SMOOTH APPROXIMATION OF DISPERSION WITH STRONG SPACE CHARGE *

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Abstract

We apply the Venturini-Reiser envelope-dispersion equations [1] to a continuous beam in a uniform focusing/bending lattice to study the combined effects of linear dispersion and space charge. Within this simple model we investigate the scaling of average dispersion and the effects on beam dimensions; we also introduce a generalization of the space-charge intensity parameter defined in [2] and apply it to the University of Maryland Electron Ring (UMER) and other machines. In addition, we present results of calculations to test the smooth approximation by solving the Venturini-Reiser original equations and also through simulations with the code ELEGANT [3].

INTRODUCTION

The combined effects of dispersion and space charge in circular accelerators are relevant especially to hadron machines. Examples of these are the proton storage rings for spallation neutron sources and the heavy-ion rings envisioned as drivers for heavy-ion fusion. One important issue, as in most high-energy machines, is the control of beam size and excursions to minimize beam degradation and avoid particle losses that lead to machine activation.

The problem of linear dispersion in the presence of strong space charge in continuous, i.e. unbunched, coasting beams has been addressed by Venturini and Reiser [1] and, independently, by Lee and Okamoto [4]. Both papers lead to generalized envelope-dispersion equations that differ in details of the emittance terms. With dispersion, the standard RMS emittance is not conserved; although the change is ordinarily very small, a new generalized conserved emittance can be defined that permits also the generalization of the RMS envelope equations. The new envelope-dispersion equations in the Venturini-Reiser theory (V-R equations for short) have been used by a number of authors to study dispersion matching, resonances, and halo formation [5].

In this paper, we revisit the uniform focusing/dispersion approximation of the V-R equations to investigate the dependence of average dispersion and beam dimensions on beam current and RMS fractional momentum error. Details of the derivations and additional calculations and general considerations will appear in a separate paper [6]. We apply the theory to beam transport with strong space charge at the University of Maryland Electron Ring (UMER) [7] and compare the results with solutions of the V-R equations and calculations with the code ELEGANT [3].

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SMOOTH APPROXIMATION OF THE V-R EQUATIONS

The uniform focusing approximation is a very useful design and theoretical tool for studying space charge dominated beam transport. The standard theory is discussed in its simplest form in [2], without including the effects of bending and energy spread, i.e. dispersion. In a similar way, it's straightforward to derive a set of algebraic equations for the average dispersion D and the beam semi-axes dimensions, a and b, from the envelope-dispersion equations in the V-R theory [1, 5, 6]:

$$\left[k_{x0}^2 - \frac{2K}{a(a+b)}\right] D - \frac{1}{\rho} = 0, \tag{1}$$

$$k_{x0}^2 a - \frac{2K}{(a+b)} - \frac{4\Delta^2}{a}\frac{D}{\rho} - \frac{\epsilon_x^2}{a^3\left(1-\frac{4\Delta^2 D^2}{a^2}\right)} = 0, \tag{2}$$

$$k_{y0}^2 b - \frac{2K}{(a+b)} - \frac{\epsilon_y^2}{b^3} = 0, \tag{3}$$

where K is the generalized beam perveance [2], ρ is the average machine radius, $k_{x0,y0}$ represent external focusing (constants in the smooth approximation), $\Delta \equiv \sqrt{\langle(\delta p/p_0)^2\rangle}$ is the RMS fractional momentum error or spread, a, b stand for the horizontal (bending plane) and vertical 2×RMS semi-axes dimensions, respectively, of an equivalent K-V beam [2], and $\epsilon_{x,y}$ are the 4×RMS unnormalized transverse emittances. Two major assumptions are made in deriving Eqs. 1-3: the emittances $\epsilon_{x,y}$ are conserved, and the beam envelope and dispersion functions are RMS-matched. We emphasize also that the term $(1-4\Delta^2D^2/a^2)$ that appears in the emittance part of Eq. 2 is important as it permits to obtain the correct beam size in the limit of zero current with non-zero dispersion [6].

