Transverse Beam Transfer Functions via the Vlasov Equation

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Introduction

- Stability and Response properties are encapsulated in Beam Transfer Functions (BTFs).
- Measurements are generally straightforward but can be challenging.
- The data can be quite rich.
- Reliable predictions and improvements to the model can result from an adequate theory.
- The problem is important enough to warrant more than one approach.

Calculating BTFs using simulations

Starter problem: Consider a kicker giving a kick f(τ) as the bunch passes on turn 0. Let $D_f(k, \tau)$ be the beam response (dipole moment) to this kick on turns k=0,1,…

Now take the forcing function $F(n,\tau) = \cos(2\pi n Q) f(\tau) e^{gn}$

Assuming linear response

$$D(n,\tau) = \sum_{k=-\infty}^{\infty} D_f(n-k,\tau)\cos(2\pi k Q)e^{gk}$$

$$= \sum_{m=0}^{\infty} D_f(m,\tau)\cos[2\pi Q(n-m)]e^{g(n-m)}$$

$$= \mathrm{Re}\left\{ e^{2\pi j Q n + gn} \sum_{m=0}^{\infty} D_f(m,\tau) e^{-2\pi j Q m - gm} \right\}$$

Hence, only one simulation is needed for all Q.

Calculating BTFs using simulations

In a real BTF with $|\omega_1| < \omega_0/2 = \pi/T_{rev}$

$$F(k,\tau) = A\cos[(n\omega_0 + \omega_1)(kT_{rev} + \tau)]$$

$$= A\cos[n\omega_0\,\tau + \omega_1 kT_{rev} + \omega_1\tau], \qquad |\,\omega_1\,\tau\,| \le \pi / 2h$$

$$\approx A\cos[n\omega_0\,\tau + \omega_1 kT_{rev}]$$

$$= A\cos(\omega_1\,kT_{rev})\cos(n\omega_0\tau) - A\sin(\omega_1 kT_{rev})\sin(n\omega_0\tau)$$

Hence we need the impulse response to a sine kick and a cosine kick to get the response for all frequencies.

Also, BTF is one complex number and not a function of τ.

Take the Fourier component of the beam response at the drive frequency.

$$BTF(\omega_1) = \sum_{m=0}^{\infty} e^{-(j\omega_1+\varepsilon)mT_{rev}} \int_{-\tau_b}^{\tau_b} e^{-jn\omega_0\tau}\left\{D_{\cos}(m,\tau) + jD_{\sin}(m,\tau)\right\}d\tau$$

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Calculations with the Vlasov equation

Single particle equations

$$\frac{dx}{d\theta} = p$$

$$\frac{dp}{d\theta} = -Q_\beta^2(\varepsilon)x + 2Q_\beta \Delta Q_{sc}(\tau)[x - \bar{x}(\theta,\tau)]$$

$$+ F_e(\theta,\tau) - \frac{q}{2\pi P_0 \omega_0} \int_0^{2\tau_b} W_\perp(\tau_1) D(\theta, \tau - \tau_1) d\tau_1$$

Vlasov

$$\frac{\partial F(x,p,a,\psi,\theta)}{\partial \theta} + \frac{dx}{d\theta}\frac{\partial F}{\partial x} + \frac{dp}{d\theta}\frac{\partial F}{\partial p} + Q_s \frac{\partial F}{\partial \psi} = 0$$

$$\tau = a \sin\psi \qquad \varepsilon = a\cos\psi$$

Take moments $\begin{pmatrix} X(\psi,a,\theta) \\ P(\psi,a,\theta) \\ \Psi(a) \end{pmatrix} = \iint dxdp \begin{pmatrix} x \\ p \\ 1 \end{pmatrix} F(x,p,\psi,a,\theta)$

$$\frac{dX}{d\theta} = \frac{\partial X}{\partial \theta} + Q_s \frac{\partial X}{\partial \psi} = P$$

$$\frac{dP}{d\theta} = -Q_\beta^2(\varepsilon)X + 2Q_\beta\kappa\left\{X\int \Psi d\varepsilon - \Psi \int X d\varepsilon\right\}$$

$$+ F_e(\theta,\tau)\Psi - \frac{q^2\Psi}{2\pi P_0\omega_0}\int_0^{2\tau_b} W_\perp(\tau_1)d\tau_1 \int X(\theta,\tau-\tau_1)d\varepsilon$$

$$\int X d\varepsilon = \int adad\psi X(\psi,a,\theta)\delta(\tau - a\sin\psi)$$

Head-tail phase $X = X_1 \exp(-iQ\theta - i\xi\omega_0\tau/\eta)$ $\quad Q = Q_0 + \Delta Q$

