EIGENMODE COMPUTATION FOR CAVITIES WITH PERTURBED GEOMETRY BASED ON A SERIES EXPANSION OF UNPERTURBED EIGENMODES*

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Abstract

The geometry of an accelerator cavity determines its eigenmodes and thereby its performance characteristics. Therefore, accelerating performance and wakefield characteristics may be improved by an intentional modification of the geometry. However, undesired geometry perturbations due to manufacturing tolerances and operational demands can likewise impair it. To analyze the effects of geometry variations on the eigenmodes, parameter studies are to be undertaken. Using common eigenmode solvers it usually is necessary to perform a full eigenmode computation for each variation step, even if the geometry is only slightly altered. Parameter studies for cavity perturbations thus tend to be computationally extensive and inefficient. In this paper, we present the fundamentals of an efficient eigenmode computation method for varying cavity geometries. Knowing a set of initial eigenmodes of an unperturbed geometry, the method allows expanding the eigenmodes of any geometry that is part of the unperturbed one as a series of the initial eigenmodes. Thereby the computation effort may be reduced significantly. The method is demonstrated by means of analytically evaluable cavity geometries.

INTRODUCTION

The shift of the resonant frequency of a specific eigenmode arising from a deformation of a cavity's geometry can be calculated using Slater's Theorem [1]

$$\frac{\widetilde{\omega}_\mathrm{i}-\omega_\mathrm{i}}{\omega_\mathrm{i}} = \frac{\iiint_{\Delta V}\left(\mu|\mathbf{H}_\mathrm{i}(\mathbf{r})|^2-\epsilon|\mathbf{E}_\mathrm{i}(\mathbf{r})|^2\right)\,\mathrm{d}V}{4U_\mathrm{i}}. \quad (1)$$

$\mathbf{E}_\mathrm{i}(\mathbf{r})$, $\mathbf{H}_\mathrm{i}(\mathbf{r})$ and U_i are the electric and magnetic stationary field pattern and the energy of the i$^{\mathrm{th}}$ mode of the unperturbed cavity. All characteristics of the perturbed cavity are distinguished from those of the unperturbed one by an additional tilde e.g. ω_i is the unperturbed and $\widetilde{\omega}_\mathrm{i}$ the perturbed resonant frequency. ϵ and μ are the permittivity and the permeability inside the cavity. V is the unperturbed cavity volume and ΔV is the part that is removed from V by the deformation.

Slater's theorem (ST) offers a simple approximation for the frequencies of the perturbed cavity but not for the electromagnetic fields. In [2] and [3] a generalisation of ST is presented that also allows the calculation of the perturbed fields by using a set of unperturbed modes instead of a single one. For the generalisation not only the interaction of the corresponding unperturbed mode i with itself (like in (1)) is evaluated but also the interactions with (ideally) every other mode k

$$s_\mathrm{ik} = 2\iiint_{\Delta V}\left(\omega_\mathrm{i}\mu\mathbf{H}_\mathrm{i}(\mathbf{r})\cdot\mathbf{H}_\mathrm{k}(\mathbf{r})-\omega_\mathrm{k}\epsilon\mathbf{E}_\mathrm{i}(\mathbf{r})\cdot\mathbf{E}_\mathrm{k}(\mathbf{r})\right)\,\mathrm{d}V. \quad (2)$$

The complete algorithm is described in [3]. In [4] we examined the applicability of the method concluding that it provides very accurate results for the resonant frequencies and the fields of the perturbed cavity using a small set of unperturbed modes. However, it turned out that this is only the case if the order of the mode does not exceed an certain explicit limit. The results for modes beyond this order are profoundly incorrect. Seriously, the critical mode order and thereby the maximal number N_max of modes is exclusively limited by the extent of the relative geometry perturbation $\Delta\xi/\xi$

$$N_\mathrm{max} = \left\lceil\frac{\xi}{2\Delta\xi}\right\rceil - 1 \quad (3)$$

and cannot be improved by using a larger set of unperturbed modes. Consequently, we developed another method seeking to overcome this limitation. In the following, the new method is described exemplarily for the computation of electric fields. The magnetic fields may be derived from the electric ones.