From Eqs. 1-3, if $a \cong b$, $\epsilon_x = \epsilon_y$, $k_{x0} = k_{y0} \equiv k_0$, it's possible to write an (implicit) expression for a^2, the horizontal beam semi-axis squared:

$$a^2 = \frac{a_0^2}{\eta_{x\Delta}} + 4\Delta^2 D^2,\ \eta_{x\Delta} \equiv \sqrt{1-\frac{2K}{a(a+b)k_0^2}}, \tag{4}$$

where $a_0 \equiv \sqrt{\epsilon_x/k_0}$ is the horizontal beam radius in the limits of zero-current and zero-momentum error. In the limit of small momentum error but arbitrary space charge, the quantity $\eta_{x\Delta}$ reduces to the standard incoherent space charge tune depression (horizontal plane) that we denote by "η". In the same limit of small Δ we have $D \cong D_0/\eta^2$ (see Eq. 1 and Eq. 4, right-hand side).

An approximation for a^2 can be easily derived if we ignore the term $(1 - 4\Delta^2 D^2/a^2)$ in the denominator of the emittance part of Eq. 2; this would be justified only for space charge dominated transport. The result, from Eq. 2, is:

$$\frac{a^2}{{a_0}^2} \cong \frac{1-\eta^2}{2\eta} + \frac{{\xi_0}^2}{2\eta^2} + \sqrt{\left(\frac{1-\eta^2}{2\eta} + \frac{{\xi_0}^2}{2\eta^2}\right)^2 + 1}, \quad (5)$$

where $\xi_0 \equiv 2D_0\Delta/a_0$. The expression $(1-\eta^2)/2\eta$ can be recognized as the parameter "u" defined in [2].

The approximation in Eq. 5 overestimates the effect on beam radius when both space charge and momentum error are significant ($\eta \lesssim 0.3$, $\Delta \gtrsim 1\%$) and underestimates the effect for small current. However, the results are accurate within 10% or better [of the exact result from Eqs. 1-3] over a fair range of momentum errors and space charge.

DISPERSION AND SPACE CHARGE IN UMER AND OTHER MACHINES

UMER is a low energy (10 keV), high current (0.6-100 mA) electron storage ring with long bunches (100 ns or about half the ring circumference) and a high density of short magnetic quadrupoles (72) in a 11.52-m circumference. Beam transport in UMER is also made possible by 36 short magnetic dipoles and the earth's magnetic field. These features allow us to inject beams in a single turn with calculated incoherent space charge tune shifts of the order of an integer or larger values at typical operating points.

Table I contains basic parameters for a number of beams used in UMER. The operating point is $\nu_{0x} = \nu_{0y} = 6.37$, and the assumed RMS fractional momentum error is Δ = 0.01 in all cases except 6.0* mA, for which Δ = 0.015. The last three columns are based on exact solutions of the smooth approximation (SA) Eqs. 1-3 for the average horizontal beam semi-axis dimension (normalized to either the zero-current value a_0 or the full-current value -without dispersion- a_S) and the average dispersion D normalized to D_0. The parameter tabulated in the last column is defined by:

$$\chi_D \equiv \frac{\xi_0}{\xi} = \left(\frac{D_0}{D}\right)\left(\frac{a}{a_0}\right). \quad (6)$$

which we introduce as a possible *dispersion-space charge intensity parameter*. For UMER, we have $D_0 = \rho/\nu_0^2$ = 0.045 m, and $a_0(mm) = 0.54\times\epsilon^{0.5}(\mu m)$.