First order terms $\left\{\Delta Q + iQ_s \frac{\partial}{\partial\psi} + \Delta Q_{sc}(a\sin\psi)\right\} X_1(a,\psi) = \Psi(a)\widetilde{F}(\tau,[X_1])$

Extend head-tail phase. (new?)

$$\Delta Q_{sc}(a\sin\psi) = \lambda(a) + \frac{\partial\Lambda(a,\psi)}{\partial\psi} \qquad X_1 = X_2\exp(i\Lambda/Q_s)$$

Equation for X_2 is a linear ODE with "constant" coefficients.

$$\left\{\Delta Q + iQ_s \frac{\partial}{\partial\psi} + \lambda(a)\right\} X_2(a,\psi) = \exp(-i\Lambda/Q_s)\Psi(a)\widetilde{F}(\tau,[X_1])$$

Integrating factor $\exp[-i\psi(\Delta Q+\lambda)/Q_s]$

Integrate from ψ to ψ+2π employing periodicity of X_2.

Back substitute X_1 and integrate over ε.

$$D_1(\tau)=\int \hat{G}(\tau,t)F(t)dt$$

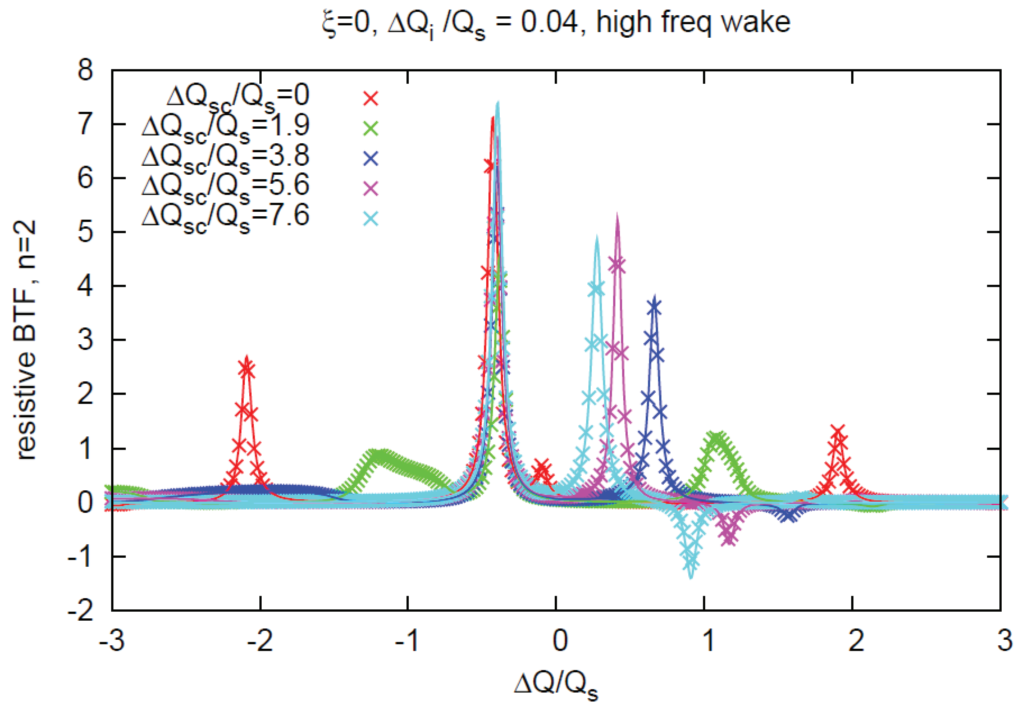
$$\hat{G}(\tau,t)=\int_0^\infty ada\int_0^{2\pi} d\psi\delta(\tau-a\sin\psi)e^{iG(a,\psi)}K(a)\int_\psi^{\psi+2\pi} d\phi\delta(t-a\sin\phi)e^{-iG(a,\phi)}$$

