TWISS PARAMETERS OF COUPLED PARTICLE BEAMS WITH EQUAL EIGENEMITTANCES

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Abstract

We show that the 1D Courant-Snyder theory can be considered as a partial case of the multidimensional theory of coupled particle beams with equal eigenemittances.

INTRODUCTION

The parametrization of coupled beam motion has been studied intensively over the past decades. Nevertheless, there is still no representation of general coupled motion that would be as complete and as widely accepted as the 1D Courant-Snyder theory. But is it really so, that the Courant-Snyder theory is, essentially, the theory which is applicable only to the 1D linear motion? Our answer is that the one-dimensionality of motion is not the main source of the elegance of the Courant-Snyder approach. It is the property of the beam matrix to be proportional to the matrix which is simultaneously symmetric positive definite and symplectic. Because this property is independent from the phase space dimensions and is the characteristic property of the particle beams with equal eigenemittances, the purpose of this paper is to show that the 1D Courant-Snyder theory can be considered as a natural partial case of the multidimensional theory of coupled particle beams with equal eigenemittances. It is also possible to say (in the opposite way), that in this paper we extend the 1D Courant-Snyder formalism to the multidimensional theory of beams with equal eigenemittances. Due to space limitation, we consider only the most basic theoretical questions and the more detailed study will be presented in a separate publication.

BEAM MATRIX AND EIGENEMITTANCES

Let us consider a collection of points in $2n$-dimensional phase space (a particle beam) and let, for each particle,

$$z \,=\, (q,p)^\top \,=\, (q_1,\ldots,q_n,p_1,\ldots,p_n)^\top \tag{1}$$

be a vector of canonical coordinates and momenta. Then, as usual, the beam (covariance) matrix is defined as

$$\Sigma = \Big\langle \big(z - \langle z\rangle\big)\cdot\big(z - \langle z\rangle\big)^\top \Big\rangle \stackrel{\text{def}}{=} \begin{pmatrix} \Sigma_{qq} & \Sigma_{qp} \\ \Sigma_{pq} & \Sigma_{pp} \end{pmatrix}, \tag{2}$$

where the brackets $\langle\,\cdot\,\rangle$ denote an average over a distribution of the particles in the beam and $n \times n$ submatrices of the $2n \times 2n$ matrix Σ satisfy

$$\Sigma_{qq} \,=\, \Sigma_{qq}^\top, \quad \Sigma_{pp} \,=\, \Sigma_{pp}^\top, \quad \Sigma_{pq} \,=\, \Sigma_{qp}^\top. \tag{3}$$

One says that the beam is uncoupled if all four submatrices of the matrix Σ are diagonal matrices. By definition, the matrix Σ is symmetric positive semidefinite and in the following we will restrict our considerations to the situation when this matrix is nondegenerated and therefore positive definite. For simplification of notations and without loss of generality, we will also assume that the beam has vanishing first-order moments, i.e. $\langle z\rangle \,=\, 0$.

Let s be the independent variable (time or path length along design orbit) and let $M(\tau)$ be the linear transfer matrix which propagates particle coordinates from the state $s = 0$ to the state $s = \tau$, i.e let

$$z(\tau) \,=\, M(\tau)\, z(0). \tag{4}$$

Then the beam matrix Σ evolves between these two states according to the rule

$$\Sigma(\tau) \,=\, M(\tau)\,\Sigma(0)\, M^\top(\tau). \tag{5}$$

If the matrix $M(\tau)$ is symplectic and satisfies the relations

$$M^\top(\tau)\, J M(\tau) \,=\, M(\tau)\, J M^\top(\tau) \,=\, J, \tag{6}$$

where

$$J \,=\, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{7}$$

is the $2n \times 2n$ symplectic unit matrix and I is the $n \times n$ identity matrix, then the transport equation (5) can be transformed into the following equivalent form

$$(\Sigma J)(\tau) \,=\, M(\tau)\cdot(\Sigma J)(0)\cdot M^{-1}(\tau). \tag{8}$$

From this form of the equation (5) we see that the eigenvalues of the matrix ΣJ are invariants, because (8) is a similarity transformation. The matrix ΣJ is nondegenerated and is similar to the skew symmetric matrix $\Sigma^{1/2} J\, \Sigma^{1/2}$

$$\Sigma J \,=\, \Sigma^{1/2}\cdot(\Sigma^{1/2} J\, \Sigma^{1/2})\cdot\Sigma^{-1/2}, \tag{9}$$

which means that its spectrum is of the form

$$\pm i\epsilon_1,\,\ldots,\,\pm i\epsilon_n, \tag{10}$$

where all $\epsilon_m > 0$ and i is the imaginary unit.

