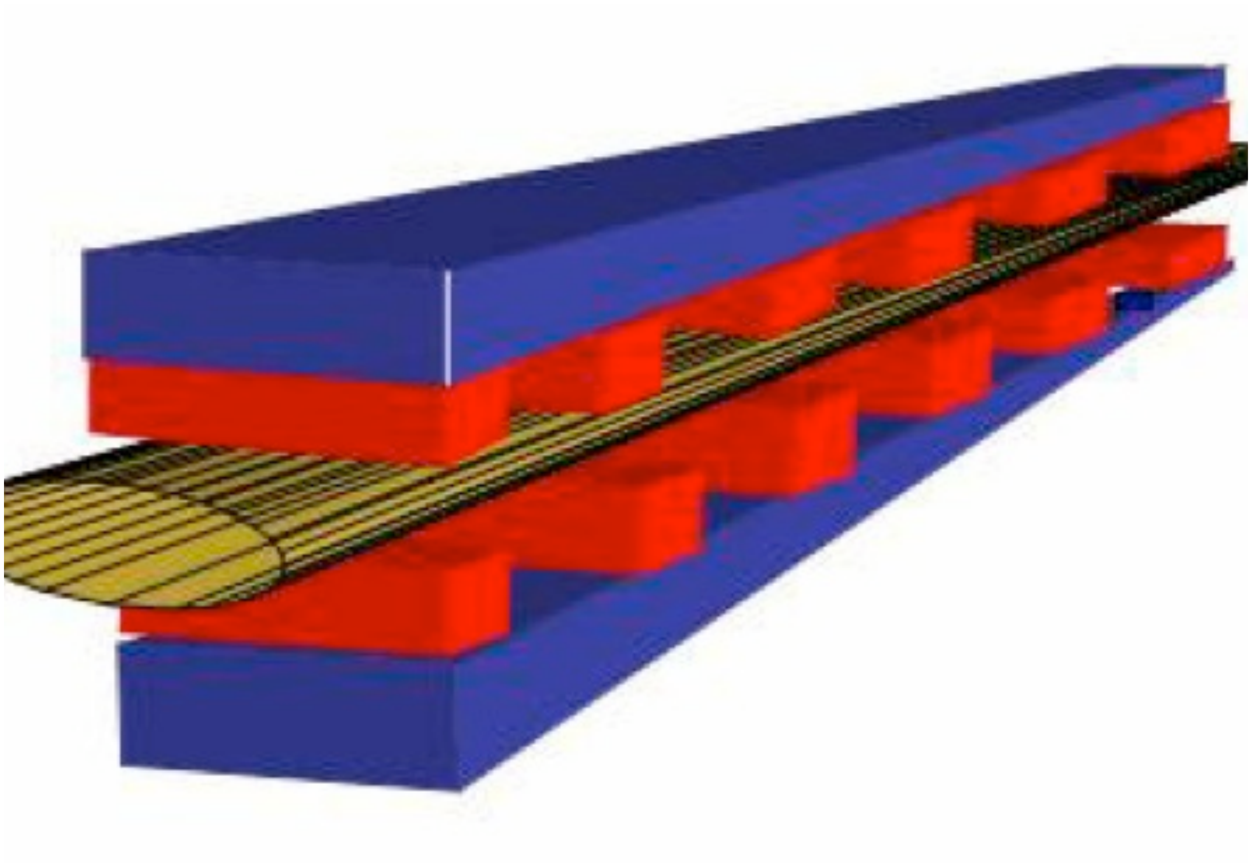
- Consider, for example, the case of magnetic elements. Analogous results hold for electrostatic and RF elements.
- Need to know the H_m, which entails knowledge of the vector potential and its derivatives.
- But, in general, B field is only known numerically (with the aid of some 3-D solver) on a collection of grid points. How can this data be used to find the H_m?

Surface Methods to the Rescue

- Cannot apply numerical differentiation to the grid data because numerical differentiation is intolerably sensitive to (amplifies) numerical noise.
- But, Maxwell's equations are smoothing: Interior data in a volume V can be computed from data on a surrounding surface S and, thanks to smoothing, this interior data and its derivatives are relatively insensitive to noise in the surface data. The smoothing provided by the use of surface methods overcomes the amplification of noise associated with differentiation.