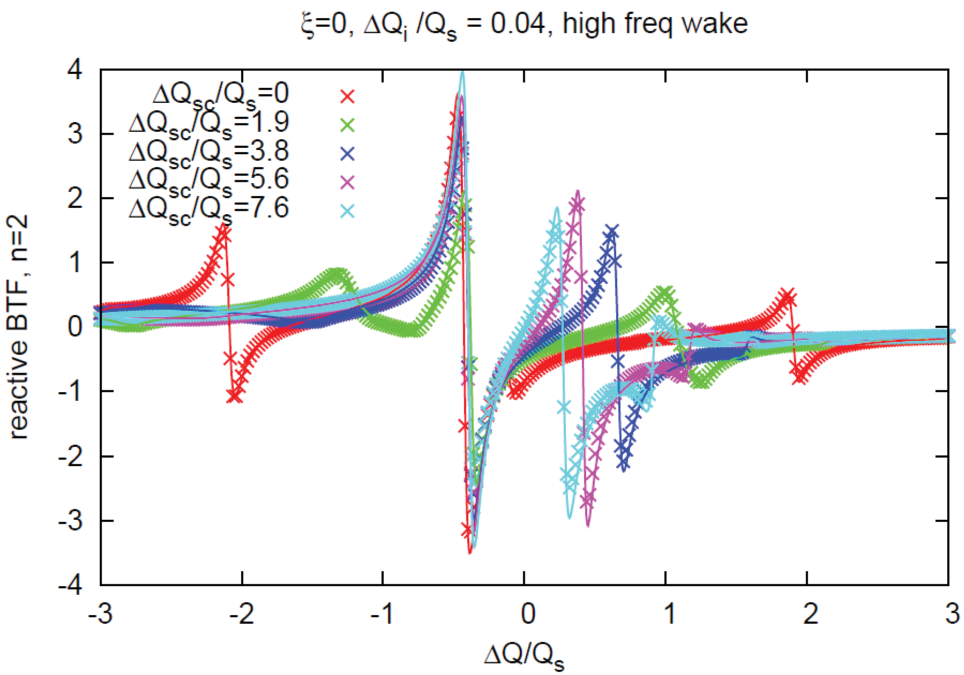
$$K(a)=i\frac{\Psi(a)/Q_s}{1-\exp\left[-2\pi i(\lambda(a)+\Delta Q)/Q_s\right]}$$

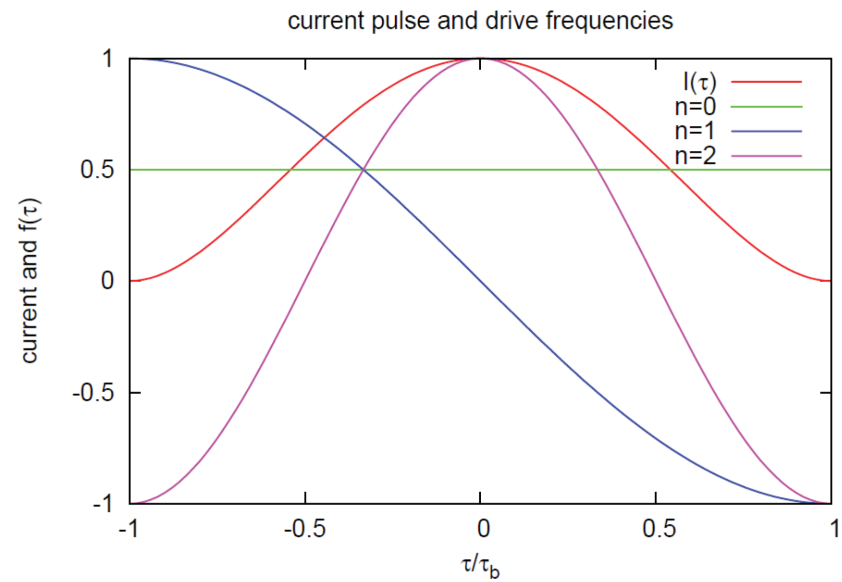
$$G(a,\psi)=\int_0^\psi d\varphi\frac{\Delta Q+\Delta Q_{sc}(a\sin\varphi)}{Q_s}$$

$$F(\tau)=\kappa D_1(\tau)+\int_0^\infty \hat{W}_\perp(\tau_1)e^{i\xi\omega_0\tau_1/\eta}D_1(\tau-\tau_1)-F_e(\tau)e^{i\xi\omega_0\tau/\eta}$$

