

ANALYTICAL ANALYSIS OF PARTICLE-CORE DYNAMICS*

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Abstract

Particle-core interaction is a well-developed model of halo formation in high-intensity beams [1-5]. In this paper, we present an analytical solution for averaged, single particle dynamics, around a uniformly charged beam. The problem is analyzed through a sequence of canonical transformations of the Hamiltonian, which describes nonlinear particle oscillations. A closed form expression for maximum particle deviation from the axis is obtained. The results of this study are in good agreement with numerical simulations and with previously obtained data.

PARTICLE-CORE MODEL

The model is based on a simultaneous solution of an equation for oscillation of beam envelope, $r=R/R_o$, around an equilibrium beam radius R_o , and an equation for particle deviation from the axis $u=x/R_o$:

$$\frac{d^2 r}{d\tau^2} + r - \frac{1}{(1+b)r^3} - \frac{b}{(1+b)r} = 0, \quad (1)$$

$$\frac{d^2 u}{d\tau^2} + u = \begin{cases} \frac{b}{(1+b)} \frac{u}{r^2}, & |u| \leq r \\ \frac{b}{(1+b)u}, & |u| > r \end{cases}, \quad (2)$$

where $\tau = \Omega_r t$ is the dimensionless time, Ω_r is the frequency of particle oscillation in a uniform focusing channel, b is the dimensionless space charge parameter

$$b = \frac{2}{\beta\gamma} \frac{I}{I_c} \frac{R_o^2}{\epsilon^2}, \quad (3)$$

I is the beam current, $I_c = 4\pi\epsilon_o mc^3 / q$ is the characteristic beam current, ϵ is the normalized beam emittance, β is the particles velocity, and γ is the particle energy.

Fig. 1 illustrates typical single-particle trajectories around an oscillating core, resulting from numerical integration of Eqs. (1), and (2). Depending on initial conditions, a particle can oscillate (a) inside the core, (b) around the core, (c) far away from core with constant amplitude, and (d) resonantly with the core. Fig. 2 illustrates the same trajectories as stroboscopic images in phase space, where particle positions and velocities are measured after each core oscillation period.

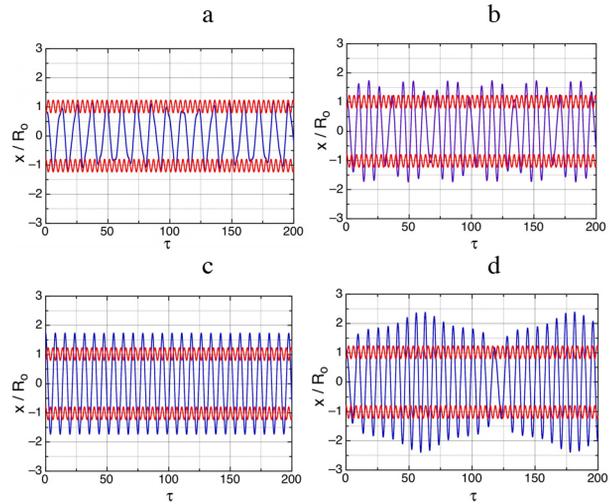


Figure 1: (Red) envelope oscillations of the beam with space charge parameter $b = 3$, amplitude $\Delta = 0.2$, and (blue) single particle trajectories with initial conditions (a) $x_o/R_o=0.8$, (b) $x_o/R_o = 1.071$, (c) $x_o/R_o = 1.728$, (d) $x_o/R_o = 1.082$.

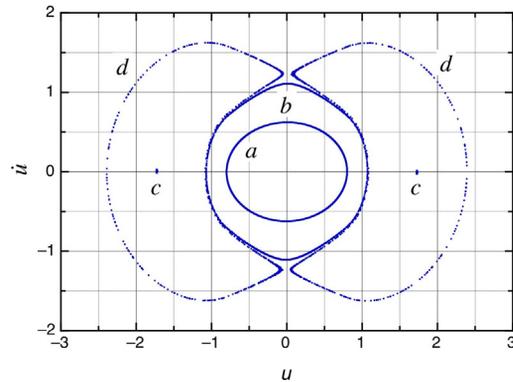


Figure 2: Stroboscopic particle trajectories at phase plane ($u, du/d\tau$) taken consequently after each envelope oscillation period: (a) $x_o/R_o=0.8$, (b) $x_o/R_o = 1.071$, (c) $x_o/R_o = 1.728$, (d) $x_o/R_o = 1.082$.

