STUDY AND COMPARISON OF THE METHOD OF MOMENTS AND THE SINGLE LEVEL FAST MULTIPOLE METHOD FOR 2D SPACE CHARGE TRACKING

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Abstract

Strong space charge is a significant impediment in charged particle beam physics, particularly at the high intensity frontier. For future applications, where particles must occupy the smallest region possible, quickly and accurately and efficient modeling space charge modeling is essential, for instance, to minimize the space charge contribution to beam dispersion. In this paper, we study and compare the performance for the method of moments (MoM) and the single-level fast multipole method (SLFMM) in 2D. The method of moments has been widely used to solve computational electromagnetic problems but assumes a series-expandable smooth distribution function, limiting its reliability in some cases. The fast multipole method was more recently developed and shows remarkable accuracy with difficult beam distributions. We demonstrate these methods using a simplified version of the University of Maryland electron ring (UMER). We present some multi-particle tracking results obtained using these methods. Future work will study the space charge inclusive transfer maps calculated from these methods.

INTRODUCTION

The general N body problem, although simple in principle, is computationally very challenging. The simplest method uses direct pair-wise interaction computations, where the complexity behaves like $O(N^2)$, and gives a machine precision solution for each pair. The mean field methods almost always involve some kind of numerical approximation. Two methods in particular were developed that approximate our solution to a significant degree of accuracy and reduce the scaling to $O(NlogN)$ or even $O(N)$ [1, 4, 5]. The method of moments (MoM) has been successfully used in several cases [1, 2]. As expected, it is most accurate for smooth unimodal distributions, where the relatively low order moments reliably approximate the distribution [4]. The fast multipole method (FMM), recently developed by Greengard and Rokhlin in 1987 [5], shows remarkable accuracy with arbitrary distributions, overcoming some weaknesses of the MoM. We are mainly interested in efficiently and accurately modeling the nonlinear effects present in a space charge dominant beam. The purpose of this paper is to evaluate and compare the MoM and FMM performance for a test case involving a simplified model of UMER.

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IDEALIZED UMER

The University of Maryland electron storage ring has been designed and built specifically for the study of space charge dominated, high intensity beams in a localized, inexpensive setting. Some of the design specifications are shown in Table 1, with our specific settings in the lower section [3]. The design is such that the electron bunches fill the ring, allowing a quasi-2D approximation. We included only the main recirculation (RC) sections, each consisting of two equivalent FODO sections with a guiding dipole in the middle of each FODO section.

Table 1: The UMER Design Specifications are Listed in the Top Half and Our Relevant Simulation Conditions are in the Bottom Half

Circumference	11.52 m
Electron energy	10 keV
$\beta(=\frac{v}{c})$	0.2
Current	1–100 mA
Aperture	2.95 cm
Pulse Length	40 ns
Lap time	197 ns
Pulse repetition rate	60 Hz
Turns	30
Simulated current	6 mA
FODO period	0.32 m
# of FODO sections	36
Dipole bend	10°
Dipole effective length	3.82 cm
Quadrupole current	1.826 A
Quadrupole effective length	5.164 cm
Initial emittance (X,Y)	(5.28×10^{-8},4.94×10^{-8})
Integration region	5.25σ

RESULTS

MoM

Reference [1] explains the details of our implementation of the MoM and FMM. For the MoM, there are three separate orders to which we will refer. The space charge order (SO) is the order of computation for the space charge map. The map order is the order of computation for the regular transfer map or equivalently the order of the

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differential algebra vectors (DAO).The moment order (MO) is the order of the sample moments used to model the particle distribution. The main purpose for the development of the MoM was the ability to extract self-consistent transfer maps that include space charge. However, the same method can be used for tracking space charge dominated beams. On the other hand, it is not clear how accurate is the MoM for tracking. Therefore, we focus on tracking capabilities in this paper. We started with a spatially uniform, angularly Gaussian beam that almost fills the ring acceptance.

In principle, the map order and space charge order do not have to be equal. We performed a study on the effects of mismatched DAO and SO, up to $DAO = SO + 3$, with overall maximum order at 18. We also constrained the SO and MO to be equal. We first examined the final emittances of our 6 mA electron beam. Figure 1 shows our final emittances. We calculated the emittances in X and Y using the standard deviations in position and angle of the beam and plot the total emittance. The results show that high order space charge effects are important, as expected, while no other high order effects are present in the ring.

