A Model Ring With Exactly Solvable Nonlinear Motion

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Table of Contents

1. Introduction

Recently, a concept of accelerator lattices with nonlinear transverse motion possessing two analytic invariants has been proposed [1]. Based on further studies [2], the Integrable Optics Test Accelerator (IOTA) was designed and is being constructed at the Fermi National Accelerator Laboratory. Such a nonlinear lattice may be helpful in suppression of the collective instabilities by introducing a relatively large tune spread in a beam, while reducing phase-space area occupied by chaotic trajectories.

[1] V. Danilov and S. Nagaitsev, *"Nonlinear Accelerator Lattices with One and Two Analytic Invariants"*, **Phys. Rev. ST Accel. Beams 13**, 084002 (2010).
http://prst-ab.aps.org/abstract/PRSTAB/v13/i8/e084002

[2] P. Piot, V. Shiltsev, S. Nagaitsev, M. Church, P. Garbincius, S. Henderson and J. Leibfritz, *"The Advanced Superconducting Test Accelerator (ASTA) at Fermilab: A User-Driven Facility Dedicated to Accelerator Science & Technology,"*, **arXiv:1304.0311 [physics.acc-ph]**.
http://arxiv.org/pdf/1304.0311

[3] T. Zolkin, Y. Kharkov, I. Morozov and S. Nagaitsev, *"Accelerator with Transverse Motion Integrable in Normalized Polar Coordinates"*, **Conf. Proc. C 1205201**, 1116 (2012).
http://accelconf.web.cern.ch/accelconf/IPAC2012/papers/tueppb003.pdf

Consider a 1D transverse motion in a model ring with a single thin nonlinear lens:

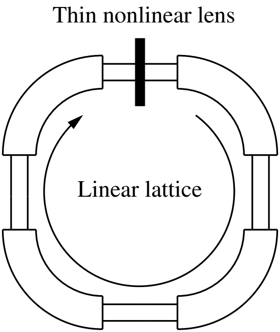
$$\begin{bmatrix} q \\ p \end{bmatrix}_{n+1} = \begin{bmatrix} \cos 2\pi\nu & \sin 2\pi\nu \\ -\sin 2\pi\nu & \cos 2\pi\nu \end{bmatrix} \times \left(\begin{bmatrix} q \\ p \end{bmatrix}_n + \begin{bmatrix} 0 \\ \delta p(q_n) \end{bmatrix} \right)$$

◦ **Hénon's quadratic twist map**

$$\delta p(q_n) = -3\epsilon q_n^2$$

◦ **McMillan map**

$$\delta p(q_n) = -\epsilon \frac{a q_n}{1 + b q_n^2}$$



The Hénon map is equivalent to 1D sextupole and is known to be nonintegrable.

Video: Stroboscopic Poincare surface of section for Hénon's quadratic twist map.

The McMillan map is integrable for $\nu = 0.25$, but it is not easy to generalize for 2D. rot

Video: Stroboscopic Poincare surface of section for McMillan map.

2. Concept of Nonlinear Integrable Transverse Optics

Desired spread of frequencies can be achieved by adding to the Hamiltonian of linear lattice an additional nonlinear potential:

$$\mathcal{K}[p_x, p_z, x, z; s] = \underbrace{\sum_{q=x,z} \left[\frac{p_q^2}{2} + g_q(s) \frac{q^2}{2} \right]}_{\mathcal{K}_0[p_x, p_z, x, z; s] \text{ — linear lattice}} + \boxed{V(x, z, s)}.$$

In general, the new equations of motion do not necessarily provide two (and even a one) analytic invariants.
Below, we will consider one of the possible ways of how to modify a Hamiltonian $\mathcal{K}_0$ **preserving the integrability** at the same time **[1]**.

2.1 First integral of motion

Step 1: Lattice with linear axially symmetric focusing

The use of betatron phase advance as a new independent variable with subsequent canonical transformation to normalized coordinates, **moves the time dependence into the nonlinear term**.

$[p, q; s] \rightarrow [\mathcal{P}_q, \eta_q; \psi(s)]$:

$$\begin{cases} \eta_q = q/\sqrt{\beta_q} \\ \mathcal{P}_q = p_q\sqrt{\beta_q} - q\dfrac{\beta_q'}{2\,\beta_q^{3/2}} \end{cases},$$

where $' \stackrel{\text{def}}{=} d/d\psi$.