The beam envelope performs oscillations in a continuous focusing channel around an equilibrium value $r = 1 + \Delta \cos(2\Omega\tau)$ where the envelope oscillation frequency is

$$2\Omega = \sqrt{2\left(\frac{2+b}{1+b}\right)}. \quad (4)$$

The space charge field of a uniform beam consists of a linear oscillating part as well as non-oscillating nonlinear part (see Eq. (2)). Following Ref. [1], we approximate the nonlinear part by a cubic term. With this approximation in place, the equation of particle motion becomes:

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$$\frac{d^2u}{d\tau^2} + u - \left(\frac{b}{1+b}\right) \left[\frac{u}{(1+\Delta \cos 2\Omega\tau)^2} - \frac{u^3}{4} \right] = 0. \quad (5)$$

Then, using the expansion

$$\frac{1}{(1+\Delta \cos 2\Omega\tau)^2} \approx 1 - 2\Delta \cos 2\Omega\tau, \quad (6)$$

equation of particle motion takes the form:

$$\frac{d^2u}{d\tau^2} + u \left(\frac{1}{1+b} \right) (1 + 2b\Delta \cos 2\Omega\tau) + \left(\frac{b}{1+b} \right) \frac{u^3}{4} = 0. \quad (7)$$

Equation (7) corresponds to the Hamiltonian, which can be written as:

$$H = \frac{(du/d\tau)^2}{2} + \varpi^2 \frac{u^2}{2} (1 - h \cos 2\Omega\tau) + \alpha \frac{u^4}{4}, \quad (8)$$

where the following notations are used:

$$\varpi^2 = \frac{1}{1+b}, \quad h = -2b\Delta, \quad \alpha = \frac{b}{4(1+b)}. \quad (9)$$

The Hamiltonian, Eq. (8), describes an anharmonic oscillator with parametric excitation. Presence of nonlinear term $\sim u^4$ limits amplitude of particle oscillation around the core.

AVERAGED HAMILTONIAN

Let us change the variables (u, \dot{u}) in the Hamiltonian, Eq. (8), to new variables (Q, P) using a generating function:

$$F_2(u, P, \tau) = \frac{uP}{\cos \Omega\tau} - \left(\frac{P^2}{2\varpi} + \varpi \frac{u^2}{2} \right) \text{tg} \Omega\tau. \quad (10)$$

The relationships between the variables are given by $Q = \partial F_2 / \partial P$, $\dot{u} = \partial F_2 / \partial u$ which results in the following transformation [6]:

$$\begin{cases} u = Q \cos \Omega\tau + \frac{P}{\varpi} \sin \Omega\tau \\ \dot{u} = -\varpi Q \sin \Omega\tau + P \cos \Omega\tau \end{cases}. \quad (11)$$

Transformation (11) is a representation of a solution of the equation of a single particle motion as a combination of fast oscillating terms, which have the frequency Ω , and the slow variables, Q, P . The new Hamiltonian is given by:

$$\begin{aligned} K = & \frac{P^2}{2} + \varpi^2 \frac{Q^2}{2} - \frac{\varpi^2 h}{2} \left(Q \cos \Omega\tau + \frac{P}{\varpi} \sin \Omega\tau \right)^2 \cos 2\Omega\tau \\ & + \frac{\alpha}{4} \left(Q \cos \Omega\tau + \frac{P}{\varpi} \sin \Omega\tau \right)^4 - \frac{P^2 \Omega}{2\varpi} - \frac{\Omega \varpi}{2} Q^2. \end{aligned} \quad (12)$$

