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SKEW-QUAD PARAMETRIC-RESONANCE IONIZATION COOLING: THEORY AND MODELING*

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Abstract

Muon beam ionization cooling is a key component for the next generation of high-luminosity muon colliders. To reach adequately high luminosity without excessively large muon intensities, it was proposed previously to combine ionization cooling with techniques using a parametric resonance (PIC). Practical implementation of PIC proposal is a subject of this report. We show that an addition of skew quadrupoles to a planar PIC channel gives enough flexibility in the design to avoid unwanted resonances, while meeting the requirements of radially-periodic beam focusing at ionization-cooling plates, large dynamic aperture and an oscillating dispersion needed for aberration corrections. Theoretical arguments are corroborated with models and a detailed numerical analysis, providing step-by-step guidance for the design of Skew-quad PIC (SPIC) beamline.

INTRODUCTION

Experiments at energy-frontier colliders require high luminosities, of order 10^{34} cm^{-2} sec^{-1} or more, in order to obtain reasonable rates for events having point-like cross sections. High luminosities require intense beams, small transverse emittances, and a small beta function at the collision point. For muon colliders, high beam intensities and small emittances are difficult and expensive to achieve because muons are produced diffusely and must be cooled drastically within their short lifetimes. Ionization cooling is a major first step toward providing adequate luminosity without large muon intensities, and its 6D-implementation with a helical cooling channel was described in Ref. [1]. Further reduction of emittances requires anomalously large magnetic fields in ionization-cooling channels, but the use of parametric resonance allows relaxing this requirement, while providing significant beam cooling effects [2].

PIC CONCEPT

In the PIC technique the resonant approach to particle focusing can achieve equilibrium transverse emittances that are at least an order of magnitude smaller than in conventional ionization cooling. The main principle is similar to half-integer parametric resonant extraction from a synchrotron, except for targeting different variables of the phase space [2]. Briefly speaking, parametric resonance provides focusing of the muon beam at periodic locations down the beamline; the beam angular spread is naturally maximized at these locations, therefore unw anted angular smearing (or "heating") due to multiple Coulomb scattering in ionization-cooling plates has the least effect.

While the concept of PIC has been around for a few years, its practical implementation faced several difficulties that were addressed by an Epicyclic PIC proposal [3] and proposed epicyclic twin-helix magnetic structure [4]. The latter uses a superposition of two opposite-helicity equal-period equal-strength helical dipole harmonics and a straight normal quadrupole. Here we propose and develop a technique that will help to avoid (integer) resonances in such a system that requires periodic focusing, making the important step toward its engineering design.

COUPLING RESONANCE IN PIC

Our proposal is based on inducing a linear betatron coupling resonance in PIC transport line between the horizontal (x) and vertical (y) planes. Previously developed (and described in Ref. [4]) twin-helix magnetic system is supplemented with skew quads that generate coupling between horizontal and vertical betatron motion.

Let us consider the effect of such coupling, first in a simplified model. The equations for coupled betatron oscillations are

$$\begin{aligned} x_b'' + k_x^2 x_b + g y_b = 0 \\ y_b'' + k_y^2 y_b + g x_b = 0 \end{aligned}$$

where $k_x^2 = K^2 - n$, $k_y^2 = n$, with K^2 and n denoting the curvature function and quad strength, respectively. Coupling g is provided by 45^o-skewed quadrupoles.

In the case of constant coefficients, the above system has an analytic solution described by superposition of two normal-mode oscillations with wave vectors $k_{1,2}^2 = \frac{1}{2}(k_x^2 + k_y^2 \pm \sqrt{(k_x^2 - k_y^2)^2 + 4g^2})$.

$$\begin{aligned} x_b(s) &= C_1 \cos(k_1 s) + C_1' \sin(k_1 s) + G \\ &\qquad \cdot (C_2\, cos(k_2 s) + C_2'\, sin(k_2 s)) \\ y_b(s) &= -G \cdot (C_1 \cos(k_1 s) + C_1' \sin(k_1 s)) \\ &\qquad + C_2\, cos(k_2 s) + C_2'\, sin(k_2 s), \end{aligned}$$

where $G = (k_2^2 - k_y^2)/g = -(k_1^2 - k_x^2)/g$, and s is the path length along the reference trajectory. The constants are obtained from initial conditions at s=0:

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$$C_1 = \frac{x_0 - Gy_0}{1+G^2},\; C_1' = \frac{(x_0' - Gy_0')/k_1}{1+G^2}$$
$$C_2 = \frac{Gx_0 + y_0}{1+G^2},\; C_2' = \frac{(Gx_0' + y_0')/k_2}{1+G^2}$$