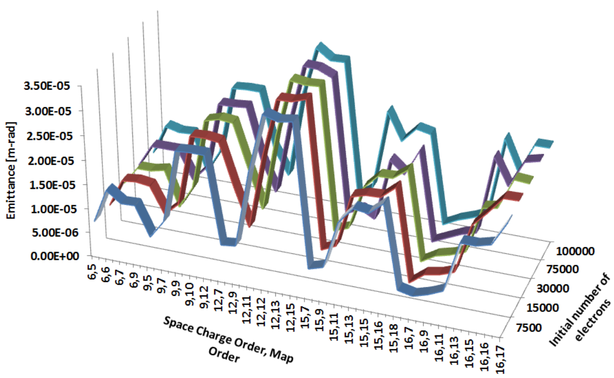

Figure 1: Comparison of Final Emittance [m-rad] vs. Space Charge Order, Map Order vs. N_p using the moment method for the UMER model.

In Figure 1, we compare the final total emittance with the space charge order, map order and number of particles, N_p. The emittance grows with map order and space charge order. The growth is mainly in the Y emittance. The variation in the plateau heights reflect the well-known fact about the optimal truncation order, i.e. best space charge order. As detailed in [1], there are two competing effects: increasing the space charge order improves the modeling of the distributions in the ideal case, but the finite number of particles introduce noise, which affect the sample moments that in turn decrease the accuracy due to the large condition number of the matrix that embodies the linear relationship between the moments and the coefficient set of the distribution function expansion. As shown in Table 2, the final emittances also reflect this slight instability of the moment method tracking, with optimal order seen around 12. The evolution of the particle distribution showed significant formation of halos and chaotic regions. The differences are minor between N_p. Our beam lifetimes showed similar behavior to the emittance, low survival at low map order, and approximately same survival once DAO equals SO. The timing results for the idealized UMER model are shown in Figure 2. We only show three representative order sets for each N_p.

Table 2: Averages and Standard Deviations of the Final Emittances at Plateau for SO 12, Sampling DAO 12,13,15 for $N_p = 7500, 15000, 30000, 75000, 100000$ After 30 Turns. Convergence is achieved in map order.

N_p	Average [m-rad]	σ
7500	3.399×10^{-5}	4.376×10^{-7}
15000	3.433×10^{-5}	5.79×10^{-7}
30000	3.420×10^{-5}	1.996×10^{-7}
75000	3.347×10^{-5}	7.651×10^{-7}
100000	3.308×10^{-5}	1.147×10^{-6}

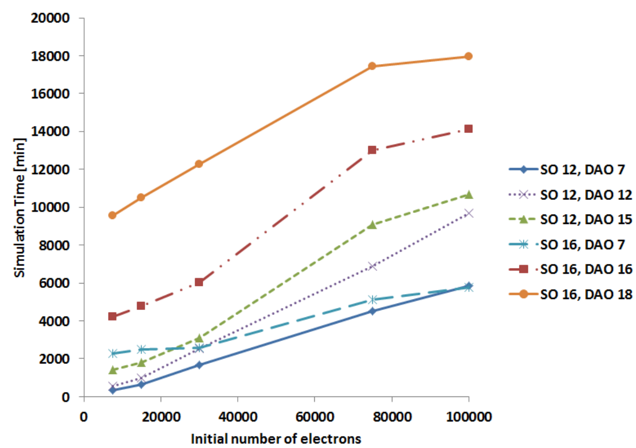

Figure 2: Comparison of Simulation time for 30 turns in mins using the moment method for the UMER model.

As shown in Figure 2, the simulation time exhibits roughly linear behavior. At low map, high space charge order, the number of particles decreases due to higher order space charge terms being truncated which would cancel strong lower order terms. This significantly decreases simulation time and leads to the kinks. Overall, we can achieve accurate results at high orders but expect unreliable predictions for low space charge order, showing the importance of higher order space charge terms.

