A MATRIX PRESENTATION FOR A BEAM PROPAGATOR INCLUDING PARTICLES SPIN

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Abstract

This approach based on the matrix formalism for Lie algebraic tools provides a constructive method for a beam propagator in magnetic and electrical fields. The beam propagator is evaluated in according to the well known Lie algebraic tools. But in contrast to traditional approaches matrix presentation for Lie propagators bases on two dimensional matrices.

INTRODUCTION

High-energy and nuclear physics using accelerators has reached a point where a very large fraction of the experiments require polarized beams. One of the the greatest triumph is the recent successful installation and commissioning of 'Siberian Snakes' and spin rotators at RHIC,the Relativistic Heavy-Ion Collider at BNL. RHIC is the world's first polarized proton collider. There are several ongoing works for creation different types of accelerators with polarized beams usage (for example NICA machine JINR, Dubna, Russia). The spin program is an important and integral part also for the NICA project. Indeed, ever since the "spin crisis" of 1987, the composition of the nucleons spin in terms of the fundamental constituents - quarks and gluons - remains in the focus of attention of many physicists. This section contains the discussion of the physics goals and perspectives of the spin program at NICA. The highlights of the NICA spin program include the measurements of Drell-Yan processes with longitudinally polarized proton and deuteron beams, spin effects in the inclusive and exclusive production of baryons, light and heavy mesons and direct photons, and the studies of helicity amplitudes and double spin asymmetries in elastic scattering. This section also addresses the issue of the competitiveness of the NICA spin program - it appears that the SPD detector at NICA would allow to contribute significantly to the current and planned international program in spin physics.

EQUATIONS

In general case particle motion equations

$$d\mathbf{X}/dt = \mathbf{F}(\mathbf{X}, t) = \sum_{k=1}^{\infty} \mathbb{P}^{1k}\mathbf{X}^{[k]} \qquad (1)$$

can be presented using so called matrix formalism for differential equations [1]. The solution of Eq. 1 can be written in the following form

$$\mathbf{X}(t) = \sum_{k=1}^{\infty} \mathbb{R}^{1k}(t|t_0)\mathbf{X_0}^{[k]}, \quad \mathbf{X_0} = \mathbf{X}(\mathbf{t_0}),$$

where $\mathbf{X_0}^{[k]}$ is so called Kronecker power of k-the order of the initial phase vector $\mathbf{X_0}$.

The matrices $\mathbb{R}^{1k}(t|t_0)$ can be evaluated using the matrix formalism for Lie algebraic tools [1]. In linear approximation one obtains well known linear solution

$$\mathbf{X} = \mathbb{R}^{11}(t|t_0)\mathbf{X}_0, \mathbf{X}_0 = \mathbf{X}(t_0).$$

The vector of spin components $\mathbf{S}$ changes with the time t of the laboratory frame according to the Thomas-Bargmann-Michael-Teledi (T-BMT) equation (see i.e. [2]). The spin precession equation can be written in two following forms

$$\frac{d\mathbf{S}}{dt} = \mathbf{W_s} \times \mathbf{S} = \mathbb{W} \cdot \mathbf{S},$$

where $\mathbb{W}$ is a skew-symmetric matrix

$$\mathbf{W_s} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \text{ and } \mathbb{W} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}.$$

We have to understand that $\mathbf{W_s}$ depends of the particle position in phase space $\mathbf{W_s} = \mathbf{W_s}(\mathbf{X})$ and lose some advantages of matrix formalism representation. However we can courageously build it in linear approach. In introduction we already mentioned about EDM-machine which will have only electrostatic elements. But in this paper neglect the electric fields (do'nt generality) and leave only components of magnetic field $\mathbf{B}$. It is used to work in accelerator coordinate system and all differentiation are by the independent variable s measured along a reference orbit. The process of converting the spin precession equation to accelerator coordinates described as well in [2, 3]. In this coordinate system one can write

$$\mathbf{W_s} = -\frac{e}{pc}(1+hx)\Big[(a\gamma+1)(B_x\mathbf{e_1}+B_s\mathbf{e_2}+B_y\mathbf{e_3}) - \\ - \frac{a\gamma^2\beta^2}{\gamma+1}\frac{B_x x' + B_s(1+h) + B_y y'}{(1+hx)^2 + x'^2 + y'^2} \times \\ \times (x'\mathbf{e_1} + (1+hx)\mathbf{e_2} + y'\mathbf{e_3})\Big] \qquad (2)$$

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This expression can be used for evaluation of components for the matrix $\mathbb{W}$. For example, for the element w_1 one can write

$$w_1 = -\frac{e}{pc}(1+hx)(a\gamma+1)B_x - \\ -\frac{a\gamma^2\beta^2}{\gamma+1}\frac{B_x x' + B_s(1+h) + B_y y'}{(1+hx)^2 + x'^2 + y'^2} y'^2 x' \quad (3)$$

The equalities (2) and (4) are used for forming our skew-symmetric matrix $\mathbb{W}$.