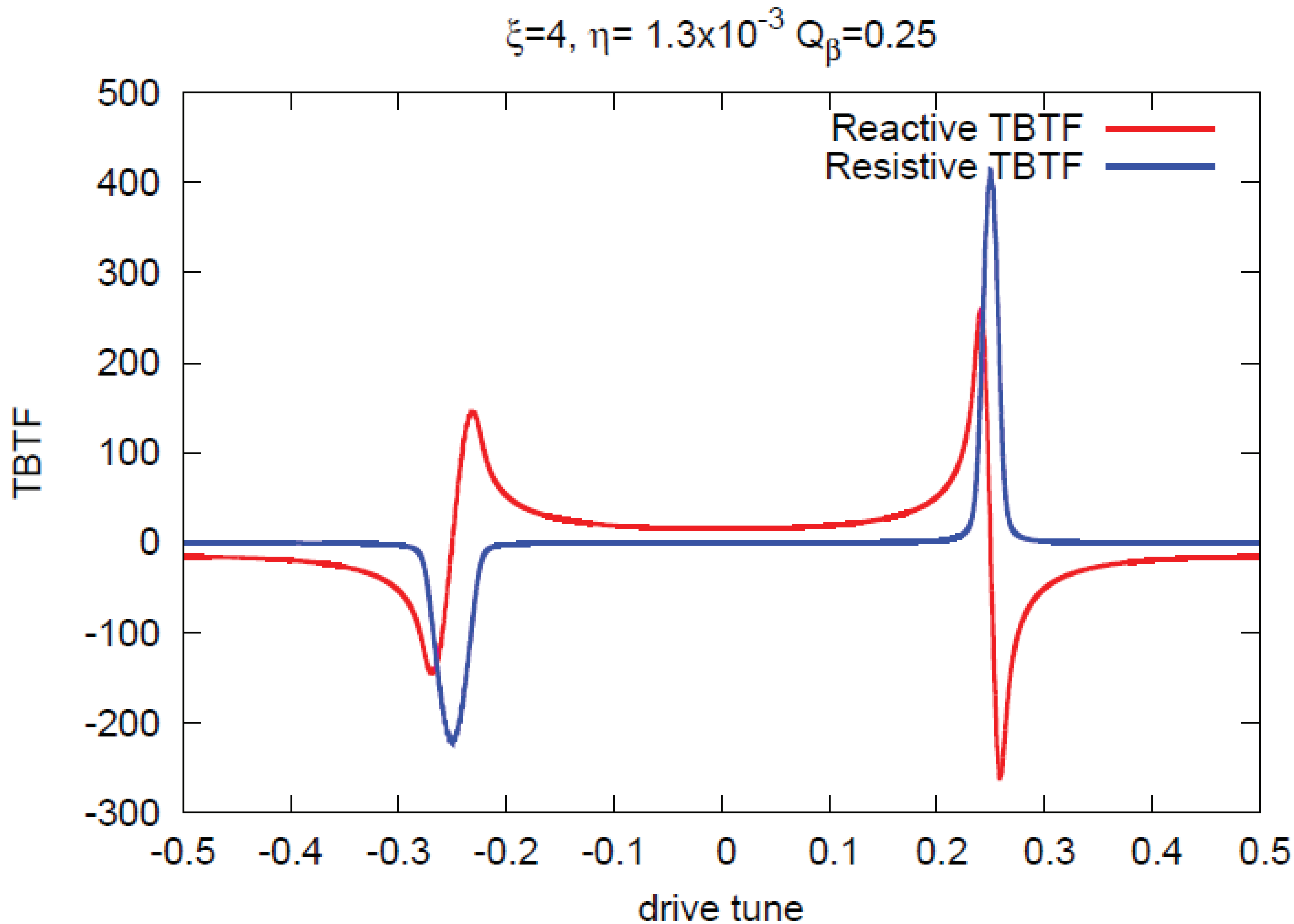
BTF with $f(\tau)$ for n=2. Space charge and a Q=5 resonator wake with $f_r\tau_b$=6. Solid lines are tracking results, crosses from Vlasov.