THEORY

Since the electric field patterns of a cavity form a system of mutually orthogonal functions (δ_ik is the Kronecker Delta)

$$\delta_\mathrm{ik} = \frac{\epsilon}{2U_\mathrm{i}}\iiint_{V}\mathbf{E}_\mathrm{i}(\mathbf{r})\cdot\mathbf{E}_\mathrm{k}(\mathbf{r})\,\mathrm{d}V \quad (4)$$

$$\delta_\mathrm{ik} = \frac{\epsilon}{2\widetilde{U}_\mathrm{i}}\iiint_{\widetilde{V}}\widetilde{\mathbf{E}}_\mathrm{i}(\mathbf{r})\cdot\widetilde{\mathbf{E}}_\mathrm{k}(\mathbf{r})\,\mathrm{d}V \quad (5)$$

it follows that the perturbed fields can be expanded in terms of the unperturbed ones

$$\widetilde{\mathbf{E}}_\mathrm{i}(\mathbf{r}) = \sum_{\mathrm{k}=1}^{N} a_\mathrm{ik}\cdot\mathbf{E}_\mathrm{k}(\mathbf{r}). \quad (6)$$

Here the a_ik are the sought weighting factors. Substituting (6) into (5) and normalizing $\frac{2\widetilde{U}_\mathrm{i}}{\epsilon}$ to 1 (disregarding the unit

* Work supported by Federal Ministry for Research and Education BMBF under contracts 05H09HR5 and 05K10HRC

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ISBN 978-3-95450-115-1

V^2m) the following expression can be derived

$$\delta_{ik} = \sum_{m=1}^{N}\sum_{n=1}^{N} a_{im}\cdot a_{kn} \iiint_{\widetilde{V}} \mathbf{E}_m(\mathbf{r})\cdot\mathbf{E}_n(\mathbf{r})\,\mathrm{d}V. \quad (7)$$

Since $\widetilde{V} = V + \Delta V$, the integral in (7), denoted as the interaction term g_{mn}, can also be written as

$$g_{mn} = \iiint_{\widetilde{V}} \mathbf{E}_m(\mathbf{r})\cdot\mathbf{E}_n(\mathbf{r})\,\mathrm{d}V \quad (8)$$

$$= \iiint_{\Delta V} \mathbf{E}_m(\mathbf{r})\cdot\mathbf{E}_n(\mathbf{r})\,\mathrm{d}V + \delta_{mn} \quad (9)$$

by using (4) and normalizing $\frac{2U_i}{\epsilon}$ also to 1. As can be seen, it is only necessary to evalute the unperturbed fields inside the (small) deformed volume ΔV to determine g_{mn}. Regarding the interactions of a set of N unperturbed modes (7) can be written in matrix form

$$I = A\,G\,A^T. \quad (10)$$

Examing the eigensystem of the interaction matrix G leads to the conclusion that the eigenvectors $\mathbf{p}_k$ form an orthogonal matrix P ($I = P\,P^T$, each row of P contains one $\mathbf{p}_k$) and that the major part ($\widehat{N}$) of the eigenvalues λ_k is 1 (the rest lies between 0 and 1). Since the diagonal matrix D_λ, containing the eigenvalues, is approximately identical to the identity matrix I and due to the orthogonality of P the eigenvalue problem can be transformed to

$$\begin{aligned} P^T\,D_\lambda &= G\,P^T \\ P^T\,I &\approx G\,P^T \\ I &\approx P\,G\,P^T. \end{aligned} \quad (11)$$

Comparing (10) and (11) shows that P approximately satisfies the orthogonal condition for the weighting factor matrix A. To achieve full validity all eigenvalues unequal to 1 are removed from the $N\times N$–matrix D_λ resulting in a new $\widehat{N}\times\widehat{N}$–matrix $\widehat{D}_\lambda = I$. Likewise, the corresponding eigenvectors are removed from P to create a $\widehat{N}\times N$–matrix $\widehat{P}$ for which $I = \widehat{P}\,\widehat{P}^T$ is still valid. (11) than becomes

$$I = \widehat{P}\,G\,\widehat{P}^T. \quad (12)$$

However, $\widehat{P}$ is not the desired matrix A. The fields computed thereby are orthogonal functions but no proper field patterns. Expanding (12) by an arbitrary orthogonal matrix W

