SHIELDING OF A HADRON IN A FINITE e-BEAM*

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Abstract

The thorough study of coherent electron cooling, the modern cooling technique capable to deal with accelerators operating in the range of few TeVs [1], raises many interesting questions. One of them is a shielding dynamics of a hadron in an electron beam. Now this effect is computed analytically in the infinite beam approximation [2]. Many effects are drastically different in finite and infinite plasmas. Here we propose a method to compute the dynamical shielding effect in a finite cylindrical plasma[1] - the realistic model of an electron beam in accelerators.

INTRODUCTION

The problem of shielding of a charge in plasma is represented by the Maxwell-Vlasov [3] system of equations. For an infinite plasma this system is exactly solvable [2, 4], for a finite plasma the extra term appears in the Vlasov equation, which makes the methods developed for an infinite plasma inapplicable. First time the Vlasov equation for a finite KV (Kapchinskij-Vladiirskij) beam [5] was considered by Gluckstern [6], where he found analytical solutions for its excitations without any external fields or charges, his ideas were further developed by Venturini in [7], where they studied beam's response to an external field of a special form. Here we propose a method to solve dynamical shielding of a test charge in a finite beam, we consider Maxwell and KV beam distributions, 2D and 3D cases.
We consider an electron beam with Hamiltonian [8]

$$H_0 = \frac{p_\perp^2}{2m_0\gamma} + \alpha x_\perp^2, H_z = \frac{p_z^2}{2m_0\gamma^3}, \qquad (1)$$

where α-contribution incorporates focusing and space-charge field of equilibrium distribution for KV and for Maxwell in the limiting case. Our interest is a perturbation of the electron density $n_1(\vec{x},t) = \int f_1(\vec{x},\vec{v},t)\,d\vec{v}$ caused by a moving particle with charge Q, where $f_1(\vec{x},\vec{v},t)$ is a perturbation of the density in a phase space:

$$f(\vec{x},\vec{v},t) = f_0(\vec{x},\vec{v}) + f_1(\vec{x},\vec{v},t), \qquad (2)$$

$f_0(\vec{x},\vec{v})$ is an equilibrium distribution, we consider KV in 2D and Maxwell for 3D case. The linearized Vlasov equation looks as follows:

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{x}} - \frac{1}{m_0\gamma}\frac{\partial H_0}{\partial \vec{x}}\cdot\frac{\partial f_1}{\partial \vec{v}} = \frac{e}{m_0\gamma}\frac{\partial U}{\partial \vec{x}}\cdot\frac{\partial f_0}{\partial \vec{v}}. \qquad (3)$$

The third term in the left hand side is an extra term spoiling the methods for an infinite plasma. The method of solving the Maxwell-Vlasov system via integral equation is discussed in the following sections. The term in the right hand side, incorporating potentials of the electron density perturbation and test charge, may cause singularities in the method. The possible ways to handle them are considered.

MAXWELL DISTRIBUTION IN 3D

Noticing that the left hand side is a full time derivative if we plug solutions of the Hamilton equations and than doing integrals over time and velocity we can rewrite equation (3) as an integral equation:

$$n_1(\vec{x},t) = \frac{e}{m_0\gamma}\int_0^t\int \frac{\partial U}{\partial \vec{x}}\cdot\frac{\partial f_0}{\partial \vec{v}}\Big|_{\substack{\vec{x}=\vec{X}_0(t_1)\\ \vec{v}=\vec{V}_0(t_1)}} d\vec{v}dt_1, \qquad (4)$$

where $\vec{X}_0(t_1)$ and $\vec{V}_0(t_1)$ are unperturbed orbits - solutions of the Hamilton's equation with H_0 and initial conditions $\vec{x}=\vec{X}_0(t)$, $\vec{v}=\vec{V}_0(t)$:

$$\begin{cases} \vec{X}_{0_\perp}(t_1) = \vec{x}_\perp\cos(\omega(t-t_1)) - \frac{\vec{v}_\perp}{\omega}\sin(\omega(t-t_1)) \\ \vec{V}_{0_\perp}(t_1) = \vec{v}_\perp\cos(\omega(t-t_1)) + \vec{x}_\perp\omega\sin(\omega(t-t_1)), \\ X_{0_z}(t_1) = x_z + v_z(t-t_1), \;\; V_{0_z}(t_1) = v_z, \end{cases} \qquad (5)$$

