Numerical and analytical studies of matched kinetic quasi-equilibrium solutions for an intense charged particle beam propagating through a periodic focusing quadrupole lattice.*

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\[ B'_{q}(z) \]

\[ B'_{q}(x) = B'_{q}(z)(y\hat{e}_x + x\hat{e}_y) \]

\[ F_{foc}(x) = -\kappa_{q}(z)(x\hat{e}_x - y\hat{e}_y) \]

\[ \kappa_{q}(z) = \frac{ZeB'_{q}(z)}{\gamma m\beta c^2} \]

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Particle trajectories are fast oscillations on top of slow motion of guiding centers.
Motivations

- Identifying regimes for quiescent propagation of intense beams over long distances is of the great interest in accelerator research.

- In particular, the development of systematic theoretical approaches that are able to treat self-consistently the applied oscillating force and the nonlinear self-field force of the beam particles simultaneously has been a major challenge of modern beam physics.

- To determine matched-beam quasi-equilibrium distribution functions one needs to determine a dynamical invariant for the beam particles moving in the combined applied and self-generated fields.

- Typically, it is advantageous to eliminate fast oscillations from formalism and describe complex beam particle motion is in a new non-oscillating coordinates.

- Standard Hamiltonian techniques are cumbersome due to use of mixed oscillating and non-oscillating independent variables.
Abstract

- New Hamiltonian averaging technique has been developed, which incorporated both the applied periodic focusing force and the self-field force of the beam particles.

- Newly developed technique is specially designed to avoid use of oscillating independent variables. The method is analogous to the Lie transform methods in using only non-oscillating independent variables. At the same time the new approach retains the advantages of simplicity of Hamiltonian methods.

- Using the particle’s vacuum phase advance $\epsilon = \sigma_v/2\pi$ treated as a small parameter, the perturbative series are constructed to transform away the fast particle orbit oscillations and obtain the average Hamiltonian accurate to order $\epsilon^3$.

- The average Hamiltonian is an approximate invariant of the original system, and can be used to determine self-consistent beam quasi-equilibrium solutions that are matched to the focusing channel.

- Making use of this new method equations determining the average self-field potential for general boundary conditions has been obtained for the first time by taking into account the average contribution of the charges induced on the boundary.
Vlasov-Poisson system of equations

- The transverse dynamics of the intense charged particle beam can be described by the nonlinear Vlasov-Poisson system of equations for the beam distribution function \( f(x, p, s) \) and the normalized self-field potential \( \Psi(x, s) \).

- Here \( s = v_b t \) is the longitudinal coordinate, and \( v_b \) is the directed beam velocity. The function \( f(x, p, s) \) satisfies the nonlinear Vlasov equation

\[
\frac{df}{ds} = \frac{\partial f}{\partial s} + \sum_{\alpha=1}^{2} \frac{dx^\alpha}{ds} \frac{\partial f}{\partial x^\alpha} + \sum_{\alpha=1}^{2} \frac{dp^\alpha}{ds} \frac{\partial f}{\partial p^\alpha} = 0,
\]

- where the particle equations of motion are give by

\[
\frac{dx^\alpha}{ds} = \frac{\partial H}{\partial p^\alpha}, \quad \frac{dp^\alpha}{ds} = -\frac{\partial H}{\partial x^\alpha},
\]

- The Hamiltonian \( H(x, p, s) \) describes the particle motion in a force field that is the sum of a linear, externally applied, transverse focusing force with components \( F^{\alpha}_{\text{foc}} = -\kappa(s)\eta^\alpha x^\alpha \) and the normalized self-field potential \( \Psi(x, s) \) is calculated self-consistently using Poisson’s equation.

- \( \kappa(s) \) is the focusing field strength \( \eta^1 = 1, \eta^2 = -1. \)
The Hamiltonian $H$ for the particle motion

- It is convenient to introduce the re-normalized variables $\bar{x} = x/a$, $\bar{s} = s/S_0$, $\bar{\kappa}(s) = \kappa(s)/\kappa_0$, $\bar{p} = p/(a\kappa_0 S_0)$ and $\bar{f} = (f/N)a^4(\kappa_0 S_0)^2$, where $S_0$ is the characteristic period of the applied focusing force, $a$ is the characteristic transverse beam dimension, and $\kappa_0$ is the characteristic value of the lattice function $\kappa(s)$.