So, combining the motion equation for particles phase coordinates and spin components one can write the full motion equations system for space phase vector $\mathbf{X}$ and spin vector $\mathbf{S}$ in the following form

$$\frac{d\mathbf{X}}{dt} = \sum_{k=1}^{\infty} \mathbb{P}^{1k}\mathbf{X}^{[k]}, \\ \frac{d\mathbf{S}}{ds} = \mathbb{W}(\mathbf{X}) \cdot \mathbf{S}. \quad (4)$$

It should be note that the spin part of this system is linear by spin, but the matrix $\mathbb{W}$ is a function of phase space vector $\mathbf{X}$. This dependence leads us to necessity usage of special methods for evaluation of our equation system (4).

SPIN-MATCHED SYSTEM EVALUATING

The $\mathbf{X}$-dependence of the matrix $\mathbb{W}$ leads us to necessity to use some special methods for solution our problem. This problem reminds the well known problem of solution of self-consistent equations (see i. e. [1]). Similar to this problem we call the system (4) "self-spin matched" system. Introduce for more usability function $\mathbf{F}^{spin} = \mathbb{W}(\mathbf{X}) \cdot \mathbf{S}$. For the solution of this system we realize the next two steps.

At first, according to the paper [4] we should introduce the time-average spin vector $\langle \mathbf{S}(\mathbf{X}, s)\rangle_T$ in according to the following equality

$$\langle \mathbf{S}(\mathbf{X}, s)\rangle_T = \lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \mathbf{S}(\mathbf{X}, s)ds,$$

For the next we also introduce the ensemble-average spin-vector in according to the following equality

$$\langle \mathbf{S}(\mathbf{X}, s)\rangle_{\mathfrak{M}} = \frac{1}{mes(\mathfrak{M})}\int\limits_{\mathfrak{M}} \mathbf{S}(\mathbf{X}, s)dX,$$

where $\mathfrak{M}$ is the space phase set occupied by beam particles and $mes(\mathfrak{M})$ is its measure. It is not difficult to show that our dynamical system (4) is a ergodic system. It is known that for similar system it is true the following equality

$$\langle \mathbf{S}(\mathbf{X}, s)\rangle_T = \langle \mathbf{S}(\mathbf{X}, s)\rangle_{\mathfrak{M}}. \quad (5)$$

This equality allows us to use the approach described in the book [1].

In according to this approach one should change the right part of the second equation of Eq. 4 by the following equation:

$$\frac{d\langle \mathbf{S}\rangle_{\mathfrak{M}}(s)}{ds} = \langle \mathbb{W}\rangle_{\mathfrak{M}} \cdot \langle \mathbf{S}\rangle_{\mathfrak{M}}(s), \quad (6)$$

where $\langle \mathbf{S}(s)\rangle_{\mathfrak{M}}$ means that distribution function of particles spin includes in $\mathbf{F}^{spin}$ by integral way and integrating are by $\mathfrak{M} = \mathfrak{M}(s)$.

Let us introduce an operator and a function in according to the following rules: an evolution operator

$$\mathcal{M}(s|s_0): \ \mathbf{S}(s_0) \to \mathbf{S}(s)$$

and the function

$$\mathbf{F}^{spin} = \langle \mathbb{W}\rangle_{\mathfrak{M}} \cdot \langle \mathbf{S}\rangle_{\mathfrak{M}}(s).$$

These operator and function allows us to write the following equation

$$\frac{d\mathcal{M}(s, s_0, \mathcal{V})}{ds} = \mathcal{V} \circ \mathcal{M}(s, s_0, \mathcal{V}),$$

where

$$\mathcal{V} = \mathbf{F}^{spin}\frac{\partial}{\partial s}.$$

and initial state condition is

$$\mathcal{M}(s, s_0, \mathcal{V}) = \mathcal{I}d,$$

where $\mathcal{I}d$ is an identity operator. In according to [1] we can write the following algorithm for solution of our problem