where $\omega = \sqrt{\frac{\alpha}{\beta}}$ and $\beta = m_0\gamma/2$. Assuming boundary conditions at infinity we have for the potential

$$U(\vec{x},t_1) = \int_G \frac{n_1\left(\vec{x}',t_1\right)}{|\vec{x}-\vec{x}'|}d\vec{x}' + U_2(\vec{x},t_1), \qquad (6)$$

where G is an electron bunch and $U_2(\vec{x},t_1)$ is a potential of a test charge moving along th trajectory $Y(t_1)$, the potential in the equation (4) appears in the form $U\left(\vec{X}_0(t_1),t_1\right)$, which makes the equation functional-integral equation, which is generally unsolvable either analytically or numerically, the substitution (6) allows to rewrite the equation as a standard Fredholm-Volterra equation of second type, solving the Volterra equation by the Laplace transform we have for Laplace image of $n_1(\vec{x},t)$:

$$F(\vec{x},s) = \tilde{n}_1(\vec{x},s) - \lambda\int_G \tilde{n}_1\left(\vec{x}',s\right)K\left(\vec{x},\vec{x}',s\right)d\vec{x}'_s, \qquad (7)$$

where the kernel and other components are

$$K\left(\vec{x},\vec{x}',s\right) = x'_\perp\mathcal{L}\int\frac{\partial}{\partial\vec{X}_0(0)}\frac{1}{\left|\vec{X}_0(0)-\vec{x}'\right|}\cdot\frac{\partial f_0}{\partial\vec{V}_0(0)}d\vec{v}, \qquad (8)$$

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[1] By *plasma* we mean a collisionless single-species electron plasma

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$$\lambda = \frac{e^2}{\epsilon_0 m_0 \gamma}, \quad F(\vec{x}, s) = \frac{e}{m_0\gamma} \times$$
$$\times \mathcal{L}\left(\int_0^t \int \frac{\partial}{\partial \vec{X}_0(t_1)} U_2\left(\vec{X}_0(t_1), t_1\right) \cdot \frac{\partial f_0}{\partial \vec{V}_0(t_1)} d\vec{v} dt_1 \right). \quad (9)$$

Integration in the equation (7) is done over rectangular domain in which spherical coordinates $\left(x'_\perp, \hat{x}'_\perp, x'_z\right)$ of $\vec{x}'$ are considered as Cartesian coordinates of a new vector $\vec{x}'_s$ and $d\vec{x}'_s \equiv dx'_\perp d\hat{x}'_\perp dx'_z$. The equaiton (7) is a Fredholm equation of the second type, there are well-developed numerical methods for such equations if the kernel has a weak singularity [9], meaning that the integral $\int K\left(\vec{x}, \vec{x}', s\right) d\vec{x}$ exists. As it was mentioned in the introduction, the straightforward substitution of the KV distribution to the kernel (8) is not possible - the singularity will not be weak - the derivative of the KV distribution, which is a delta-function, is defined by integration by parts and this results in a second derivative of the potential, which is of the order of $1/|\vec{x}|$ in 3D, and this second derivative is not integrable.The way to proceed here is to consider Maxwell distribution in transverse direction $f_{0_\perp} = \rho\delta_\mu(H_0 - H_c) = \rho\frac{1}{\mu\sqrt{\pi}} e^{-\frac{(H_0 - H_c)^2}{\mu^2}}$, which approaches KV distribution with $\mu \to 0$. The expression for $F(\vec{x}, s)$ is divergent for a point charge, so some finite charge distribution modeling a hadron should be considered. However in 2D case we can solve with KV distribution and a point charge.