Note that $k_x^2 + k_y^2 = k_1^2 + k_2^2 = K^2$. Inverse relation between the oscillation wave vectors with and without coupling has a form:

$$k_{x,y}^2 = \left(k_1^2 + k_2^2 \pm \sqrt{(k_1^2 - k_2^2)^2 - 4g^2}\right)/2.$$

This expression is useful for finding parameters K^2 and n for the desired wavelengths of normal modes. It also sets an upper limit for $|g|$ for given values of $k_{1,2}$:

$$(k_1^2 - k_2^2)^2 \geq 4g^2$$

In the limit $g \to 0$, we also have $G \to 0$ and the solution is separated into two plane waves with wave vectors k_x and k_y, respectively. For any nonzero value of g the solution for x and y is a superposition of two plane waves with wave vectors k_1 and k_2. One can identify normal modes, each described by the wave vector k_1 or k_2, by constructing linear combinations: $x_b - Gy_b$ and $y_b + Gx_b$. Introducing a parameter θ defined as $G=\tan(\theta)$, transformation into normal coordinates (X_b,Y_b) may be represented as rotation by an angle θ:

$$X_b = x_b \cos\theta - y_b \sin\theta$$
$$= X_b(0)\cos(k_1 s) + X_b'(0)\sin(k_1 s)$$
$$Y_b = x_b \sin\theta + y_b \cos\theta$$
$$= Y_b(0)\cos(k_2 s) + Y_b'(0)\sin(k_2 s)$$

For each normal mode X_b, Y_b, the oscillation is described by a single wave vector k_1 and k_2, respectively.

The following relation relates the rotation angle θ and the coupling strength:

$$g = -\frac{1}{2}(k_1^2 - k_2^2)\sin(2\theta)$$

If coupling g is piece-wise constant, we can still use the above solutions for x_b and y_b, but with re-defined coefficients $C_{1,2}$ and $C_{1,2}'$ for the corresponding intervals of s.

Example 1: $k_1 = 2\pi, k_2 = \pi$, and $g(s)$ a periodic step-like function with a period $\lambda_g = 2$, as shown in Fig. 1.

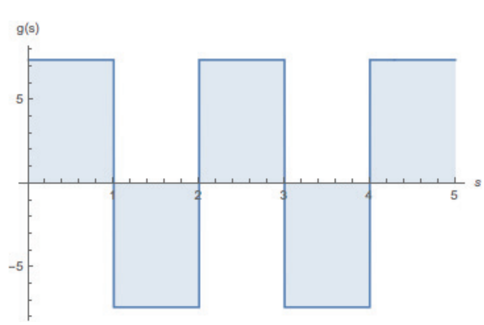

Figure 1: Step-like periodic coupling function g(s).

Prior to constructing a general solution, we obtain an expression for R-matrix that transforms the state vector (x_b,y_b,x_b',y_b') over a period $\lambda_g = 2$: $R(s = 0 \to 2)$.

The matrix is a product of two matrices, each for the corresponding section of constant g:

$$R(s = 0 \to 2) = R(s = 1 \to 2)R(s = 0 \to 1)$$

To find the matrix elements of R-matrix we use the above analytic solution for xb, yb to get:

$$x_b(s=1) = \frac{1-G^2}{1+G^2}x_0 - \frac{2G}{1+G^2}y_0$$
$$y_b(s=1) = -\frac{1-G^2}{1+G^2}y_0 - \frac{2G}{1+G^2}x_0$$
$$x'_b(s=1) = \frac{1-G^2}{1+G^2}x'_0 - \frac{2G}{1+G^2}y'_0$$
$$y'_b(s=1) = -\frac{1-G^2}{1+G^2}y'_0 - \frac{2G}{1+G^2}x'_0$$

Using the previously introduced parameterization $G=\tan(\theta)$ we find:

$$\frac{1-G^2}{1+G^2} = \cos 2\theta, \qquad \frac{2G}{1+G^2} = \sin 2\theta$$

with R-matrix taking the form:

$$R(s = 0 \to 1) = \begin{pmatrix} M_1 & 0 \\ 0 & N_1 \end{pmatrix},$$
$$M_1 = N_1 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{pmatrix}$$

We see that for both the coordinates *(x_b, y_b)* and the derivatives *(x_b',y_b')* the corresponding transformation corresponds to rotation by angle θ in *xy*-plane and reflection with respect to *xz*-plane. Such a transformation is known as an *improper rotation* or *rotary reflection*, with $\det M_1 = \det N_1 = -1$.

The matrix $R(s = 1 \to 2)$ is obtained from $R(s = 0 \to 1)$ by a replacement $\theta \to -\theta$:

$$R(s = 1 \to 2) = \begin{pmatrix} M_2 & 0 \\ 0 & N_2 \end{pmatrix},$$
$$M_2 = N_2 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

and the resulting R-matrix per period is obtained from their product:

$$R(s = 0 \to 2) = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix},$$
$$M = N = \begin{pmatrix} \cos 4\theta & -\sin 4\theta \\ \sin 4\theta & \cos 4\theta \end{pmatrix}$$

We recognize that R-matrix describes rotation by an angle 4θ for both the coordinates and derivatives per one period of function g. Note that R(s=1→3) is obtained from R(s=0→2) by switching the sign of $\theta \to -\theta$.