The quantities ϵ_m are called eigenemittances and are generalizations of the usual 1D rms emittances

$$\varepsilon_m \,=\, \sqrt{\langle q_m^2\rangle\langle p_m^2\rangle - \langle q_m p_m\rangle^2} \tag{11}$$

to the fully coupled case [1]. It is not difficult to prove that the set of all eigenemittances $\{\epsilon_m\}$ and the set of all projected emittances $\{\varepsilon_m\}$ of a given beam matrix Σ coincide if and only if the beam is uncoupled.

The other approach to the concept of eigenemittances is the way pointed out by Williamson's theorem (see, for example, references in [1]). This theorem tells us that one can diagonalize any positive definite symmetric matrix Σ by congruence using a symplectic matrix T

$$T\,\Sigma\, T^\top \,=\, D, \tag{12}$$

and that the diagonal matrix D has the very simple form

$$D \,=\, \text{diag}(\Lambda,\Lambda), \quad \Lambda \,=\, \text{diag}(\epsilon_1,\,\ldots,\,\epsilon_n) > 0, \tag{13}$$

where the diagonal elements ϵ_m are the moduli of the eigenvalues of the matrix ΣJ. The matrix T in (12) is not unique, but the diagonal entries of the Williamson's normal form D (eigenemittances) are unique up to a reordering.

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TWISS PARAMETERS OF THE BEAM MATRIX WITH EQUAL EIGENEMITTANCES

Let us assume that the beam matrix Σ has all eigenemittances equal to each other and equal to the value $\epsilon > 0$. Then, according to the Williamson's theorem, there exists a symplectic matrix T such that

$$\Sigma \,=\, \epsilon T^{-1} T^{-\top} \stackrel{\text{def}}{=} \epsilon W. \tag{14}$$

The matrix W in (14) is independent from any particular choice of the diagonalizing matrix T in (12) and is simultaneously symmetric positive definite and symplectic. We will call it the Twiss matrix and will parametrize it as follows

$$W \,=\, \begin{pmatrix} \beta & -\alpha \\ -\alpha^\top & \gamma \end{pmatrix}, \tag{15}$$

where the $n \times n$ submatrices $\beta = \beta^\top$, α and $\gamma = \gamma^\top$ are the natural matrix generalizations of the corresponding 1D scalar Twiss parameters. Due to symplecticity of the matrix W the matrix Twiss parameters satisfy the relations

$$\beta\gamma \,=\, I + \alpha^2, \tag{16}$$

$$\alpha\beta \,=\, \beta\alpha^\top, \tag{17}$$

$$\gamma\alpha \,=\, \alpha^\top\gamma. \tag{18}$$

Because the matrix W is positive definite, both its submatrices β and γ are also positive definite and, therefore, nondegenerated. From this it follows that not all relations (16)-(18) are independent. For example, the relations (17) or the relations (18) can be omitted.

The inverse of the Twiss matrix W is given by the formula

$$W^{-1} \,=\, \begin{pmatrix} \gamma & \alpha^\top \\ \alpha & \beta \end{pmatrix}, \tag{19}$$

and, therefore, the natural multidimensional analogy of the positive definite 1D Courant-Snyder invariant

$$I_{cs} \,=\, z^\top W^{-1} z \tag{20}$$

can be written as follows

$$I_{cs} = q^\top\gamma\, q + q^\top\alpha^\top p + p^\top\alpha\, q + p^\top\beta\, p. \tag{21}$$

In general, the eigenemittances of the matrix Σ have to be found as solution of the eigenvalue problem for the matrix ΣJ. But if one wants only to know if all of them are equal to each other or not, then the solution of the eigenvalue problem is not necessary. It can be done by simple matrix multiplication as explained in the following proposition.

Proposition 1 *The beam matrix Σ has all eigenemittances equal to each other and equal to the value $\epsilon > 0$ if and only if the equality*

$$(\Sigma\, J)^2 \,+\, \epsilon^2\, I \,=\, 0 \tag{22}$$

holds, i.e. if and only if the matrix $(\Sigma\, J)^2$ is a negative scalar matrix.

