QUADRUPOLE SHAPES*

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Abstract

The usual practice of constructing quadrupoles from truncated cylindrical hyperbolae is put into question. A new shape is proposed. This shape has an analytic potential function. The exact shape of the analytic quadrupole may be impractical, but in the short case where aspect ratio ≈ 1, pole shapes can be spherical. The optimal spherical radius is found to be 1.65 times the aperture radius. In the long quad limit, the aberrations of order 5 and higher are much lower for the optimized shape.

INTRODUCTION

The multipole elements commonly used to control charged particle beams correspond to solution terms of the Laplace equation $\nabla^2 V = 0$, namely, in polar coordinates (r, θ), $r^n \cos n\theta$ in the system where the potential on axis is zero. Thus $n = 2$ for a quadrupole, 3 for a sextupole, etc. This implicitly assumes the elements are infinitely extended in the axial (z) direction, and of course in real beamlines, they are not. For $n = 2$, the intended linear dependence of the fields upon transverse coordinate is thus broken by the finiteness of the quadrupole. This results in nonlinear force terms and aberrations.

It is not obvious how to terminate the poles of a quadrupole. Often, they are simply truncated. Does the shape in the longitudinal direction matter? And if so, what shape is optimal? For very long quadrupoles, it can be argued that hyperbolic equipotential surfaces given by $r^2 \cos 2\theta =$ constant are optimal. However, this is only true sufficiently far from the ends; for quadrupoles whose length is comparable to or shorter than the aperture, the 2-D hyperbolic shape is clearly not optimal. What then is the optimal shape of quadrupoles in the short limit? What is the optimal shape in the long limit? Answering these questions is the subject of this paper.

Hardness of the Fringe Field

Let the strength function of the quadrupole be $k(z)$. Rigourously, this means $\partial_{xx} V = -\partial_{yy} V = k(z)$ along the axis $x = y = 0$, so that

$$V(x, y, z) \to \frac{k(z)}{2}(x^2 - y^2) \text{ as } (x, y) \to (0, 0) \qquad (1)$$

In the "hard-edge" limit, k is a step function. But using a discontinuous step function instead of an analytic function to calculate the optics leads to dramatically incorrect results. It is thus regrettable that almost all the major higher order optics codes allow calculation of third order optics in the "no fringe field" case. This case is unphysical because it brings a particle from the field-free region outside the quadrupole instantaneously into the region where $k \neq 0$ without traversing intermediate fields. For example, for electrostatic quadrupoles, this violates conservation of energy as the potential energy is thereby incremented without changing the kinetic energy (in magnetic quadrupoles, angular momentum conservation is violated).

Once the neophyte beamline designer has learned that the third order aberrations calculated without fringe fields are incorrect, he/she is still left with the impression that the fringe field is at fault and customizing it in some way will improve the third order optics. Further, of quadrupoles with the same effective length, those with short fringe fields are erroneously thought to be superior even though this often means they have smaller aperture. In fact, such quadrupoles are inferior, as their fifth and higher order aberrations are worse.

Simplified Fringe Fields

We are thus drawn towards fringe fields that are "soft" as opposed to "hard". In this limit, in the case of short quadrupoles, we dispense with the idealized 2-D hyperbolic shape, as there is anyway no region of the quadrupole that is sufficiently far from the fringe field. In other words, the quadrupole has no flat area where $k(z) = k_0$, a constant.

A common calculational technique is to fit the fringe field to a so-called Enge function[4]:

$$k_{\rm f}(z) \equiv \frac{k_0}{1 + \exp\left[\sum_{m=0}^{N-1} a_m \left(\frac{-z}{D}\right)^m\right]} \qquad (2)$$

where D is the aperture diameter. Generally one uses up to $N = 6$ coefficients. This function has the advantages that it is analytic, and tends realistically (exponentially) to asymptotic values. On the other hand, the Enge coefficients a_m are not directly related to any physical parameter. For example, varying any one of the a_m changes the quadrupole's effective length. Further, the hardness of the fringe field is not given by a single parameter but rather by a combination of all 6. This makes it difficult for a designer to learn for example that the third order aberration is insensitive to fringe field hardness.