Application to Straight Beam-Line Elements

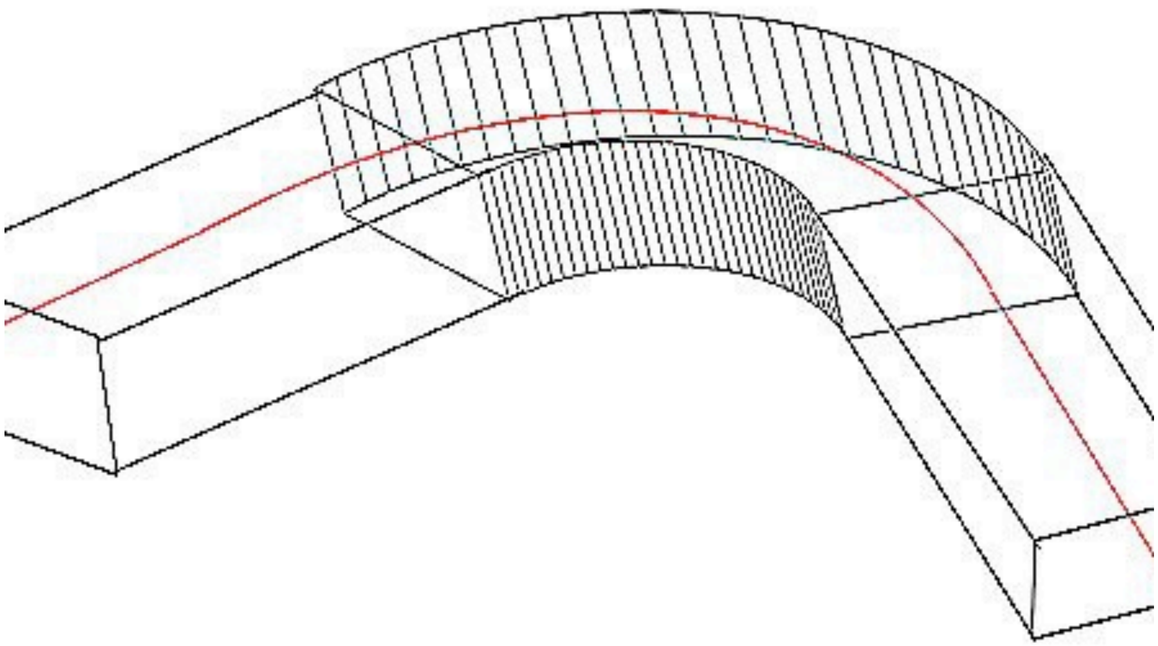
- Surround the beam with an imaginary cylinder fitting within the beam-line element and extending beyond the fringe-field regions at the ends.
- Extrapolate the grid data for the interior B field onto the surface of the cylinder, and from this surface data compute the interior vector potential and its derivatives to yield the H_m.
- Compute M from these H_m.



An elliptical cylinder, centered on the z-axis, fitting within the bore of a wiggler, and extending beyond the fringe-field regions at the ends of the wiggler.

Application to Curved Beam-Line Elements, e.g., Dipoles with Large Sagitta

- Surround the beam with an imaginary bent box fitting within the dipole and having straight legs extending beyond the fringe-field regions.
- Extrapolate grid data (now for B and the related scalar potential for B) onto the surface of the box and its straight ends, and from this surface data compute the interior vector potential and its derivatives to yield the H_m.
- Compute M from these H_m.



A bent box, fitting within a dipole, and having straight end legs that extend beyond the fringe-field regions at the entry and exit of the dipole.

Conclusion

- Lie methods provide a powerful unified approach to both linear and nonlinear behavior.
- Using Lie and surface methods, it is now possible, for the first time, to compute accurate high-order transfer maps for realistic beam-line elements based on 3-D field data provided by a 3-D numerical field solver. So doing includes all fringe-field and multipole error effects.