$$I = W^T\,\widehat{P}\,G\,\widehat{P}^T\,W \quad (13)$$

leads to the conclusion that further solutions for (10) exist

$$A = W^T\,\widehat{P}. \quad (14)$$

To get a unique and particularly the correct matrix A an additional condition is needed. Therefore the Helmholtz equation is used, which has to be valid for every unperturbed and perturbed mode

$$\Delta\mathbf{E}_i(\mathbf{r}) = -\epsilon\mu{\omega_i}^2\cdot\mathbf{E}_i(\mathbf{r}) \quad (15)$$

$$\Delta\widetilde{\mathbf{E}}_i(\mathbf{r}) = -\epsilon\mu{\widetilde{\omega}_i}^2\cdot\widetilde{\mathbf{E}}_i(\mathbf{r}). \quad (16)$$

Substituting (6) and (15) into (16) results in

$$\sum_{k=1}^{N}\widetilde{\omega}_i^2\cdot a_{ik}\cdot\mathbf{E}_k(\mathbf{r}) = \sum_{k=1}^{N}{\omega_k}^2\cdot a_{ik}\cdot\mathbf{E}_k(\mathbf{r}). \quad (17)$$

which also can be expressed in matrix form

$$D_{\widetilde{\omega}^2}\,A = A\,D_{\omega^2} \quad (18)$$

after cancelling the field vectors. Further transformation of (18) using (14) and $I = \widehat{P}\,\widehat{P}^T$ leads to an expression which represents another eigenvalue problem

$$W\,D_{\widetilde{\omega}^2} = (\widehat{P}\,D_{\omega^2}\,\widehat{P}^T)\,W. \quad (19)$$

The matrix $(\widehat{P}\,D_{\omega^2}\,\widehat{P}^T)$ can be calculated from known quantities. Its eigenvalues provide the resonant frequencies $\widetilde{\omega}_i$ of the first $\widehat{N}$ perturbed modes. Its eigenvector matrix W^T can be inserted into (14) to determine the weighting factors a_{ik} which finally yield the perturbed electric fields using (6).

RESULTS

The new perturbation method was applied to coaxial cavity structures. Since both the unperturbed and the perturbed eigenmodes of such a (seemingly simple) structure are analytically known it is well suited to analyse the applicability and accuracy of the method excluding numerical errors. In the following, selected results of a coaxial cavity subjected to length variations are shown.

The most important fact is that the number of computable modes $\widehat{N}$ is no longer limited by the extent of the perturbation. In case of a relative perturbation of 10% the prior method only provided 4 correct modes no matter how many unperturbed modes were used (see (3)). Our investigations of the new method show that

$$\widehat{N} = (1 - \Delta\xi/\xi)N \quad (20)$$

i.e. by using 100 unperturbed modes for a 10% perturbation a total of 90 perturbed modes can be computed.

Furthermore, enlarging the set of unperturbed modes simultaneously increases the number of computable modes and their accuracy as fig. 1 shows. Based on a set of 300 modes the resonant frequencies of 270 perturbed modes were determined with a relative deviation of about 10^{-3} for a length reduction of 10%. Even for the extreme perturbation of 50% (which is beyond any genuine case) 50 perturbed frequencies were computed with an error smaller than 10^{-2} using only 100 unperturbed modes. So the method not only overcomes the limitation of the mode order but also provides results with a good accuracy. This

ISBN 978-3-95450-115-1

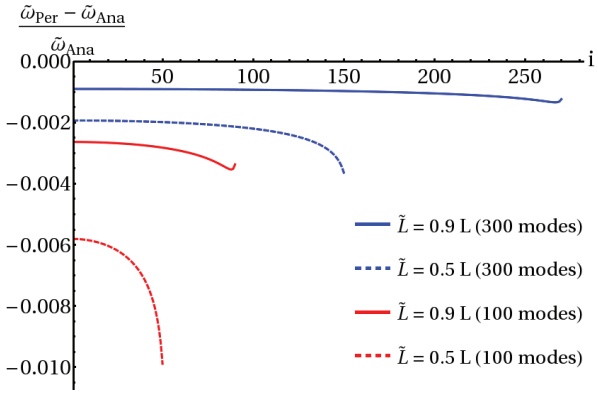

Figure 1: Relative deviation of the numerically approximated perturbed resonant frequency from the analytical one depending on the mode index i (length reduction of 10% / 50%, set of 100 / 300 unperturbed modes).

is also evident for the perturbed field patterns. The approximated fields of lower order modes and the actual ones match excellently. Even the higher order modes (HOMs) of an exceedingly perturbed cavity may be expanded in terms of a reasonable number of unperturbed modes with a very good agreement (like it is demonstrated for the TEM_{10} mode in fig. 2).