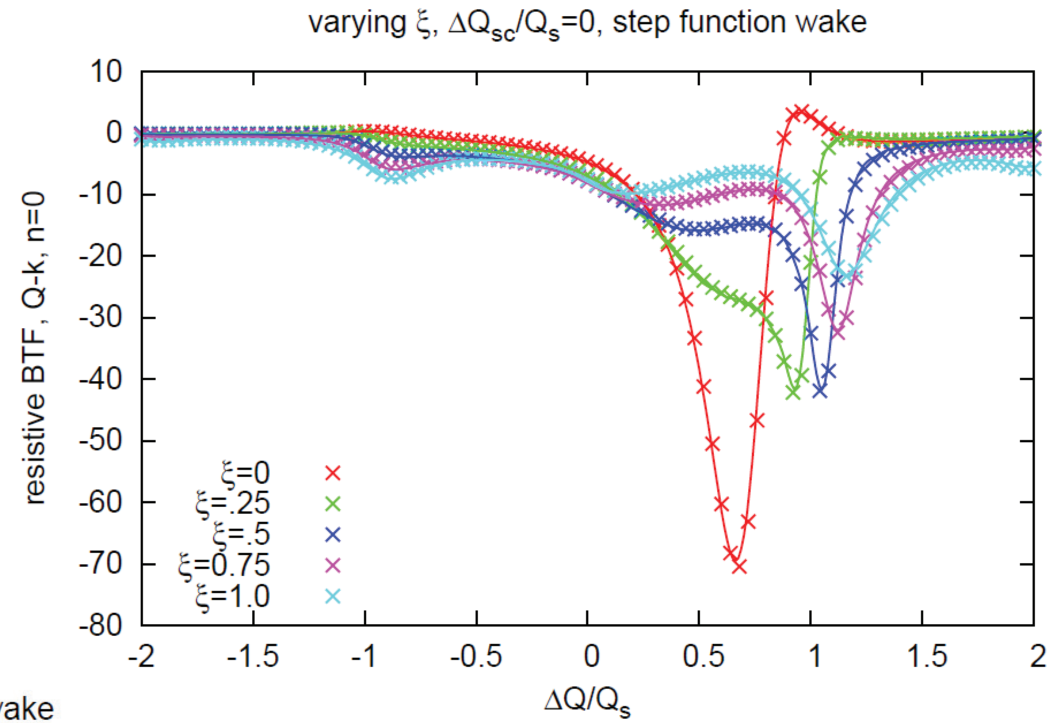


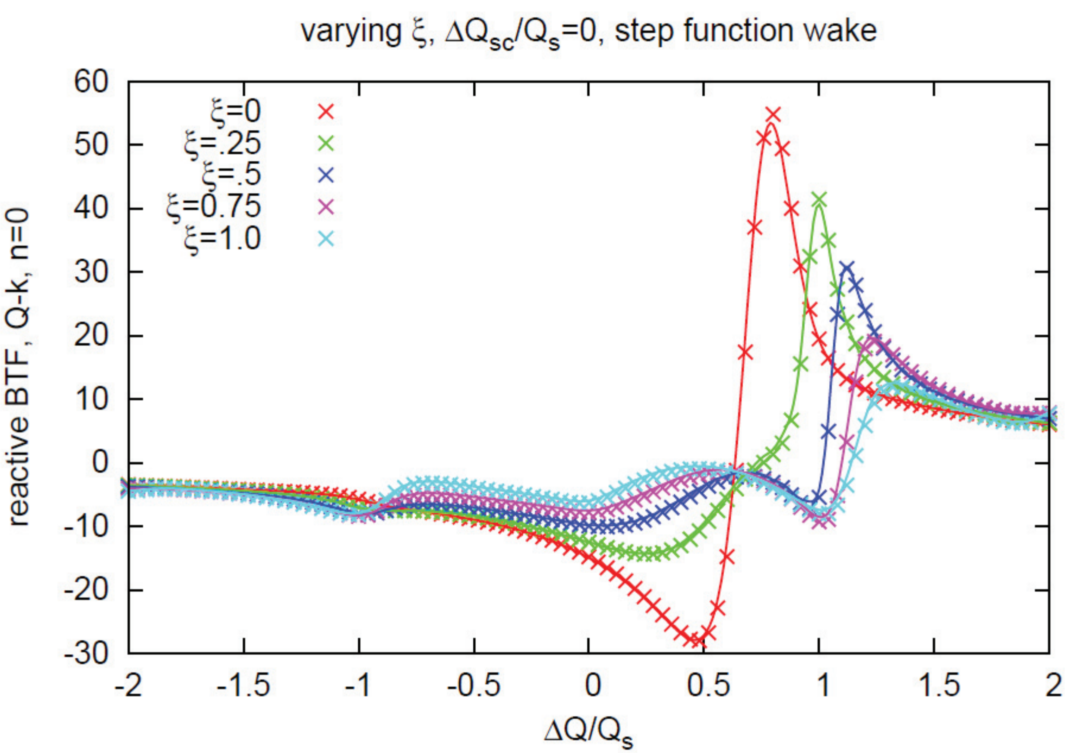


Low frequency resolution BTF taken at frequency equal to inverse bunch length (n=2).