After averaging all time-dependent terms over a period of $T=2\pi/\Omega$, the Hamiltonian, Eq. (12), takes on a simple form:

$$\bar{K} = \frac{\varpi^2 \bar{Q}^2}{2} \left(1 - \frac{\Omega}{\varpi} - \frac{h}{4} \right) + \frac{\bar{P}^2}{2} \left(1 - \frac{\Omega}{\varpi} + \frac{h}{4} \right) + \frac{3\alpha}{32} \left(\bar{Q}^2 + \frac{\bar{P}^2}{\varpi^2} \right)^2, \quad (13)$$

where \bar{Q}, \bar{P} are the average values of variables over a period T . Let us make an additional canonical transformation and change variables (\bar{Q}, \bar{P}) to action-angle variables (J, ψ) utilizing generating function

$$F_1(\bar{Q}, \psi) = \frac{\varpi \bar{Q}^2}{2 \text{tg} \psi}. \quad (14)$$

Transformation here is given by:

$$\bar{Q} = \sqrt{\frac{2J}{\varpi}} \sin \psi, \quad \bar{P} = \sqrt{2J\varpi} \cos \psi. \quad (15)$$

Utilizing transformation, Eqs. (15), the Hamiltonian, Eq. (13) can be rewritten as:

$$\bar{K} = \nu J + \kappa J^2 + 2\chi J \cos 2\psi, \quad (16)$$

with the following notations:

$$\nu = \frac{\sqrt{2} - \sqrt{2+b}}{\sqrt{2(1+b)}}, \quad \kappa = \frac{3}{32} b, \quad \chi = -\frac{1}{4} \frac{b\Delta}{\sqrt{1+b}}. \quad (17)$$

Hamiltonian, Eq. (16), describes the excitation of a nonlinear parametric resonance. Phase space trajectories, corresponding to constant values of Hamiltonian (16) are presented in Fig. 3. There are two types of separatrices in phase plane. The inner separatrix corresponds to particle motion around the beam core. Two other separatrices correspond to resonant particle oscillations outside the beam core. As stated before, presence of nonlinear term limits the maximum amplitude of particle oscillations, and the width of the resonance defines the maximum deviation of the particle from the axis.

NONLINEAR PARAMETRIC RESONANCE

Let us now determine the maximum deviation of the particle from the axis. There are specific fixed points (stable, J_s , and unstable, J_u) present in the phase plane of averaged motion, see Fig. 3. The locations of the fixed points follow directly from Hamiltonian, Eq. (16):

$$\frac{dJ}{d\tau} = -\frac{\partial \bar{K}}{\partial \psi} = 4\chi J \sin 2\psi = 0, \quad (18)$$

$$\frac{d\psi}{d\tau} = \frac{\partial \bar{K}}{\partial J} = \nu + 2\kappa J + 2\chi \cos 2\psi = 0. \quad (19)$$

The first equation has a solution $\sin 2\psi = 0$. Under this condition, the unstable points are determined by equations

$$\cos 2\psi = 1, \quad \psi = 0, \pi, \quad J_u = -\frac{v + 2\chi}{2\kappa}, \quad (20)$$

while stable points are determined as

$$\cos 2\psi = -1, \quad \psi = \frac{\pi}{2}, \frac{3\pi}{2}, \quad J_s = \frac{-v + 2\chi}{2\kappa}. \quad (21)$$

Particle with initial conditions, that are close to an unstable point J_u , can perform either small amplitude oscillations inside the inner separatrices, or large - amplitude oscillations with a maximum amplitude up to J_{max} . Let us now define the value of J_{max} .

The value of the Hamiltonian is the same at the internal and at the external separatrices. At the internal separatrix, the value of Hamiltonian $\bar{K}_u = \bar{K}(J_u)$ can be obtained from Eqs. (16), (20):

$$\bar{K}_u = -\frac{(v + 2\chi)^2}{4\kappa}. \quad (22)$$

The outer separatrix touches the inner one at the unstable point J_u . Particle with the value of Hamiltonian $\bar{K}(J_u)$ can reach the point J_{max} having $\psi = \pi/2$. The value of J_{max} is defined by the substitution $\bar{K} = \bar{K}_u$ and $\cos 2\psi = -1$ into equation (16):

$$\kappa J_{max}^2 + J_{max}(v - 2\chi) - \bar{K}_u = 0. \quad (23)$$

Thus, the solution for J_{max} reads:

$$J_{max} = \frac{(-v + 2\chi) + \sqrt{8|v\chi|}}{2\kappa}. \quad (24)$$

The value of J is related to the variables ($u, du/d\tau$) via Eqs. (11), and (15) as:

$$J = \frac{1}{2} \left(u^2 \bar{\omega} + \frac{\dot{u}^2}{\bar{\omega}} \right). \quad (25)$$