PARAMETRIZATION OF BEAM TRANSFER MATRIX, NORMALIZED VARIABLES AND PHASE ADVANCES

Due to nonuniques of the diagonalizing matrix T in the Williamson's theorem the relation (14) can be considered as a multi-valued function which maps any particular matrix Σ (and/or W) into some subset of symplectic matrices. Let us select some single-valued branch of this function which associates one, and only one, output to any particular input. We will call any such branch as T-algorithm and the examples of the T-algorithms will be given below. So, let us fix some particular T-algorithm and let us substitute the representation (14) into the equation (5). Then, after some straightforward manipulations, we obtain

$$\left(T(\tau)M(\tau)T^{-1}(0)\right) \cdot \left(T(\tau)M(\tau)T^{-1}(0)\right)^\top = I, \tag{23}$$

which means that the $2n \times 2n$ matrix

$$R(\tau) \,=\, T(\tau)\, M(\tau)\, T^{-1}(0) \tag{24}$$

is orthosymplectic (i.e. orthogonal and symplectic simultaneously). The equality (24), when written in the form

$$M(\tau) \,=\, T^{-1}(\tau)\, R(\tau)\, T(0), \tag{25}$$

gives us a (familiar in 1D) parametrization of the beam transfer matrix $M(\tau)$, and if we will introduce normalized variables z_n by the equation

$$z(s) \,=\, T^{-1}(s)\, z_n(s), \tag{26}$$

then the dynamics in the normalized variables

$$z_n(\tau) \,=\, R(\tau)\, z_n(0) \tag{27}$$

is simply a rotation and the multidimensional Courant-Snyder invariant (21) takes on the form $I_{cs} \,=\, z_n^\top \cdot z_n$.

Although the motion in the normalized variables (27) is not, in general, uncoupled motion, but from the point of view of the theory of beams with equal eigenemittances no additional simplifications are required, because

$$\Sigma_n \stackrel{\text{def}}{=} \left\langle z_n \cdot z_n^\top \right\rangle \,=\, \epsilon\, I \tag{28}$$

is already a diagonal matrix.

Let us turn now our attention to the concept of phase advances, which does not have an unique choice even in the 1D case (see, for example, discussion in [2]). The phase advances are the quantities which should be associated with the eigenvalues of the matrix R. This matrix is orthosymplectic and can be partitioned into the form

$$R \,=\, \begin{pmatrix} C & S \\ -S & C \end{pmatrix}, \tag{29}$$

where the $n \times n$ submatrices C and S satisfy

$$CS^\top = SC^\top, \qquad CC^\top + SS^\top = I. \tag{30}$$

All eigenvalues of the matrix $R(\tau)$ lie on the unit circle in the complex plane, i.e. are of the form

$$\exp(\pm i\mu_1(\tau)),\; \ldots,\; \exp(\pm i\mu_n(\tau)), \tag{31}$$

and $\mu_m(\tau)$ are the quantities which we will interpret as (fractional part of) phase advances.

If the beam transport in (5) is periodic (i.e. if $\Sigma(\tau) = \Sigma(0)$ and, therefore, $T(\tau) = T(0)$), then the equality (25) tells us that the eigenvalues of the matrix $R(\tau)$ are the same as the eigenvalues of the matrix $M(\tau)$. It means that for the periodic beam transport the phase advances are uniquely defined independently from any particular choice of the T-algorithm. It is a very pleasant property, but it seems that it is the only property of the phase advances which does not depend from the choice of the T-algorithm. Let us, for illustration, consider three 1D T-algorithms defined by the requirement for the matrix T in (14) to be in one of the following special forms

$$T_1 = \begin{pmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{pmatrix}, \; T_2 = \begin{pmatrix} \sqrt{\gamma} & \alpha/\sqrt{\gamma} \\ 0 & 1/\sqrt{\gamma} \end{pmatrix}, \quad (32)$$

$$T_3 = \frac{1}{\sqrt{(\beta+1)+(\gamma+1)}} \begin{pmatrix} \gamma+1 & \alpha \\ \alpha & \beta+1 \end{pmatrix}. \quad (33)$$