A useful approximation is to set all Enge coefficients except a_1 to zero[2]. In that case, varying a_1 changes fringe field hardness without shifting the effective field boundary. The edge function becomes:

$$k_{\rm f}(z) \equiv \frac{k_0}{1 + \exp\left(\frac{-a_1 z}{D}\right)} = \frac{k_0}{2}\left[1 + \tanh\left(\frac{a_1 z}{2D}\right)\right] \qquad (3)$$

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To simplify notation, let us measure z in units of $2D/a_1$ (effectively, the fringe field thickness). In this way, we can write the strength function of a full quadrupole of effective length L as

$$k(z) = k_{\mathrm{f}}(z) - k_{\mathrm{f}}(z-L) = \frac{k_0}{2}\left[\tanh(z) - \tanh(z-L)\right] \tag{4}$$

Comparing quadrupoles of differing lengths, we wish the integrated strength $K = \int k dz = k_0 L$ to remain constant, so write k as:

$$k(z) = \frac{K}{2L}\left[\tanh(z) - \tanh(z-L)\right] \tag{5}$$

In the short limit, we have simply that k is the derivative of tanh:

$$k(z) = \frac{K}{2}\,\mathrm{sech}^2 z \tag{6}$$

PROPERTIES OF THE SECH-SQUARED QUAD

From this strength function, we can derive the potential V for all of space using the technique of analytic continuation given by Derevjankin[3]:

$$V(x,y,z) = -\Re\left\{\int_{z+iy}^{z+ix} dt \int_0^t k(\zeta)d\zeta\right\} \tag{7}$$

We find:

$$V = \frac{K}{2}\Re\left\{-\log[\cos(x-iz)] + \log[\cos(y-iz)]\right\} \tag{8}$$

See Fig. 1 where equipotential contours are plotted using as scaling potential $V_0 \equiv V(\frac{\pi}{4},0,0) = \frac{K}{4}\log 2$. The properties of V are perhaps more readily apparent if it is written in real form:

$$V(x,y,z) = \frac{K}{4}\log\frac{\cos^2 y\cosh^2 z + \sin^2 y\sinh^2 z}{\cos^2 x\cosh^2 z + \sin^2 x\sinh^2 z} \tag{9}$$

From the symmetry and Laplace's equation, it can be shown that the following expansion holds:

$$V(x,y,z) = \tag{10}$$
$$= \frac{k}{2}(x^2-y^2) - \frac{k''}{24}(x^4-y^4) + \frac{k''''}{720}(x^6-y^6) - \ldots$$

Taking derivatives to find the fields, and integrating for constant x, y, we see that in spite of V being nonlinear with x, y, the integral of quad strength is linear for any choice of $k(z)$.

The potential V of the sech^2 quadrupole is periodic with period π in the x and y directions. At large z, the cancellation of this grid of alternating sign potentials ensures the rapid exponential falloff of the field. This is somewhat realistic. In the case of electrostatic quadrupoles, the ground planes dashed in Fig. 1 can be thought of as some approximation of the beam pipe. In magnetic quadrupoles the yoke takes the place of the ground surfaces.

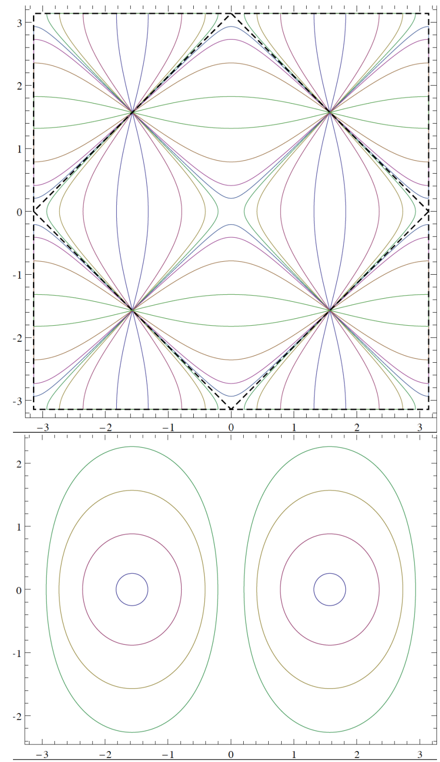

Figure 1: Equipotentials of eqn. 8. Upper: in $z = 0$ plane, contours $V = 4V_0$ (blue), $V = V_0$ (purple), $V = \frac{V_0}{4}$ (beige), $V = \frac{V_0}{16}$ (green), $V = -\frac{V_0}{16}$ (blue), $V = -\frac{V_0}{4}$ (purple), $V = -V_0$ (beige), $V = -4V_0$ (green). Lower: in $y = 0$ plane, contours $V = 4V_0$ (blue), $V = V_0$ (purple), $V = \frac{V_0}{4}$ (beige), $V = \frac{V_0}{16}$ (green).

Four choices of equipotential surfaces are shown in Fig. 2, oriented so that the quadrupole axis is vertical. Note the top left case is most like a long conventional quadrupole; the most significant difference being that the inside diameter varies along its length, as indicated by the $V = \pm\frac{V_0}{16}$ curves in the plot of Fig. 1. The lower right case in Fig. 2 would not give the correct fields without the 4 ground planes as the boundaries given by the 4 slender rods alone are insufficient. But the longer quadrupole (upper left in Fig. 2) case would work quite accurately without the ground planes.