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The conventional transfer matrix M_T that maps the state vector (x_b, x_b', y_b, y_b') over a period $\lambda_g = 2$ has a form:

$$M_T = \begin{pmatrix} \cos 4\theta & 0 & \sin 4\theta & 0 \\ 0 & \cos 4\theta & 0 & \sin 4\theta \\ -\sin 4\theta & 0 & \cos 4\theta & 0 \\ 0 & -\sin 4\theta & 0 & \cos 4\theta \end{pmatrix}$$

with two non-identical eigenvalues equal $e^{-i4\theta}$ and $e^{i4\theta}$, where one may recognize a betatron tune that equals 4θ. This is the main result we achieved: skewed quads induce coupling that results in radially-periodic beam motion, while the betatron tune is controlled by the coupling strength and allows to avoid integer resonances in such a transport channel.

NUMERICAL ANALYSIS

The results of the above calculations are illustrated in Fig. 2, where the value of g was chosen arbitrarily such that θ=1/4 radian.

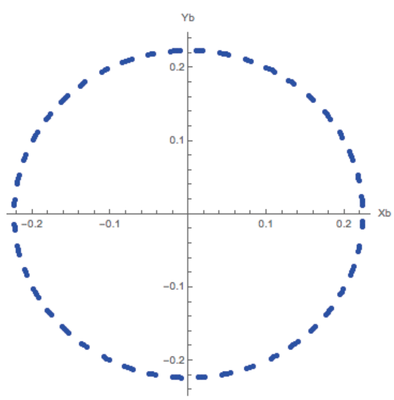

Figure 2: Particle positions calculated at the end of each period of function g (s=0, 2, 4,...) for 300 periods. One dot corresponds to one position. The magnitude of g is such that θ=1/4 rad (value chosen for illustration only). For other choices of initial position s the (x,y) locations follow an ellipse.

For small values of rotation angle 4θ that correspond to weak coupling g we can see from above analysis that after each period particle positions in (x,y) plane undergoes rotation, defining piece-wise constant coefficients of the solution for $x_b(s)$, $y_b(s)$. As a result, the magnitude of the oscillations effectively becomes modulated with factors cos(2θs) and sin(2θs), making an appearance of "beating" coming from a coupling resonance in the oscillator. This is illustrated in Fig. 3.

Formally the effect may be viewed as superposition of two oscillations with close frequencies, but actually it originates from beam rotation induced by the alternating-sign coupling function. Other functional forms for the periodic coupling have been studied and showed similar effects, with the difference that circular beam rotation becomes elliptic. We also obtained required alternating-sign behavior of the dispersion function in both planes.

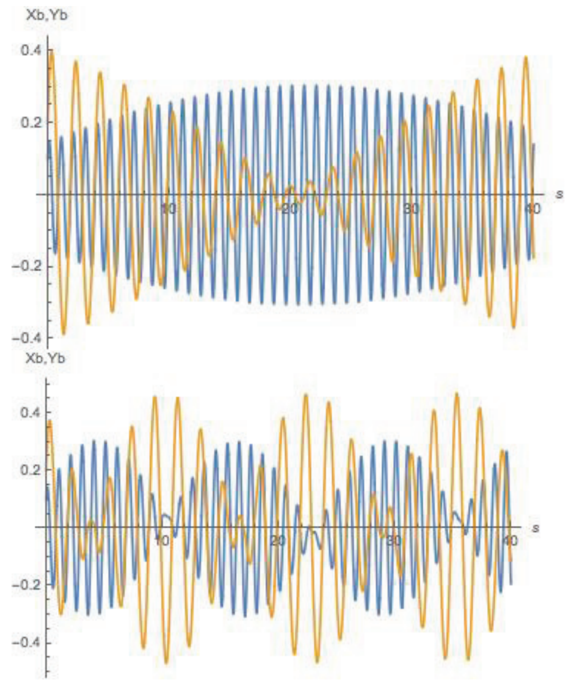

Figure 3: Beating effect in the coupled betatron oscillations. (a) $\boldsymbol{4\theta}$=0.1rad, (b) $\boldsymbol{4\theta}$=0.5rad.

SUMMARY

We have demonstrated that inducing a linear betatron coupling resonance in PIC transport line between the horizontal (x) and vertical (y) planes results in a transport line suitable for PIC implementation that allows to avoid integer resonances. Further simulations aimed at engineering design of this system are being performed using MAD-X software, as reported in Ref. [5].

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