- After the normalization, the Hamiltonian becomes

$$\bar{H}(\bar{x}, \bar{p}, \bar{s}) = \bar{\kappa}(\bar{s}) \left[ \eta^\alpha \bar{x}^\alpha \bar{x}^\alpha \right] + \epsilon \left\{ \frac{[\bar{p}^\alpha \bar{p}^\alpha]}{2} + \int L(\bar{x}, \bar{x}') \bar{f}(\bar{x}', \bar{p}', s') D\bar{x}' D\bar{p}' \right\},$$

where $\epsilon \equiv S_0^2 \kappa_0$ and $\int d\bar{x} d\bar{p} \bar{f} = 1$.

- We adopt the notation $[x^\alpha x^\alpha] \equiv \sum_{\alpha=1}^{2} x^\alpha x^\alpha$ and $\int dx_1 dx_2 Z = \int Dx Z$.

- Green’s function $L(\bar{x} - \bar{x}')$ satisfies the equation

$$\left[ \frac{\partial}{\partial \bar{x}^\alpha} \frac{\partial}{\partial \bar{x}^\alpha} \right] L(\bar{x} - \bar{x}') = -s_b \delta(\bar{x} - \bar{x}').$$

Here, $s_b = 2K/(\kappa_0 S_0)^2 a^2 = (4\pi q^2 N/ a^2 \gamma_b^3)/(\kappa_0 S_0 v_b)^2$ is a measure of the beam space-charge intensity.

- For the beam confined by the external focusing force the maximum value of $(s_b)^{max} \sim 1$. 

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Perturbative Hamiltonian Transformation Method

- We search for a time-dependent canonical transformation of the form
  \((x^\alpha, p^\alpha, H, s) \rightarrow (Q^\alpha, P^\alpha, K, s)\)
  \[x^\alpha = x^\alpha(Q, P, s),\]
  \[p^\alpha = p^\alpha(Q, P, s),\]
  with time-independent transformed Hamiltonian \(K(Q, P)\).
- For every canonical transformation there is a function \(S\) that satisfies the differential relation
  \[\[p^\alpha dx^\alpha\] - Hds = dS + [P^\alpha dQ^\alpha] - Kds.\]
- Express \(S = U + p_0(Q, P, s)^\alpha(x - Q)^\alpha\), where \(U(Q, P, s)\) and \(p_0(Q, P, s)\) are functions of the new phase-space variables.
  \[\left[(x - Q)^\alpha \frac{\partial p_0^\alpha}{\partial P^\beta}\right] = \left[(p - p_0)^\alpha \frac{\partial(x - Q)^\alpha}{\partial P^\beta}\right] - \frac{\partial U}{\partial P^\beta},\]
  \[(p - P)^\beta = -\left[(p - p_0)^\alpha \frac{\partial(x - Q)^\alpha}{\partial Q^\beta}\right] + \frac{\partial U}{\partial Q^\beta} + \left[(x - Q)^\alpha \frac{\partial p_0^\alpha}{\partial Q^\beta}\right],\]
  \[K - H = -(p - p_0)^\alpha \frac{\partial(x - Q)^\alpha}{\partial s} + \frac{\partial U}{\partial s} + (x - Q)^\alpha \frac{\partial p_0^\alpha}{\partial s}.\]
The distribution function in the new coordinates

- Particle conservation in the phase-space volume $DxDP$ under the transformation

$$F(Q, P, s) \ DQDP = f(x, p, s) \ DxDp.$$  

- For a canonical transformation, the phase-space volume is conserved according to $DxDP = DQDP$, and therefore $F(P, Q, s) = f[x(Q, P, s), p(Q, P, s), s]$.

- The new distribution function satisfies the Vlasov equation $dF/ds = 0$.

- For a time-independent Hamiltonian, there exists a trivial solution to the Vlasov equation, $F = G[K(Q, P)]$ for arbitrary function $G$.

- The matched solution can be found from $f(x, p, s) = G\{K_G[Q_G(x, p, s), P_G(x, p, s)]\}$.