As will be shown, this design has smaller aberrations than conventional designs, i.e. poles having constant xy cross section, truncated at each end. The only disadvantage is that the shape is rather more difficult to fabricate, having curvature on all directions.

POLE SHAPES

Short Limit

In the lower left of Fig. 2 (where the surfaces are strikingly similar in shape to four American regulation footballs) the potential of the shown surfaces is $\pm V_0$ and the curvature of the pole in the longitudinal direction is the same as in the transverse direction. This is an attractive

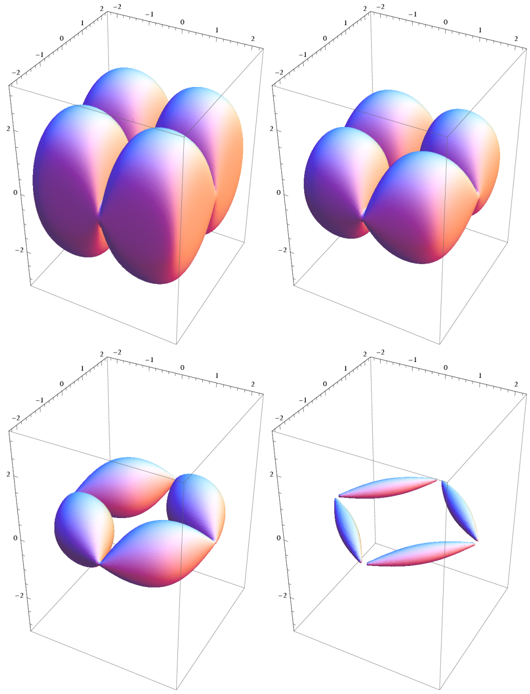

Figure 2: The coloured surfaces are 4 sets of equipotential surfaces of the potential (8). The quadrupole axis is vertical. In each case, the sides of the "box" containing the axes are also the 4 ground planes. All 4 give identical fields and the same sech^2 on-axis strength function if they are given the following potentials (left to right, top and then bottom), $\pm V_0/16, \pm V_0/4, \pm V_0, \pm 4V_0$ (adjacent surfaces have opposite sign).

feature because it allows as a good approximation for the pole to be symmetric along its axis, terminating in a spherical shape. At the normalized aperture radius $x = \pi/4$, we find the curvatures are equal and both equal 1. Or, in other words, the curvature of the pole-tip is to be

$$\text{pole-tip curvature} = \frac{4}{\pi}\,(\text{aperture radius}). \tag{11}$$

This $(4/\pi)$ is substantially different from the ratio of 1.145 used for quadrupoles whose poles are circular in cross section across the axis. The latter is derived by reducing the 12-pole (duodecapole) to zero for the 2D case (infinitely long quadrupoles).

As the xy cross section is nearly hyperbolic, while the xz cross section is nearly a circle, it is clear that choosing a hyperboloid of revolution as shape is no better than choosing a spherical shape. The latter has the advantage that it is simpler to specify to the machinist.

The shape used for practicality, namely cylindrical poles terminating in a spherical pole-face, omits important parts of the "football". This will have two effects: the quadrupole strength function $k(z)$ will not precisely follow a sech^2 law, and there will be some integrated 12-pole. The former is of little consequence, but the latter can cause aberration. However, just as with the case of the 2D quadrupole, we can alter the radius of curvature of the pole face to compensate the 12-pole. (20-pole and higher are not of course compensated in this technique but made slightly worse.)

In order to find the curvature radius that zeroes out the integrated 12-pole, Laplace's equation was solved for a 3D boundary model, using a uniform rectangular mesh in xyz of $100 \times 100 \times 300$ increments for $1/16$ of the quadrupole. The 12-pole integrated strength was found from a polynomial fit. The speed of convergence of the relaxation calculation was markedly increased when initialized with the function eqn. 8.

The result found is that the radius of curvature in units of the aperture radius is 1.65 ± 0.05:

$$\text{spherical pole-tip radius} = 1.65 \times (\text{aperture radius}). \tag{12}$$

The uncertainty arises from the grid coarseness and also from the variation due to surfaces "behind" the pole; surfaces which the engineer would be free to optimize for practicality. The potential on these surfaces also depends upon the insulator design in the case of electrostatic quadrupoles and the coil layout in the magnetic case.

Longer Quads

In principle, sech^2 quads can be built to any aspect ratio. Consider the equipotentials in the upper left of Fig. 2. The distinguishing feature would be that the aperture varies continuously along the axis, flaring out from a minimum at centre. What would be the advantage of such a complication compared with quads of constant aperture? It can be shown that quadrupoles of equal integrated strength and equal integrated strength-squared, will have equal third order aberration. However, for fifth and higher order, the sech^2 quadrupole has dramatically lower aberration than the conventional quad. An example is given and aberrations explored in a longer report[1].

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