FMM

Our implementation of the fast multipole method is one of the few using the differential algebraic framework. There are some key concepts to the FMM calculation worth noting here, but the full details may be found in a number of references [1, 5, 6]. We will refer to the number of boxes as N_{box}. The current box of interest, the nth box, will be designated our target box and all others will be designated sources. We use the single level FMM where we subdivide the space only once, into N_{box} equal size boxes. For the potential calculation, each nonempty source box is Taylor expanded with respect to the target box. Thus, the simulation time depends on N_{box}, N_p, SO, and the number of coefficients for the expansion in the FMM akin to the moment order, which we will call the FMM order (FO).

Due to time constraints, we decided to only run the FMM in tracking mode for 1 turn rather than 30 as for the MoM and at $N_p = 7500, 15000$ to compile the results in this paper. Preliminary tests showed the emittances from the FMM stabilize extremely quickly, usually within 1 turn, and the time per turn behaves consistently. Beam lifetime exhibits similar behavior, but we are hesitant to assume consistent lifetime for all FMM orders and N_p. For this study, our simulation has $N_{box} = 40^2$. A future study will test the dependence on N_{box}. Our results are shown in Figure 3 and Figure 4.

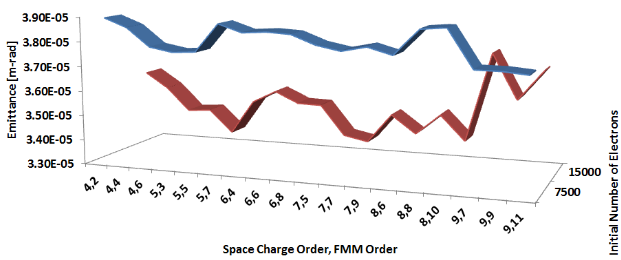

Figure 3: Final emittances [m-rad] after 1 turn vs. Space Charge Order, FMM Order using the fast multipole method.

Table 3: Averages and Standard Deviations of the Final Emit-tances at SO 9, Sampling FO 7,9,11, and $N_p = 7500$, 15000 After 1 Turn for the UMER Model Using the Fast Multipole Method

N_p	Average [m-rad]	σ
7500	3.746×10^{-5}	5.315×10^{-8}
15000	3.691×10^{-5}	9.185×10^{-7}

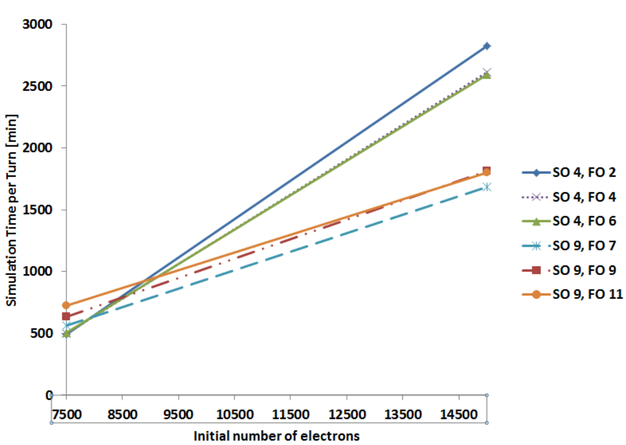

Figure 4: Simulation time per turn [mins] vs. initial number of electrons for select space charge orders, FMM orders using the fast multipole method.

In Figure 4, the simulation times per turn using the FMM tracking mode slightly depend on the orders. We reserve judgment on its N_p dependence at this time. In Figure 3, the final emittance after 1 turn for the idealized UMER model exhibits a small variation. Table 3 shows the averages and standard deviations for the final emittances at SO 9 as a function of FO for $N_p = 7500, 15000$. The relatively erratic emittance at SO 9, FO 7-11, $N_p = 15000$ is due to requiring more turns for the emittance to stabilize. The evolution of the distributions showed similar behavior to what was shown with the moment method, but the variation is no longer significant, thus it may be attributed to the chaotic nature of space charge.

CONCLUSIONS

Our study shows the various orders strongly affect our MoM results. Convergence in terms of moment and map order are achieved. There is no convergence in space charge order. Our optimum SO is 12 for SVD cutoff parameter of 1×10^{-12}. The MoM shows little dependence on N_p. From its emittances, The FMM shows slightly more dependence on N_p but stabilizes for large N_p.

Our FMM shows stable results. There is a hybrid mode implemented using FMM for tracking and MoM for map extraction but has not been studied thoroughly yet. More detailed comparisons will be forthcoming. We plan to optimize our FMM implementation for reduced runtime and further improve our MoM implementation.

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