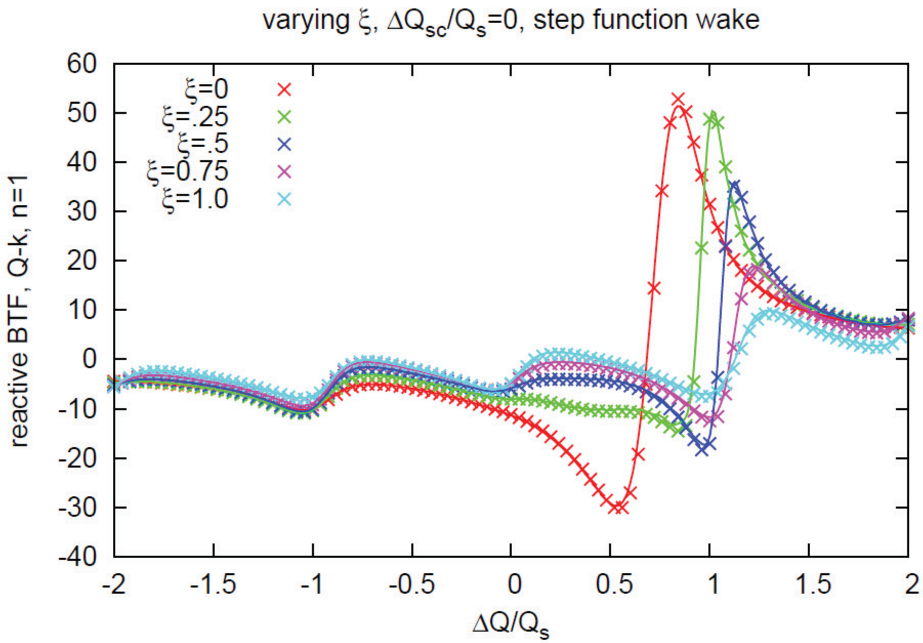
low frequency BTF (n=0). Has a step function wake for various chromaticities.