KV DISTRIBUTION IN 2D

Here we consider 2D problem of screening of a point charge in a plasma with KV distribution, which is equivalent to screening of a line charge in KV beam. We also start with the linearized Vlasov equation (3), then we do Laplace transform $\tilde{f}_1(\vec{x}, \vec{v}, s) = \mathcal{L} f_1(\vec{x}, \vec{v}, t)$ and multiply it by $e^{t_1 s}$

$$s\tilde{f}_1 e^{t_1 s} + \vec{v}\frac{\partial \tilde{f}_1}{\partial \vec{x}} e^{t_1 s} - \frac{1}{m_0\gamma}\frac{\partial H_0}{\partial \vec{x}} \cdot \frac{\partial \tilde{f}_1}{\partial \vec{v}} e^{t_1 s} =$$
$$= \frac{e}{m_0\gamma}\frac{\partial \tilde{U}(\vec{x}, s)}{\partial \vec{x}} \cdot \frac{\partial f_0}{\partial \vec{v}} e^{t_1 s}. \quad (10)$$

Plugging the trajectories in this equation and taking into account that

$$\frac{\partial}{\partial t_1}\left[\tilde{f}_1 e^{t_1 s}\right] = s\tilde{f}_1 e^{t_1 s} \quad (11)$$

we notice that its left hand side is a full derivative:

$$\frac{d}{dt_1}\left[\tilde{f}_1\left(\vec{X}_0(t_1), \vec{V}_0(t_1), s\right) e^{t_1 s}\right] =$$
$$= \frac{e}{m_0\gamma}\left.\frac{\partial \tilde{U}(\vec{x}, s)}{\partial \vec{x}} \cdot \frac{\partial f_0}{\partial \vec{v}}\right|_{\substack{\vec{x} = \vec{X}_0(t_1) \\ \vec{v} = \vec{V}_0(t_1)}} e^{t_1 s}, \quad (12)$$

the unperturbed orbits here are just transverse orbits from 3D case. Integrating over a period of the trajectories $\frac{2\pi}{\omega}$ we have

$$\tilde{f}_1\left(\vec{X}_0\left(t + \frac{2\pi}{\omega}\right), \vec{V}_0\left(t + \frac{2\pi}{\omega}\right), s\right) e^{\left(t + \frac{2\pi}{\omega}\right)s} -$$
$$-\tilde{f}_1\left(\vec{X}_0(t), \vec{V}_0(t), s\right) e^{ts} =$$
$$= \frac{e}{m_0\gamma}\int_t^{t+\frac{2\pi}{\omega}} \left.\frac{\partial \tilde{U}(\vec{x}, s)}{\partial \vec{x}} \cdot \frac{\partial f_0}{\partial \vec{v}}\right|_{\substack{\vec{x} = \vec{X}_0(t_1) \\ \vec{v} = \vec{V}_0(t_1)}} e^{t_1 s} dt_1. \quad (13)$$

Using periodicity of the trajectories and initial condition of Hamilton's equations we have

$$\tilde{f}_1(\vec{x}, \vec{v}, s) = \frac{e}{m_0\gamma}\frac{1}{e^{\frac{2\pi}{\omega}s} - 1} \times$$
$$\times \int_0^{\frac{2\pi}{\omega}} \left.\frac{\partial \tilde{U}(\vec{x}, s)}{\partial \vec{x}} \cdot \frac{\partial f_0}{\partial \vec{v}}\right|_{\substack{\vec{x} = \vec{X}_0(\tau + t) \\ \vec{v} = \vec{V}_0(\tau + t)}} e^{\tau s} d\tau, \quad (14)$$

where $\tau = t_1 - t$ and $\vec{X}_0(\tau + t)$ and $\vec{V}_0(\tau + t)$ do not depend on t, so the formula is consistent. Noticing the full time derivative of $\tilde{U}(\vec{x}, s)$ inside the integral:

$$\left.\frac{\partial \tilde{U}(\vec{x}, s)}{\partial \vec{x}} \cdot \frac{\partial f_0}{\partial \vec{v}}\right|_{\substack{\vec{x} = \vec{X}_0(\tau + t) \\ \vec{v} = \vec{V}_0(\tau + t)}} =$$
$$= 2\beta \frac{\partial \tilde{U}\left(\vec{X}_0(\tau + t), s\right)}{\partial \vec{X}_0(\tau + t)} \cdot \vec{V}_0(\tau + t)\frac{\partial f_0}{\partial H_0} = \quad (15)$$
$$= 2\beta \frac{d}{d\tau}\left[\tilde{U}\left(\vec{X}_0(\tau + t), s\right)\right]\frac{\partial f_0}{\partial H_0} \quad (16)$$