Note that here T_1 is the original Courant-Snyder choice and T_3 is symmetric (and symplectic) positive definite square root of the matrix W^{-1}. In all these cases the matrix $R(\tau) = R(\mu_m(\tau))$ in (25) is given by

$$R(\mu_m(\tau)) = \begin{pmatrix} \cos(\mu_m(\tau)) & \sin(\mu_m(\tau)) \\ -\sin(\mu_m(\tau)) & \cos(\mu_m(\tau)) \end{pmatrix}, \quad (34)$$

and, in order to see more clearly the difference between the behavior of the phase advances μ_m, let us assume that the dynamics is derived from the Hamiltonian

$$H(\tau) = (1/2) \cdot \left(p_1^2 + k(\tau)\, q_1^2 \right). \quad (35)$$

Then the phase advances μ_m obey the equations

$$\frac{d\mu_1}{d\tau} = \frac{1}{\beta}, \qquad \frac{d\mu_2}{d\tau} = \frac{k}{\gamma}, \quad (36)$$

$$\frac{d\mu_3}{d\tau} = \frac{k\,(\beta+1) + (\gamma+1)}{(\beta+1) + (\gamma+1)}. \quad (37)$$

One sees that while μ_2 stays constant in the drift spaces, μ_1 and μ_3 change; while μ_1 changes monotonously, μ_2 and μ_3 can be locally increasing and decreasing; and etc. Besides that, let us note that the multiplication of any matrix T_m from the left by a constant rotation matrix does not change the corresponding phase advance. For example, for an arbitrary constant angle ψ the T-algorithm associated with the matrix

$$\begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix} \cdot T_1 \quad (38)$$

will produce the same phase advance as in the original Courant-Snyder case, but, in general, the triangular form of the transition to the normalized variables (26), which is also important for 1D theory, will be lost. So, it seems that the correct way to extend the Courant-Snyder choice of the 1D phase advance to the multidimensional case without losing what else important comes through the direct generalization of the 1D T-algorithm defined by the matrix T_1 to many dimensions.

The matrix T_1 is a lower triangular matrix with positive diagonal elements and as its multidimensional analog we will take the lower block triangular symplectic matrix

$$T = \begin{pmatrix} w^{-1} & 0 \\ uw^{-1} & w^{\top} \end{pmatrix}, \quad (39)$$

where u and w are, respectively, a symmetric and an invertible $n \times n$ matrix. Note that any lower block triangular symplectic matrix can be represented in this form with the proper choice of the matrices u and w.

Proposition 2 *The beam matrix Σ can be diagonalized by congruence using symplectic block triangular transformation of the form (39) if and only if*

$$(\Sigma J)^2_{qp} = \Sigma_{qq} \Sigma^{\top}_{qp} - \Sigma_{qp} \Sigma_{qq} = 0, \quad (40)$$

and the condition (40) is invariant under linear symplectic transport (5) of the beam matrix Σ if and only if the matrix Σ is the matrix with equal eigenemittances.

This proposition tells us that though not only Σ matrices with equal eigenemittances can be diagonalized by the matrix (39), the consistent theory of diagonalization by the lower block triangular symplectic matrices can be created only for the beams with equal eigenemittances.

Substituting representation (39) into relation (14) we obtain the following equations for the determination of the matrices w and u as functions of the given matrix Twiss parameters β, α and γ

$$\beta = w\, w^{\top}, \quad (41)$$

$$\alpha = w\, u\, w^{-1}, \quad (42)$$

$$\gamma = w^{-\top} (I + u^2)\, w^{-1}. \quad (43)$$

The general solution of the equation (41) can be written as $w = \hat{w}\, r$, where $\hat{w}$ is any particular solution of this equation and r is an arbitrary $n \times n$ orthogonal matrix. When some solution of the equation (41) is chosen, then the matrix u is uniquely defined from the equation (42) and is symmetric, and the equation (43) is satisfied automatically due to relation (16). Thus the only remaining uncertainty in the choice of the T-algorithm lies in the nonuniques of the solution of equation (41). Unfortunately, this uncertainty cannot be resolved simply by the requirement to recover the Courant-Snyder choice in the limit of multidimensional uncoupled beams. For example, both, w taken as unique Cholesky factor of the matrix β and $w = \beta^{1/2}$, where $\beta^{1/2}$ denotes an unique positive definite symmetric square root of the matrix β, will satisfy this requirement. But if we will take into account that the Courant-Snyder matrix T_1 has positive diagonal elements and will reformulate this property in the form that 1×1 diagonal submatrices of the matrix T_1 are symmetric positive definite matrices, then the choice for w becomes unique. One takes $w = \beta^{1/2}$ and obtains for the matrix T the final (and familiar from 1D theory) form

$$T = \begin{pmatrix} \beta^{-1/2} & 0 \\ \beta^{-1/2}\alpha & \beta^{1/2} \end{pmatrix}. \quad (44)$$

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