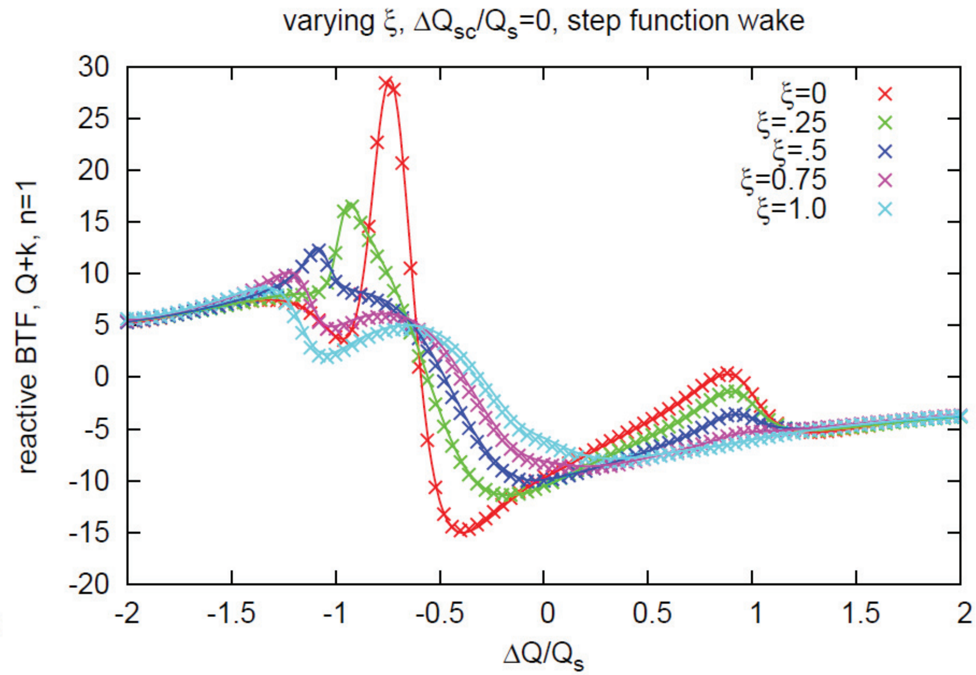


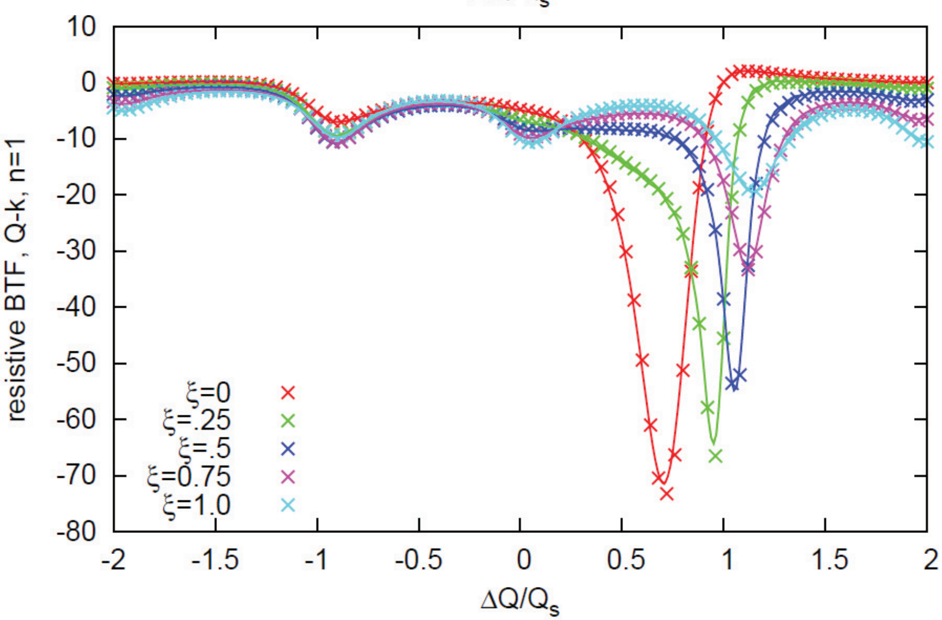


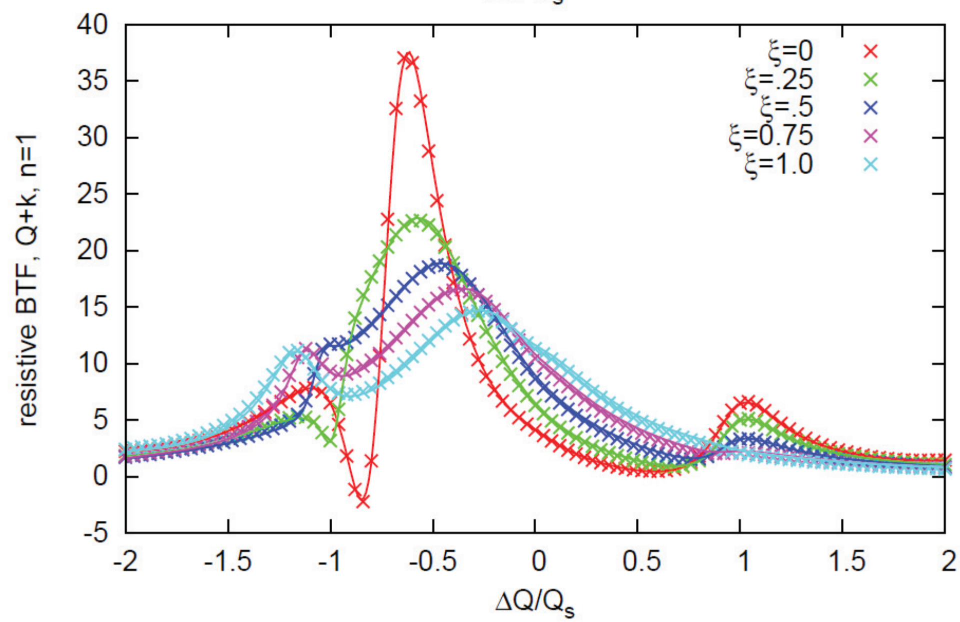
BTF with same wake and chrom but with $f=1/4\tau_b$.(n=1)

upper and lower sidebands differ.