- For solutions of this form, the Hamiltonian becomes

$$H(x, p, s) = \kappa(s)\eta^{\alpha}x^{\alpha}x^{\alpha} + \epsilon\left\{\frac{p^{\alpha}p^{\alpha}}{2}\right\} + \int L[x, x(\bar{Q}, \bar{P}, s)]G[K(\bar{Q}, \bar{P})]D\bar{Q}D\bar{P}.$$
Iterative procedure of finding the canonical transformation in terms of the small parameter 
\[ \epsilon \sim \sigma_v/2\pi \ll 1 \]

- Make an expansion

\[
p = p_0(Q, P, s) + \sum_{n=1}^{\infty} \epsilon^n p_n(Q, P, s),
\]

\[
x = Q + \sum_{n=1}^{\infty} \epsilon^n x_n(Q, P, s),
\]

\[
U = U_0(Q, P, s) + \sum_{n=1}^{\infty} \epsilon^n U_n(Q, P, s),
\]

\[
K = K_0(Q, P) + \sum_{n=1}^{\infty} \epsilon^n K_n(Q, P),
\]

where \( p_n, x_n, U_n \) and \( K_n (n = 0, 2, \ldots) \) are functions to be determined by the iterative procedure.

- We expand the function \( H \) according to

\[
H(x, p, s) \equiv H[Q + \sum_{n=1}^{\infty} \epsilon^n x_n, p_0 + \sum_{n=1}^{\infty} \epsilon^n p_n, s]
\]

\[
= H_0(Q, P, s) + \sum_{n=1}^{\infty} \epsilon^n H_n(Q, P, s).
\]
Iterative procedure up to second order in 
\[ \epsilon \sim \sigma v / 2\pi \]

- For the Hamiltonian function \( H(x, p, s) \) the functions \( Z_n \) depend only on \( p_k \) and \( x_k \), with \( k < n \).

- Because \( K_0 = 0 \), the average Hamiltonian \( K \) has the form \( K = \epsilon (K_1 + \epsilon K_2 + \epsilon^2 K_3 + \cdots) \). The \( \epsilon \) in front of the bracket renormalizes the time scale, so that the average dynamics occurs on the slow time-scale \( Q = Q(\epsilon s) \) and \( P = P(\epsilon s) \).

- Therefore, to determine the trajectories \( x(s) \) and \( p(s) \) valid to second order in \( \epsilon \), we need to determine the average Hamiltonian \( K \) valid up to the third order in \( \epsilon \).
Canonical transformation up to second order in 
\[ \epsilon \sim \sigma_\nu/2\pi \]

\[ x^\alpha = Q^\alpha - \epsilon \kappa^{(2)} \eta^\alpha Q^\alpha + \epsilon^2 \left\{ 2\kappa^{(3)} \eta^\alpha P^\alpha + (\kappa \kappa^{(2)})^{(2)} Q^\alpha \right\}, \]

and

\[ p^\alpha = \left\{ P^\alpha - \kappa^{(1)} \eta^\alpha Q^\alpha \right\} + \epsilon \left\{ \kappa^{(2)} \eta^\alpha P^\alpha + (\kappa \kappa^{(2)})^{(1)} Q^\alpha \right\} + \]

\[ + \epsilon^2 \left\{ (3 < (\kappa^{(2)})^2 > - 2(\kappa \kappa^{(3)})^{(1)} - (\kappa \kappa^{(2)})^{(2)}) P^\alpha \right\} + \]

\[ + (\kappa^{(3)} < (\kappa^{(1)})^2 > - (\kappa (\kappa \kappa^{(2)}))^{(2)}(1) \eta^\alpha Q^\alpha \right\} + \epsilon \frac{\partial}{\partial Q^\alpha} \psi(Q)^{(1)}. \]

- Here, \( < A > \equiv (1/S) \int_{s_0}^{s} ds A(\bar{s}), \) and \( \ll A \gg \equiv A - < A >. \) Here, we also introduce the notation \( A^{(0)} \equiv \ll A \gg \) and

\[ A^{(n)} \equiv \ll \int ds A^{(n-1)} \gg, \] for \( n \geq 1 \)

- The time-independent Hamiltonian correct to the third order in \( \epsilon \) is

\[ K = \epsilon \left\{ \left[ \frac{P^\alpha P^\alpha}{2} \right] (1 + 3\epsilon^2 < (\kappa^{(2)})^2 >) + \frac{[Q^\alpha Q^\alpha]}{2} ( < (\kappa^{(1)})^2 > + \epsilon^2 < (\kappa \kappa^{(2)}))^{(1)} > + < \psi(Q) > \right\} \]

\[ \psi(Q) = \int D\bar{Q}L(Q_\alpha (1 + \epsilon \eta^\alpha \kappa^{(2)}), \bar{Q}_\alpha (1 + \epsilon \eta^\alpha \kappa^{(2)})) n(\bar{Q}) \]
Approximate vs. Exact

- Plots of $\Delta Q(s) = Q(s) - Q_{tr}(x, p, s)$, $\Delta P(s) = P(s) - P_{tr}(x, p, s)$ and $\Delta K/\epsilon = [K - K_{tr}(x, p, s)]/\epsilon$ as functions of the normalized variable $s/S$ over the interval $[0, 1/\epsilon^4]$ for the choice of dimensionless parameters $s_b = 0$, $\bar{\kappa} = \kappa = 1$, $\epsilon = 0.1$ ($\sigma_v = 25^0$).