In Figure 1 we plot the results of approximate SA calculations of a/a_0 as a function of tune depression based on Eq. 5, and indicate the UMER beams by their currents. The latter points are obtained from Eqs. 1-3 (exact within the SA approximation). The point labeled "6 mA*" on the curve for $\xi_0 = 0.60$ is possible if we adjust both the emittance and the momentum error ($\Delta = 0.015$) to yield $\xi_0 = 0.60$ and $\eta = 0.50$ while keeping other parameters the same as for the standard transport of 6 mA in UMER. This example illustrates that reducing transverse emittance of a beam without at the same time controlling the momentum error spread can lead to a significant increase on the average beam dimension in the bending plane. Also noteworthy from Table 1, the average dispersion can grow by a factor of 2 or larger with significant space charge (small tune depression), and the average beam dimension in the bending plane can be from 10% to 20% larger than the expected value from space charge alone.

Table 1: Basic Parameters in UMER and Results of SA Calculations from Eqs. 1-3. The emittances are 4×RMS, unnormalized. See text for additional details.

Curr. (mA), Emitt. (μm)	**Tune Dep.**	D_0/a_0, D/D_0	a/a_0, a/a_S	χ_D
0.0, 8.05	1.00	30, 1.0	1.16, 1.16	1.16
0.6, 8.05	0.85	30, 1.3	1.30, 1.20	1.03
6.0, 26.2	0.62	16.5, 2.1	1.40, 1.10	0.65
6.0*, 17.2	0.50	20, 2.2	1.78, 1.26	0.82
21, 30.2	0.30	15, 4.6	2.02, 1.11	0.44
104, 64.4	0.14	10.5, 10.4	2.83, 1.06	0.27

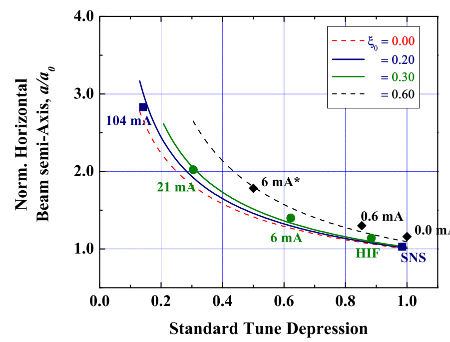

Figure 1: Approximate normalized horizontal beam semi-axis length as a function of standard tune depression (see Eq. 5) for four values of the parameter $\xi_0 = 2D_0\Delta/a_0$, with $\Delta = 0.0$ (red dashed line) and $\Delta = 0.01$ (three cases); all cases include full space charge. For comparison, the points with labels represent exact results from Eqs. 1-3.

Also shown in Fig. 1 are two special points, one corresponding to a typical proton storage ring for a spallation neutron source (e.g. SNS at Oak Ridge Natl. Lab.) and the other representing a ring envisioned for a heavy-ion fusion (HIF) driver. Although the effects of dispersion and space charge on the average beam radius for the SNS ring are not entirely negligible ($\Delta \simeq 10^{-2}$), we obtain $\chi_D \simeq 1.0$. By contrast, the ring envisioned for HIF has $\chi_D \simeq 0.8$, even after assuming a stringent $\Delta \simeq 10^{-4}$.

NUMERICAL TESTS

We have solved the original differential equations of the V-R theory [1] for the beam's envelope and dispersion functions to test the smooth approximation results in four cases (see Table I): 0.0, 0.6, 6.0 and 21 mA. In addition, we have employed the code ELEGANT [3] using an approxi-