It can be matched an integral equation in Volterr-Urison form. Write this equation in formal form

$$\mathcal{M} = \mathcal{A} \circ \mathcal{M}, \quad (7)$$

where $\mathcal{A}$ – Urison's operator. The main seal of solution existence of Eq. 7 is the method of successive approximations which helps to find out the existence of stable point of operator $\mathcal{A}$. In other words has to be build a sequence $\mathcal{M}^k = \mathcal{A} \circ \mathcal{M}^{k-1}$ by some initial value $\mathcal{M}^0$ then proof a convergence of $\mathcal{M}^k$ to some $\mathcal{M}^*$ and there is the parity $\mathcal{M}^* = \mathcal{A} \circ \mathcal{M}^*$.

Lets consider the general aspects of building a solution for Eq. 7 for motion with spin adjusted system (Eq. 4).

- *Step 0.* First of all set an interval for a solution $[s_0, s_1]$, $\Delta s = s_1 - s_0$. For the interval $[s_0, s_1]$ define a system of transportation therefore $\mathbf{X}$ – a component of $\mathbf{F}^{spin}$ function.
- *Step 1.* Then choose the distribution function $\mathbf{S}(\mathbf{X}(t), s_0) = \mathbf{S}_0(\mathbf{X})$ see [1, 4].
- *Step 2.* Calculate the evolution operator $\mathcal{M}: \mathcal{M}^0 = \mathcal{M}(s|s_0, \mathcal{V}, s \in [s_0, s_1]$.

- *Step 3.* Evaluate the current value of distribution function $\mathbf{S}^1(\mathbf{X}, s) = \mathbf{S}_0((\mathcal{M}^0)^{-1} \circ \mathbf{X_0})$.

- *Step 4.* Solve corresponding spin equations with distributing function $\mathbf{S}^1(\mathbf{X}, s)$ and retrieve $\mathbf{S}^1$.

- *Step 5.* Evaluate function $\mathbf{F}^{spin} = \mathbf{F}^{spin}(\mathbf{S}, \mathbf{X}, s)$.

- *Step 6.* One can solve the Eq. 7 and define $\mathcal{M}^1 = \mathcal{A} \circ \mathcal{M}^0$.

- *Step 7.* Retrieve a new value for $\langle f(\mathbf{X}, t_0)\rangle^1_{\mathfrak{M}}$ uses formula

$$\langle \mathbf{S}(\mathbf{X}, s_0)\rangle^1_{\mathfrak{M}_1} = (1-\alpha)\langle \mathbf{S}_0((\mathcal{M}')^{-1} \circ \mathbf{X}_0)\rangle_{\mathfrak{M}_0} + \\ + \alpha\langle \mathbf{S}_0((\mathcal{M}^{\infty})^{-1} \circ \mathbf{X}_0)\rangle_{\mathfrak{M}_0} \quad (8)$$

and $0 < \alpha < 1$.

- *Step 8.* Checking the specific criterion

$$||\mathcal{M}^k - \mathcal{A} \circ \mathcal{M}^{k-1}|| < \varepsilon, \quad k \geq 1. \quad (9)$$

If Eq. 9 is true the process of determination $\mathcal{M}$ on the interval $[t_0, t_1]$ finished. But if the condition of (9) is false the process repeating from step four with corresponding functions and operators redefinition.

Treat the Eq. 8 in general form matches for any step

$$\langle \mathbf{S}(\mathbf{X}, s_0)\rangle^k_{\mathfrak{M}_1} = \\ = (1-\alpha)\langle \mathbf{S}_0((\mathcal{M}^{\|-\infty})^{-1} \circ \mathbf{X}_0)\rangle_{\mathfrak{M}_0} + \\ + \alpha\langle \mathbf{S}_0((\mathcal{M}^{\|-\infty})^{-1} \circ \mathbf{X}_0)\rangle_{\mathfrak{M}_0}, \quad (10)$$

and $0 < \alpha < 1$, $\mathcal{M}^k = \mathcal{A} \circ \mathcal{M}^{k-1}$. This algorithm is different from some authors uses. Furthermore in most papers there are no studying a convergence problem except numerical algorithm examination.

CONCLUSION

This approach permit to apply all of matrix algebra opportunities and advantages in contrast with the tensor presentation based on multi-indexes description. The necessary computation can be realized in symbolic (using computer algebra codes as Mathematica, Mapple, Maxima and so on). The corresponding symbolic objects itself can be stored in special databases and used then in numerical computing. Parallel and distributed conception is well acceptable with the suggested matrix formalism.

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