- The approximate expression for the square of the vacuum phase advance $\sigma_v^2$ valid up to forth order in small parameter $\epsilon$, i.e.,

$$\sigma_v^2 = S^2 \epsilon^2 \left\{1 + 3\epsilon^2 < (\kappa^{(2)})^2 > \right\} \left\{ < (\kappa^{(1)})^2 > + \epsilon < \kappa (\kappa^{(2)})^2 > + \epsilon^2 < ((\kappa \kappa^{(2)})^{(1)})^2 > \right\}$$

- Plots of the normalized quantities $\sigma_v/\sigma_{sf}$ and $\sigma_v/\sigma_v^a$ versus the vacuum phase advance $\sigma_v$ for periodic step-function lattice with filling factor $\delta = 1/2$. 

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Evaluation of self-field potential

- Expanding correct to second order in the small parameter $\epsilon$, we obtain the expression for the average potential

$$< \Psi(Q) > = (1 + \epsilon^2 < v^2 >) \phi_0 + \epsilon^2 < v^2 > \left( \phi_1 - \left[ \eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] \phi_2 \right) + \frac{1}{2} \left[ \eta^{\alpha \beta} Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta} \right] \phi_0,$$

where the functions $\phi_0(Q)$, $\phi_1(Q)$ and $\phi_2(Q)$ satisfy the Poisson-type equations

$$\nabla_\perp^2 \phi_0 = -s_b n(Q),$$
$$\nabla_\perp^2 \phi_1 = -s_b \left( \left[ Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] + \frac{1}{2} \left[ \eta^{\alpha \beta} Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta} \right] \right) n(Q),$$
$$\nabla_\perp^2 \phi_2 = -s_b \left[ \eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] n(Q),$$

- In cylindrical coordinates $Q_1 = r \cos(\theta), Q_2 = r \sin(\theta)$

$$\left[ \eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha} \right] = Q^1 \frac{\partial}{\partial Q^1} - Q^2 \frac{\partial}{\partial Q^2} = \cos(2\theta) r \frac{\partial}{\partial r} - \sin(2\theta) \frac{\partial}{\partial \theta}.$$
Boundary conditions for self-field potential

- Needs to specify some boundary surface in the coordinate space \((Q^1, Q^2)\) and certain boundary conditions on this boundary.

- It is convenient to designate this boundary surface to be a surface in the coordinate space \((Q^1, Q^2)\), where the function \(L(Q, \bar{Q})\) satisfies the same boundary conditions as the function \(L(x, \bar{x})\) in the coordinate space \((x^1, x^2)\).

- In that case, the boundary conditions for \(\phi_0(Q), \phi_1(Q)\) and \(\phi_2(Q)\) in the coordinate space \((Q^1, Q^2)\) are the same as the boundary conditions for the Green’s function \(L(Q, \bar{Q})\).

- Note that this boundary surface in the coordinate space \((Q^1, Q^2)\) becomes a surface that oscillates around the boundary surface in the coordinate space \((x^1, x^2)\).

- Because the two surfaces differ, the average potential \(\bar{\Psi}(Q)\) in the coordinate space \((Q^1, Q^2)\) does not satisfy the same boundary conditions as the un-averaged potential in the coordinate space \((x^1, x^2)\).
Equilibrium self-field potential

- In equilibrium

\[ n(Q) = \int dPG(K) = \int dPG(K_{kin} + K_{ext} + \Phi_0 + \epsilon^2 < v^2 > \bar{\Psi}_1). \]