The maximum value that a particle can take on for its position given by Eq. (25) is therefore

$$u_{max} = \sqrt{2J_{max} / \bar{\omega}}, \text{ or:}$$

$$\frac{x_{max}}{R_o} = \sqrt{\frac{32}{3} \frac{\sqrt{1 + \frac{b}{2} - 1 + \frac{b|\Delta|}{2}} + \sqrt{2b|\Delta|} \left(\sqrt{1 + \frac{b}{2} - 1} \right)}{b}}. \quad (26)$$

In Fig. 4 the value of $u_{max} = x_{max}/R_o$, Eq. (26), is presented as a function of envelope amplitude Δ , for different values of the beam space charge parameter b . These dependencies show that maximum deviation of a

particle from the axis can be several times larger than the beam core size. For comparison, Fig. 4 also contains an empirical dependence

$$\frac{x_{max}}{(R_o / 2)} = A + B \ln(\mu), \quad (27)$$

where $A = B = 4, \mu = 1 + \Delta$, obtained in Ref. [3] as a generalization of numerical and experimental data. Comparison shows good agreement between analytical predictions, Eq. (26), and previously obtained data.

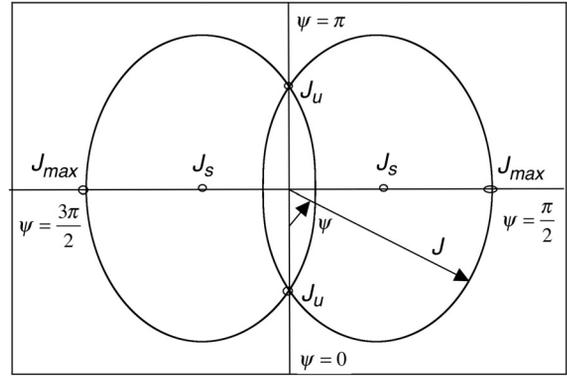


Figure 3: Phase space trajectories of averaged particle motion around the beam core, Eq. (16).

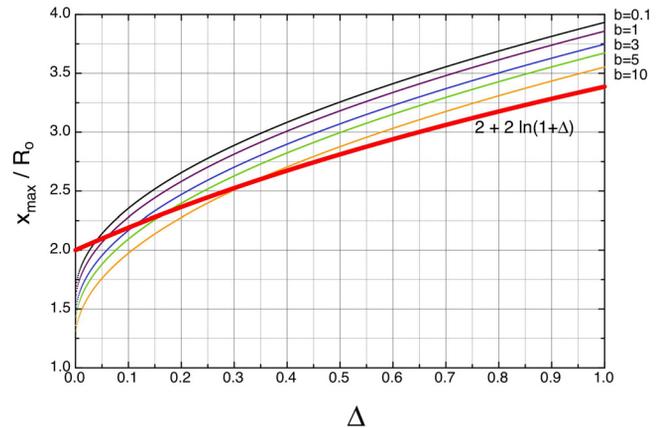


Figure 4: Maximum deviation of a particle from the axis as a function of amplitude of core oscillation, Eq. (26).

REFERENCES

- [1] R. L. Gluckstern, Phys. Rev. Lett. 73, 1247–1250 (1994).
- [2] R.A.Jameson, LA-UR-94-3753, LANL Report (1994).
- [3] T. P. Wangler, K. R. Crandall, R. Ryne, and T. S. Wang, PRSTAB, Vol. 1, 084201 (1998).
- [4] A.V.Fedotov, R.L.Gluckstern, S.S.Kurennoy, R.D.Ryne, PRSTAB, Vol. 2, 014201 (1999).
- [5] Yu.Senichev, N.Vasyukhin, NIM-A, Vol. 561, Issue 2, 166-172 (2006).
- [6] G.L.Kotkin, and V.G.Serbo, Collection of Problems in Classical Mechanics, Pergamon Press, Oxford, New York (1971).