However, a non-avoidable effect occuring for HOMs was observed. The error of the resonant frequency directly effects the error of a single wavelength. Thus, the maximal deviation of a field increases by its mode order (see fig. 3). This effect can only be diminished by reducing the frequency error. This can be done by using a larger set of unperturbed modes but at the price of a higher computational effort. For this reason it is planned to revise the method's algorithm to basically improve its accuracy.

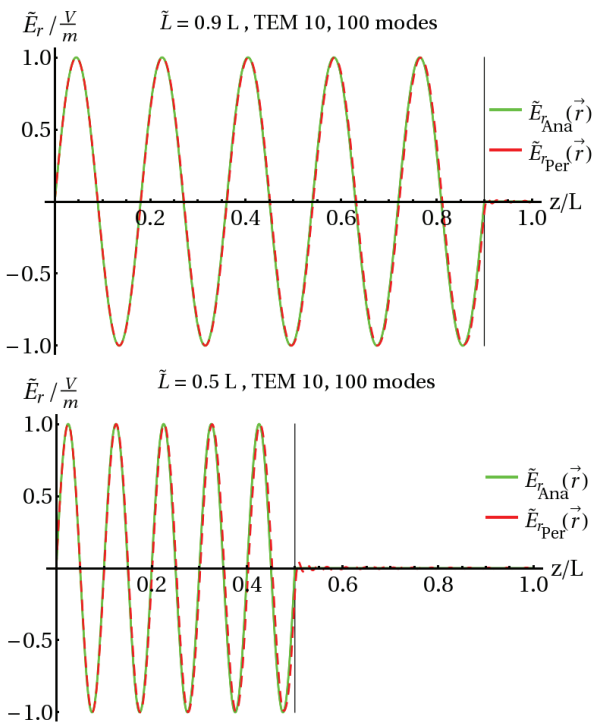

Figure 2: Radial electric field of the TEM_{10} mode along longitudinal axis (length reduction of 10% / 50%, set of 100 unperturbed modes): Approximated and analytical field both coincide so well that only one line can be observed.

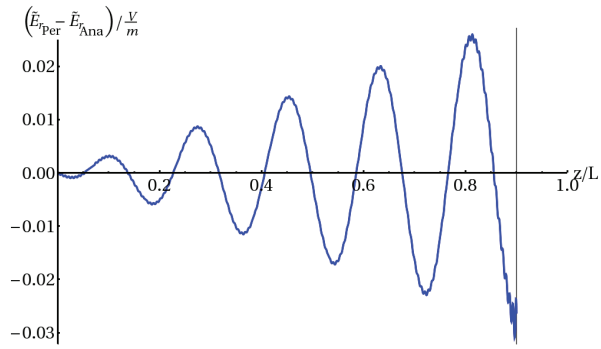

Figure 3: Radial electric field of the TEM_{10} mode along longitudinal axis (length reduction of 10%, set of 300 unperturbed modes): The deviation between the approximated and the analytical field scales with the number of wavelengths.

CONCLUSIONS

The presented algorithm is an efficient method for the computation of the eigenmodes of a perturbed cavity. It yields accurate results not only for the perturbed resonant frequencies but also for the electromagnetic field patterns. Furthermore, an arbitrary number of perturbed modes can be computed, determinable by the number of used unperturbed modes. So, the method offers a much larger range of application than other known perturbation methods ([1]: only frequencies for small perturbations determinable, [2], [3]: limitation of mode order). Since it is only necessary to evaluate the unperturbed electric fields (not the magnetic ones like in (2)) the computational effort may be further reduced. Possible improvements of the accuracy could be obtained by revising the algorithm.

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ISBN 978-3-95450-115-1