- For perfectly conducting cylindrical boundary of radius \( R_w \),

\[ \bar{\Psi}_1 = p(r) + \cos(4\theta)q(r), \]

where

\[
\begin{aligned}
\left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} + s_b n'_0(r) \right) p(r) &= -s_b \frac{8}{R_w^4} \int_0^{R_w} drr^3 n_0(r), \\
\left( \frac{1}{r} \frac{d}{dr} r \frac{d}{dr} - \frac{16}{r^2} + s_b n'_0(r) \right) q(r) &= -2s_b \left( n_0 + \frac{4}{r^2} \int_0^r d\bar{r} \bar{r}n_0(\bar{r}) - \frac{12}{r^4} \int_0^r d\bar{r} \bar{r}^3 n_0(\bar{r}) \right)
\end{aligned}
\]

- with boundary conditions

\[
\begin{aligned}
p(R_w) &= -\frac{2s_b}{R_w^2} \int_0^{R_w} drr^3 n_0(r), \quad q(R_w) = -s_b \left( \frac{2}{R_w^2} \int_0^{R_w} d\bar{r} \bar{r}^3 n_0(\bar{r}) - \frac{1}{2} \int_0^{R_w} d\bar{r} \bar{r}n_0(\bar{r}) \right).
\end{aligned}
\]

- Here, \( n_0(r) \equiv \int dPG[K_{kin} + K_{ext} + \Phi_0(r)] \) and \( n'_0 \equiv \partial n_0(r)/\partial \Phi_0 \) with \( \Phi_0(r) \) determined self-consistently through

\[
\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \Phi_0 = -s_b n_0(r),
\]
Thermal equilibrium distribution

- Zero-order thermal equilibrium average beam profile

\[ \bar{n}_0 = n_0(r)/n_0(0) = \exp\left(-\frac{k^2r^2}{2} + \Phi_0(r) \right). \]

- For thermal equilibrium distribution normalized quantities are function of two parameters

\[ \epsilon \sim (\sigma_v/2\pi) \]

and beam intensity

\[ \bar{s} = \omega_p^2/2\omega_{sf}^2 < 1, \]

where \( \omega_p \) is the plasma density at the beam center and \( \omega_{sf} = (\sigma_v/2\pi)(v_b/S) \) is the average focusing frequency of the lattice.
Maxwellian distribution averaged density profile

- Plots of density profiles $\bar{n}_0(\rho)$ for different values of intensity parameter $\bar{s} = 0.1; 0.3; 0.5; 0.8; 0.9; 0.99; 0.999$.

- For $\bar{s} \ll 1$

  \[
  \bar{n}_0(\rho) \approx \left( 1 + \frac{\rho^2}{4\bar{s}} \right)^{\bar{s}} \exp\left( -\frac{\rho^2}{4\bar{s}} \right).
  \]

- For $\Delta = 1/\bar{s} - 1 \ll 1$

  \[
  \bar{n}_0(\rho) \approx \frac{\left[ 1 + \frac{1}{2}\Delta + \frac{1}{24}\Delta^2 \right]^2}{\left[ 1 + \frac{1}{2}\Delta I_0(\rho) + \frac{1}{24}[\Delta I_0(\rho)]^2 \right]^2},
  \]

  where $I_0(\rho)$ is zero order modified Bessel function.
Corrections to density profile for Maxwellian distribution

• Define RMS radius as

\[ R^2(\theta) = \frac{\int \rho^3 \bar{n}(\rho, \theta) d\rho}{\int \rho \bar{n}(\rho, \theta) d\rho}, \quad R_0^2 = \frac{\int \rho^3 \bar{n}_0(\rho) d\rho}{\int \rho \bar{n}_0(\rho) d\rho}. \]

• Relative change in RMS radius as \( \frac{\delta R(\theta)}{R_0} = \frac{R(\theta)}{R_0} - 1 \) can be expressed as

\[ \frac{\delta R(\theta)}{R_0} = -\epsilon^2 < v^2 > \cos(4\theta) A(\bar{s}), \]

where \( A(\bar{s}) \) is given by

\[ A(\bar{s}) = \frac{1}{2} \left[ \frac{\int_0^\infty \rho^3 \bar{q}(\rho) \bar{n}_0(\rho) d\rho}{\int_0^\infty \rho^3 \bar{n}_0(\rho) d\rho} - \frac{\int_0^\infty \rho \bar{q}(\rho) \bar{n}_0(\rho) d\rho}{\int_0^\infty \rho \bar{n}_0(\rho) d\rho} \right]. \]
Corrections to density profile, cont’d

- Plots of density profile correction \( \bar{q}(\rho)\bar{n}_0(\rho) \) for different values of intensity parameter \( \bar{s} = 0.3; 0.6; 0.9; 0.99; 0.999 \).