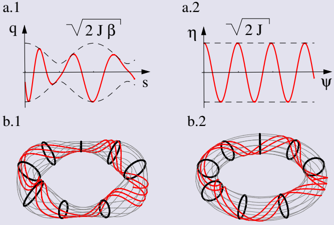


Figure: (a.1,2) Particle trajectory in old and new canonical variables. (b.1,2) Trajectory in extended phase space, (p, q, s) vs. $(\mathcal{P}_q, \eta_q, \psi)$.

2.1 First integral of motion

After the transformation $\mathcal{K}[p_x, p_z, x, z; s] \to \mathcal{H}[\mathcal{P}_x, \mathcal{P}_z, \eta_x, \eta_z; \psi]$ we have:

$$\mathcal{H}[\mathcal{P}_x, \mathcal{P}_z, \eta_x, \eta_z; \psi] = \sum_{q=x,z} \left(\frac{\mathcal{P}_q^2 + \eta_q^2}{2} \right) + \beta\,[s(\psi)]\; V(\mathbf{q}(\eta, \psi), \psi).$$

Step 2: Special "time"-dependence

At least one integral of motion, the Hamiltonian by itself, can be ensured, if the time dependence can be compensated by **special "time"-dependence** of the nonlinear potential:

$$\boxed{\beta\,[s(\psi)]\; V(\mathbf{q}(\eta, \psi), \psi) = U(\eta_x, \eta_z)}.$$

2.2 Second integral of motion

For the Hamiltonian in the form

$$\mathcal{H}[\mathcal{P}_x, \mathcal{P}_z, \eta_x, \eta_z; \psi] = \sum_{q=x,z} \left(\frac{\mathcal{P}_q^2 + \eta_q^2}{2} \right) + U(\eta_x, \eta_y),$$

a presence of a second integral can be guaranteed by the choice of new generalized coordinates where **variables can be separated**.

Harmonic condition

Additional constraint on a potential $U(\eta_x, \eta_z)$ to satisfies the Laplace equation essentially reduce the number of possible choices among the whole possible functions.

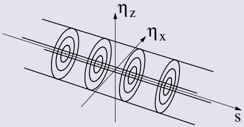
3. Separation of Variables in Polar Coordinates

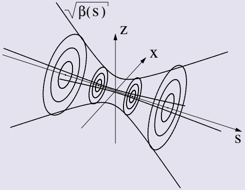
Three different families of integrable lattices were found for the invariant in the form

$$W = A(x,z)\mathcal{P}_x^2 + B(x,z)\mathcal{P}_x\mathcal{P}_z + C(x,z)\mathcal{P}_z^2 + D(x,z).$$

Normalized polar coordinates (r, θ)

$$\eta_x = r\cos\theta,$$
$$\eta_z = r\sin\theta,$$
$$\mathcal{P}_x = p_r\cos\theta - \frac{p_\theta}{r}\sin\theta,$$
$$\mathcal{P}_z = p_r\sin\theta + \frac{p_\theta}{r}\cos\theta,$$



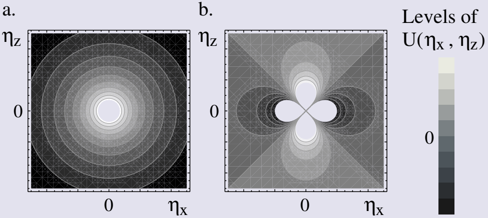


In polar coordinates the variables separation is possible for the potentials in the form:

$$U(r,\theta) = f(r) + \frac{h(\theta)}{r^2}.$$

Harmonic potentials

- $B \ln r$ — **straight wire carrying a constant current**
- $A \sin(2\theta + \varphi)/r^2$ — **point-like magnetic quadrupole**



4. Transverse Motion

Finally we have a Hamiltonian

$$\mathcal{H}[p_r, p_\theta, r, \theta; \psi] = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{r^2}{2} + \frac{A\sin(2\theta + \varphi)}{r^2},$$

with two invariants of motion:

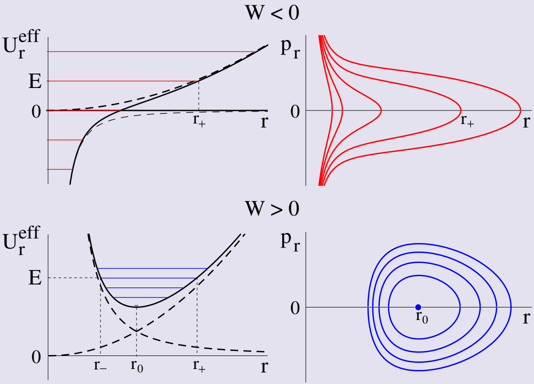
- **energy**

$$\boxed{E = \frac{p_r^2 + r^2}{2} + \frac{W}{r^2}}$$

- **effective angular momentum**

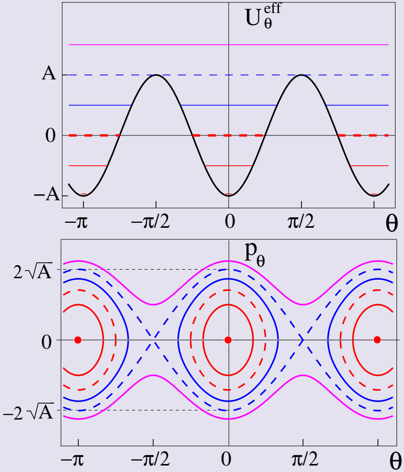
$$\boxed{W = \frac{p_\theta^2}{2} + A\sin(2\theta + \varphi)}$$

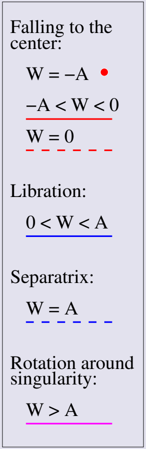
Radial motion



$$J_r(E) = \frac{1}{2\pi} \oint p_r \, dr = \frac{E - \sqrt{2W}}{2}, \qquad \omega_r = \frac{\partial \mathcal{H}}{\partial J_r} = 2$$

Angular motion





Classification of trajectories

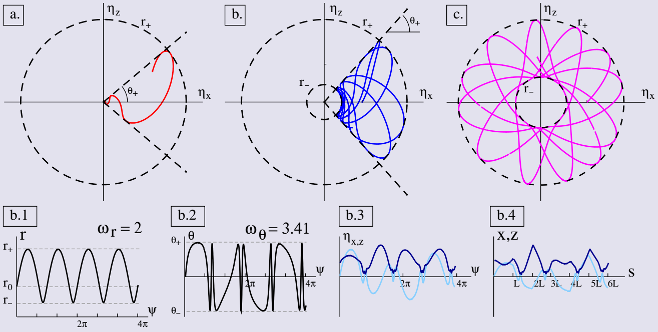
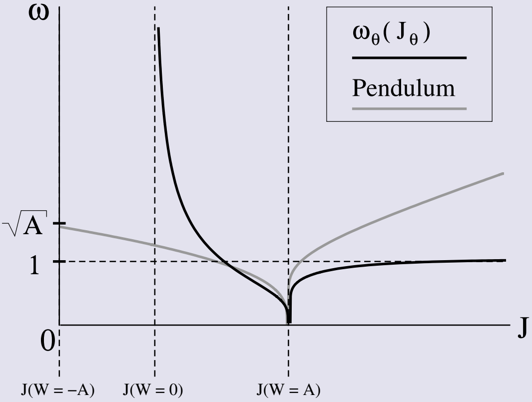


Figure: Particle trajectory in the normalized coordinates for (a.) falling to the center ($-A < W < 0$) (b.) libration ($0 < W < A$) (c.) rotation around the origin ($W > A$).

Frequency dependence of the amplitude for the angular motion



$$\omega_\theta = \frac{1}{\sqrt{2W}} \left(\frac{\partial J_\theta}{\partial W} \right)^{-1} = \frac{1}{\sqrt{2W}} \times \begin{cases} 2\ \omega_{\text{pend}} \\ \omega_{\text{pend}} \end{cases} ,$$

5. Model Ring

Guide to the design

- The absence of equilibrium orbit of motion means that under the action of friction force particles will fall towards the singularity and eventually will be lost. Thus, below we will consider the design of an accelerator ring for protons, so far as the damping of oscillations due to radiation effects is negligible for them.
- A super-period of a lattice with axially symmetric focusing can be realized with a drift space of length L, where the nonlinear lens is located, and an optics insert (so-called T-insert), which is equivalent to the thin axially symmetric lens with the focal length equal to $1/k$.