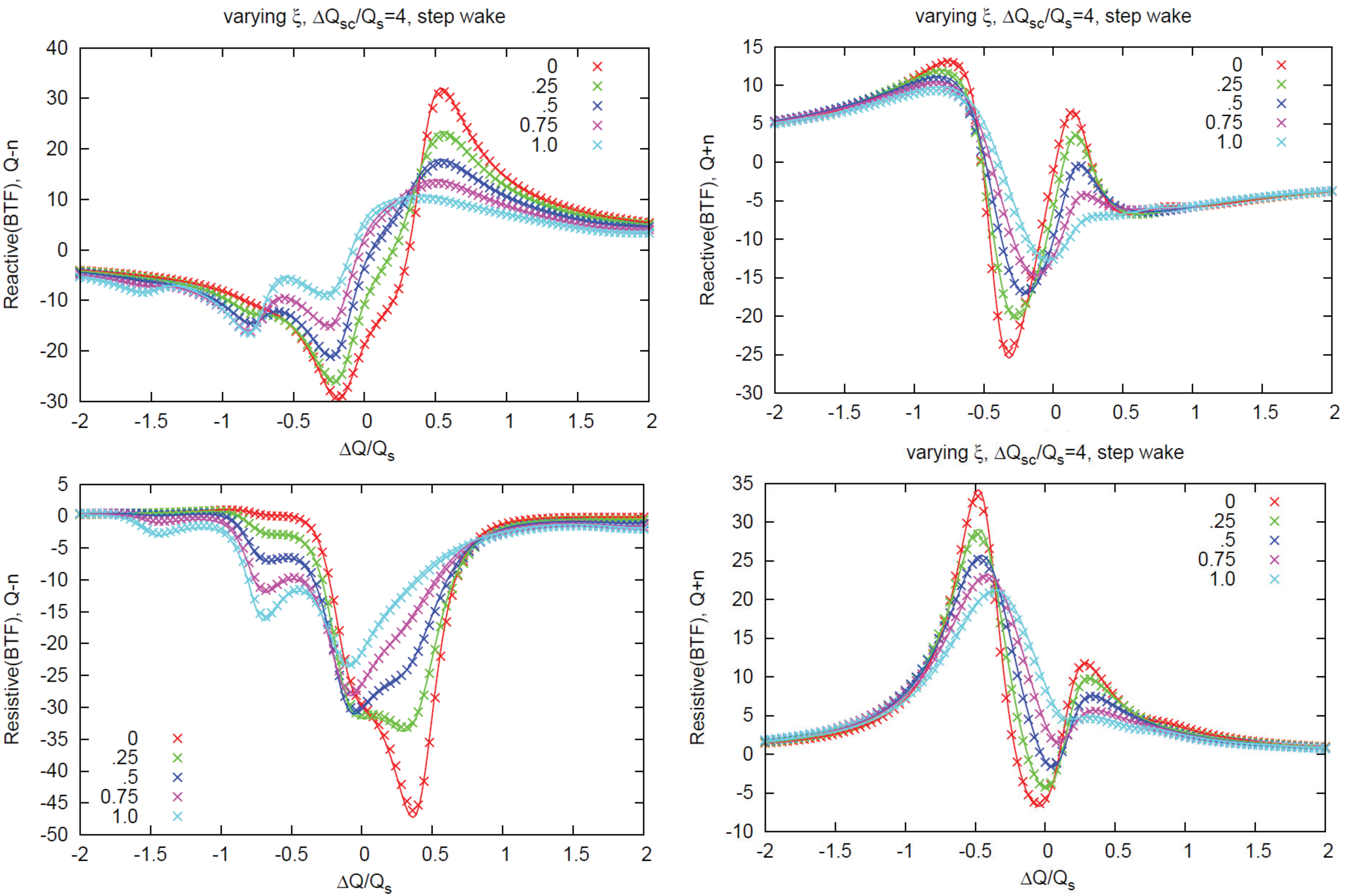








Like previous but now including space charge.



Conclusions

- Transverse beam transfer functions can be calculated reliably using either simulations or the Vlasov equation.
- Only 2 tracking runs are needed to get the full TBTF for bunches short compared to the machine circumference.
- The Vlasov approach requires more CPU power than tracking for comparable results. This is especially true with large space charge.

Discretize with linear interpolation

$$D_1(m\Delta) = \sum_{n=-N}^{N} M_{m,n} F(n\Delta)$$

$$M_{m,n} = \int_0^{\sqrt{\tau_b^2 - m^2\Delta^2}} du K(a) \sum_{p=1,2} e^{iG(a,\psi_p)} \int_{\psi_p}^{\psi_p+2\pi} e^{-iG(a,\phi)} T(n - \frac{a}{\Delta}\sin\phi) d\phi$$

$$a = \sqrt{u^2 + m^2\Delta^2}, \qquad T(x) = 1 - |x| \text{ for } |x| < 1, \qquad \sin\psi_p = m\Delta / a$$

The matrix element is left as a 2D integral. The driving terms for upper and lower sidebands are $F_0(\theta,\tau) = \exp(-iQ\theta \mp in\omega_0\tau)$

BTF with f(τ) for n=1. Solid lines are tracking results. Crosses are from Vlasov. Space charge is the collective effect.