and integrating by parts over τ we throw the derivative to $e^{\tau s}$ as $\frac{\partial f_0}{\partial H_0}$ doesn't depend on time. This trick of throwing a derivative from singular term to nonsingular is not possible for 3D KV case, because 3D KV distribution is not a function of the full 3D Hamiltonian, while a product of two functions of H_0 and H_z. Then integrating over $\vec{v}$ with polar coordinates $\left(\sqrt{\xi}, \theta\right)$ we have:

$$\tilde{n}_1(\vec{x}, s) = \frac{e}{m_0\gamma}\frac{s}{e^{\frac{2\pi}{\omega}s} - 1}\delta\left(\alpha x^2 - H_c\right) \times$$
$$\times \int_0^{2\pi}\int_0^{\frac{2\pi}{\omega}} \tilde{U}\left(\vec{X}_0(\tau + t), s\right)\Big|_{\xi=0} e^{\tau s} d\tau d\theta + \frac{e}{m_0\gamma} \times$$
$$\times \frac{s}{e^{\frac{2\pi}{\omega}s} - 1}\int_0^{2\pi}\int_0^{\frac{2\pi}{\omega}} \frac{\partial}{\partial \xi}\tilde{U}\left(\vec{X}_0(\tau + t), s\right)\Big|_{\xi=\xi_*} e^{\tau s} d\tau d\theta, \quad (17)$$

where $\frac{\partial f_0}{\partial H_0}$ were transfromed to $\frac{\partial f_0}{\partial \xi}$, integration by parts over ξ were performed, and $\xi_* = \frac{H_c - \alpha x^2}{\beta}$. For the poten-

tial in 2D we have

$$\tilde{U}_1\left(\vec{X}_0(\tau+t),s\right)= \\ =-\frac{e}{4\pi\epsilon_0}\int\tilde{n}\left(\vec{x}',s\right)\ln\left|\vec{x}'-\vec{X}_0(\tau+t)\right|d\vec{x}', \quad (18)$$

and analogously for $\tilde{U}_2$. These substitutions allow to write a weakly singular integral equation for $\tilde{n}_1(\vec{x},s)$ of the form (7) with the following kernel and left hand side:

$$K_s\left(\vec{x},\vec{x}'\right)=x'\int_0^{2\pi}\int_0^{\frac{2\pi}{\omega}}\left(\frac{\partial}{\partial\xi}\ln\left|\vec{x}'-\vec{X}_0(\tau+t)\right|\Big|_{\xi=\xi_*}+\right. \\ \left.+\ln\left|\vec{x}'-\vec{X}_0(\tau+t)\right|\Big|_{\xi=0}\delta\left(\alpha x^2-H_c\right)\right)\mathrm{e}^{\tau s}d\tau d\theta, \\ F_s(\vec{x})=\frac{Q}{e}\lambda_s\int_0^{2\pi}\int_0^{\frac{2\pi}{\omega}}\int_0^{\infty}\left(\frac{\partial}{\partial\xi}\ln\left|\vec{Y}(t_2)-\vec{X}_0(\tau+t)\right|\Big|_{\xi=\xi_*}+\right. \\ \left.+\delta\left(\alpha x^2-H_c\right)\ln\left|\vec{Y}(t_2)-\vec{X}_0(\tau+t)\right|\Big|_{\xi=0}\right)\times \\ \times\mathrm{e}^{(\tau-t_2)s}dt_2d\tau d\theta,\quad \lambda_s=-\frac{e}{4\pi\epsilon_0}\frac{e}{m_0\gamma}\frac{s}{\mathrm{e}^{\frac{2\pi}{\omega}s}-1}, \quad (19)$$

where terms proportional to $\delta\left(\alpha x^2-H_c\right)$ are responsible for surface charge perturbation.

INVERSE LAPLACE TRANSFORM AND COLLOCATION METHOD

The equations presented in the previous sections can be solved by the numerical methods from [9]. The next step is a computation of the inverse Laplace transform [10] via trigonometric series involving solution of the equations with complex Laplace variable $s_k=v+\mathrm{i}\frac{k\pi}{T}$, the expression and the meaning of the parameters, applicability conditions and uncertainties can be found in [11].