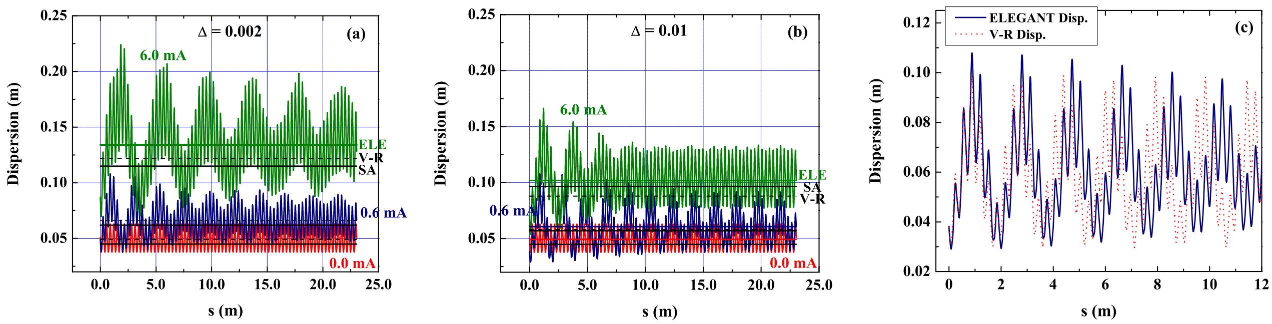

Figure 2: (a) and (b): Dispersion functions with space charge for 0.0, 0.6 and 6.0 mA at 10 keV over 2 turns in UMER from ELEGANT code. The solid dark lines represent results from the smooth approximation (Eqs. 1-3), while the broken lines indicate results from the V-R differential equations. (c) Dispersion functions (ELEGANT, solid curve; V-R equations, dotted curve) with space charge for 0.6 mA, $\Delta = 0.01$ over one turn in UMER.

mate model of space charge to track up to 250K particles over two turns (72 FODO periods) in an idealized UMER lattice. Details of the model for the magnets and the space charge in ELEGANT can be found in [6] and [8].

For approximate matching of envelopes and dispersion, we manually adjust $D(0)$, $D'(0)$, starting with $D(0)$ equal to the SA value. For the initial envelope sizes and slopes, on the other hand, we start with values obtained with TRACE3D [9]. Figures 2a-b show examples of calculations of dispersion in ELEGANT; also shown are lines indicating average values from the smooth approximation and the solutions of the V-R differential equations. Figure 2c shows a comparison of dispersion functions in ELEGANT and the numerical solution of the V-R equations.

All calculations yield a significantly larger average dispersion for a combination of small Δ ($\lesssim$ 0.002) and large current; e.g. $D = 0.47$ m for $\Delta = 0.001$ at 21 mA (ELEGANT and SA), $D = 0.40$ m (V-R Equations). However, the product $D\Delta$, which is the RMS change in the x-coordinate (RMS orbit displacement) of beam particles relative to the reference orbit, is only about 0.5 mm. As expected, for $\Delta = 0.001$ the effect on the average beam horizontal dimension is also small, $a/a_S = 1.004$, compared to $a/a_S = 1.11$ for $\Delta = 0.01$ (see Table I).

To summarize the results, for low current (0.6 mA or less) and Δ = 0.001-0.01, the smooth approximation (SA) predicts an average dispersion that agrees well with both ELEGANT calculations and solutions of the envelope-dispersion equations of the V-R theory. For space charge dominated transport around 6 mA, we obtain fair agreement among the three approaches for estimating average dispersion, also in the range Δ = 0.001-0.01. At larger current (21 mA), however, there's agreement between SA and ELEGANT only for small Δ=0.001; for larger momentum errors, SA yields values of average dispersion that are significantly larger ($\gtrsim$ 30%) than those from ELEGANT or the V-R equations. As for the average beam horizontal dimension, the V-R equations yield values that are always larger (10-20%) than SA results, while ELEGANT predicts values that fall between V-R and SA.

Although the model of smooth focusing/dispersion discussed would no be obviously applicable to short bunches or to the case of strong correlation of momentum spread with bunch's slice, we can still apply it to the core of the bunch, or, in an *ad hoc* way, to the whole bunch in an average sense. In UMER, the actual uncorrelated energy spread may correspond closer to $\Delta = 0.002$ for low current and $\Delta = 0.005$ for high current, but we have chosen to do calculations with $\Delta = 0.01$ to enhance the effects of dispersion and because this figure corresponds closely to the correlated energy spread. Much work remains to be done in this area: realistic dispersion-envelope matching involving a bending element before injection into UMER, 6D treatment of dispersion with space charge (bunched beams), improved calculations in ELEGANT, and by-the-turn measurements of beam size, emittance, and energy spread.

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