5.1 Axially symmetric focusing

IOTA ring layout

$\beta_{\min} \approx 100$ cm, $\beta_{\max} \approx 200$ cm, $\nu \in (0;2)$

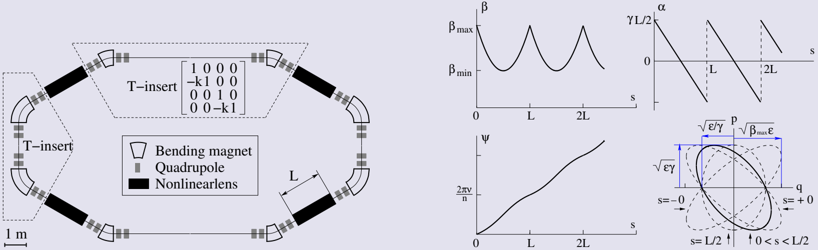


Figure: IOTA layout and behavior of optical functions in $\frac{F}{2}O\frac{F}{2}$ lattice.

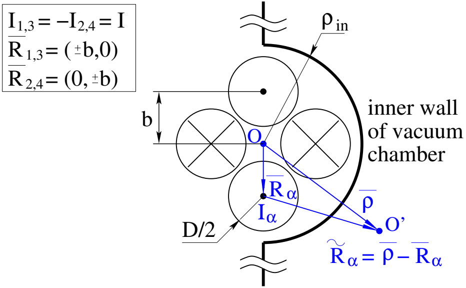
Linear Lattice Parameters		
# of super-periods		4
# of nonlinear lenses		2
Circumference, Π	(m)	38.7
Bending dipole field, B	(T)	0.7
Drift space length, L	(cm)	200
T-insert strength parameter, k	(cm^{-1})	$\in (0; 0.02)$

Beam at the Injection		
Beam kinetic energy, $\mathcal{E}_{\rm kin}$	(MeV)	1.91
Beam momentum, $P_{\rm eq}$	(MeV/c)	60
Normalized emittance, $\epsilon^{\perp}_{\rm norm}$	(cm rad)	2×10^{-5}

Table: IOTA ring parameters used in simulations and optimized for compatibility with nonlinear lens under consideration.

5.2 Nonlinear lens parameters

The design of proposed nonlinear lenses brings in two major inevitable perturbations. They are associated with the special longitudinal dependence of the field, and, the physical realization of poles of the lens.



$$A_s = -\frac{\mu_0}{2\pi}\sum_{\alpha=1,2,3,4} I_\alpha \ln\left|\widetilde{R}_\alpha\right|$$

$$= \frac{\mu_0 I}{\pi}\left(b^2\frac{\cos 2\theta}{\rho^2} + \frac{b^6}{3}\frac{\cos 6\theta}{\rho^6} + O\left[\left(\frac{b}{\rho}\right)^{10}\right]\right)$$

	Supercond.	**Water Cooling**
Beam momentum		
P_{eq}, (MeV/c)	60	30
Diametr of the wire		
D, (mm)	6	7
Current density		
ρ_I, (A/mm^2)	100	10
Total current		
I, (A)	2827	385
Inner radius of pipe		
ρ_{in}, (cm)	0.85	1
Outer radius of pipe		
ρ_{out}, (cm)	4	4

Table: Parameters of the nonlinear polar lens for two different values of current density: superconducting lens and the one with water cooling.

6. Simulation of a Monochromatic Beam Motion

Beam motion for nonlinear kick defined by the potential from 4 wires moved apart from each other. Simulation performed using 8-th order simplectic integrator. Linear lattice unperturbed frequencies $\nu_{x,z} = 0.44$.

Video: (x, z)-plane

Motion in a phase space

Video: (x, p_x)-plane

Video: (z, p_z)-plane

8. Results and Conclusions

1. The first paraxial nonlinear exactly integrable system has been studied: analytical expressions for dynamical variables change over the time as well as amplitude dependence of frequency are obtained.
2. The possibility to create such a nonlinear lens were demonstrated on the example of IOTA ring for 60 MeV protons.
3. Numerical methods for the simulation of perturbed nonlinear system were discussed (requires further study).
4. This system is of particular interest since it has an unusual feature for accelerator physics: it has no equilibrium orbit. In addition, this system is interesting in that it has the degeneracy.

Thank you for your attention.