In the collocation method domain of the equation is subdivided into indexed sub-domains $G_{\mathbf{j}}$, where $\mathbf{j}$ is a multi-index of the dimension of the equation. Then in each subdomain the numerated by multi-index $\mathbf{m}$ set $\{\boldsymbol{\xi}^{\mathbf{j},\mathbf{m}}\}$ of collocation points is defined and the solution is written in such a form

$$\begin{pmatrix}\mathrm{Re}\\ \mathrm{Im}\end{pmatrix}\{\tilde{n}(\vec{x},s_k)\}=\sum_{\vec{m}=1}^{m}\begin{pmatrix}c_{\mathbf{j}}^{\mathbf{m}}(s_k)\\ d_{\mathbf{j}}^{\mathbf{m}}(s_k)\end{pmatrix}\boldsymbol{\phi}^{\mathbf{j},\mathbf{m}}(\vec{x}), \quad (20)$$

where $\vec{x}\in G_{\mathbf{j}}$, $c_{\mathbf{j}}^{\mathbf{m}}(s_k)$ and $d_{\mathbf{j}}^{\mathbf{m}}(s_k)$ are coefficients and $\boldsymbol{\phi}^{\mathbf{j},\mathbf{m}}(\vec{x})$ are polynomials of some special form . The initial integral equation with complex s_k is equivalent to a linear system of equations

$$\begin{pmatrix}-\lambda A+\mu B+I & \lambda B+\mu A\\ -\lambda B-\mu A & \lambda A+\mu B+I\end{pmatrix}\begin{pmatrix}\vec{c}\\ \vec{d}\end{pmatrix}=\begin{pmatrix}\vec{f}\\ \vec{g}\end{pmatrix}, \quad (21)$$

where I is a unit matrix and $A,\ B$ are square matrices defined by

$$\begin{pmatrix}A\\ B\end{pmatrix}=\int_{G_{\mathbf{j}}}\begin{pmatrix}\mathrm{Re}\\ \mathrm{Im}\end{pmatrix}K_s\left(\boldsymbol{\xi}^{\mathbf{i},\mathbf{l}},\vec{y}_s\right)\boldsymbol{\phi}^{\mathbf{j},\mathbf{m}}(\vec{y}_s)\,d\vec{y}_s, \quad (22)$$

$\vec{c}$ and $\vec{d}$ are coefficients from (20) and

$$\begin{pmatrix}\vec{f}\\ \vec{g}\end{pmatrix}=\begin{pmatrix}\mathrm{Re}\\ \mathrm{Im}\end{pmatrix}F_{v+\mathrm{i}\frac{k\pi}{T}}\left(\boldsymbol{\xi}^{\mathbf{i},\mathbf{l}}\right), \quad (23)$$

in all these expressions transformation from sets of multi-indices to a single index or double index for matrices is done. Computing the inverse Laplace transform of the solutions (20) we have the time dependent perturbation of the charge density

$$n(\vec{x},t)=\frac{\mathrm{e}^{vt}}{T}\sum_{\vec{m}=1}^{m}\left[-\frac{1}{2}c_{\mathbf{j}}^{\mathbf{m}}(v,0)+\right. \\ \left.+\sum_{k=0}^{N}\left[c_{\mathbf{j}}^{\mathbf{m}}(s_k)\cos\frac{k\pi}{T}t-d_{\mathbf{j}}^{\mathbf{m}}(s_k)\sin\frac{k\pi}{T}t\right]\right]\boldsymbol{\phi}^{\mathbf{j},\mathbf{m}}(\vec{x}), \quad (24)$$

where $\vec{x}\in G_{\mathbf{j}}$.

PRESENT STATUS

Currently we have a C++ program which computes (22) and (23), solves the linear system (21) and gives the final answer via (24). Multidimensional integrals were calculated by NAG library and compared with results given by Mathematica. The program works fine in both cases: 2D with KV distribution and 3D with the Maxwell one, however, it is not yet tested enough and we will present our numerical calculations elsewhere later.

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