- Plots of function \( A(\bar{s}) \), (a) linear \( s \)-scale and (b) logarithmic \( s \)-scale.

- Here \( \rho_{1/2}(\bar{s}) \approx \ln \left[ \frac{C}{\Delta} \sqrt{2\pi \frac{C}{\Delta}} \right] \) is the beam edge radius \( \bar{n}_0(\rho_{1/2}) = 1/2 \) and \( C \approx 0.78 \).
Specific example of the beam inside of perfectly conducting cylindrical boundary

- Kapchinskij-Vladimirskij distribution \( G(K) = (\bar{n}_0/2\pi)\delta(K - K_0) \) with

\[
n_0(r) = \begin{cases} 
\bar{n}_0, & r < a, \\
0, & a < r \leq R_w.
\end{cases}
\]

- Beam radius \( r_b \) becomes weakly dependent on the angle \( \theta \) according to

\[
r_b(\theta) = a \left\{ 1 + \epsilon^2 < v^2 > \frac{\bar{s}}{(1 - \bar{s})} \left( \frac{(a/R_w)^4}{1 - \frac{2\bar{s}}{1-\bar{s}} \ln \frac{a}{R_w}} + \frac{\cos(4\theta)}{2} \frac{(a/R_w)^8}{1 + \frac{\bar{s}}{4(1-\bar{s})} \left( 1 - \left( \frac{a}{R_w} \right)^8 \right)} \right) \right\}
\]

- For \( R_w \to \infty \), \( \bar{\Psi}_1 = 0 \), and the total self-field potential inside the beam is given by zero order potential \( \Phi_0 \), which is what one would expect for a Kapchinskij-Vladimirskij distribution in free space \( (R_w \to \infty) \) which generates the constant beam density and linear self-field forces.

- For finite \( R_w \), the image charge oscillations produce additional contributions to the average self-field potential inside the beam, which lead to the octopole correction to the average beam radius.
Simulation studies

• The simulations results for a sinusoidal lattice $\kappa(s) = \kappa_0 \sin(2\pi s/S)$

• Beam intensity $s_b = 0.5$, wall radius $R_w = 5R_0$, and $\sigma_v = 25^0$ which corresponds to $\epsilon = 0.07$.

• Fourier spectrum $\sim \exp(im\theta)$ of the averaged over time relative RMS radius change $<\delta R(\theta)/R_0>$ as a function of azimuthal mode number $m$.

• Time history of $m = 4$ component of $\delta R(\theta, s)/R_0$. 

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Results and Conclusions

- New Hamiltonian averaging technique has been developed, which incorporated both the applied periodic focusing force and the self-field force of the beam particles.

- Using the particle’s vacuum phase advance $\epsilon = \sigma_v/2\pi$ treated as a small parameter, the perturbative series are constructed to transform away the fast particle orbit oscillations and obtain the average Hamiltonian accurate to order $\epsilon^3$.

- The higher order corrections allow us to extend the average formulaic results to larger vacuum phase advances approaching $\sigma_v \sim 90^0$ with accuracies to within several percent.

- The equations determining the average self-field potential have been derived for general boundary conditions by taking into account the average contribution of the charges induced on the boundary.

- The average equation can be used to find an approximate invariant of the original system, and can be used to determine self-consistent beam quasi-equilibrium solutions that are matched to the focusing channel.

- The time-dependent formulation can be used to describe collective beam dynamics which is slow in the transformed coordinates (slow compared to the period of the lattice).
Results and Conclusions, cont’d

- The corrections to the self-field are small, and therefore the "smooth-focusing" approximation for the self-field potential can be a good approximation even for moderate values of the vacuum phase advance.

- For example, for vacuum phase advance of $\sigma_v = 90^0$ the correction to the RMS radius of the beam described by a thermal equilibrium distribution arising from the corrections to the average self-field potential is of order 0.5%.

- Nonetheless, note that because the average self-field potential acquires an octupole component, the average motion of some beam particles becomes non-integrable and the trajectories become chaotic.

- This chaotic behavior of some of the beam particles may change the nature of the Landau damping (or growth) of collective excitations supported by the beam.

- Also, due to the presence of the extra non-axisymmetric terms in the equations for the self-field potential, the stability properties of different beam quasi-equilibria can change significantly.