ELECTRON-CLOUD PINCH DYNAMICS IN PRESENCE OF LATTICE MAGNET FIELDS

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Abstract

The pinch of the electron cloud due to a passing proton bunch was extensively studied in a field free region and in a dipolar magnetic field [1]. For the latter study, a strong field approximation helped to formulate the equations of motionand to understand the complex electron pinch dynamics, which exhibited some similarities with the field-free situation [2]. Here we extend the analysis to the case of electron pinch in quadrupoles and in sextupoles. We discuss the limits of validity for the strong field approximation and we evaluate the relative magnitude of the peak tune shift along the bunch expected for the different fields.

INTRODUCTION

It has recently been recognized that the pinch of the electrons during the bunch passage through an electron cloud may be responsible of long term incoherent effects on the protons [1, 2]. The studies of the induced diffusional regime have been made for simplified frozen models of the e-pinch taking place in special magnetic field configurations, namely in a field-free region and in a dipole field. Intuition suggests that the structure of the pinched electrons affects the diffusional regime, but till now no general study of the dependence of the electron pinch on a higher order magnetic field has been undertaken.

EQUATIONS OF MOTION

Here we derive the equations of motion for the electron dynamics in a quadrupolar field and under the effect of the passing proton bunch. In order to simplify the problem we make the following assumptions on the transverse electric and magnetic nonlinear fields:

1 The electric field is only transverse, i.e. $\vec{E} = (E_x, E_y, 0)$ and is given by an axisymmetric proton bunch.

2 The magnetic field is generated by an ideal quadrupole, i.e. $\vec{B} = (B_x, B_y, 0) = (g\,y, g\,x, 0)$, with g the field gradient.

For convenience we express the time variable in $s = v_p t$, where $v_p = c$ is the velocity of the protons. We also assume that $\gamma_e = \sqrt{1-\beta_e^2}$ does not vary significantly, i.e. the motion of the electrons remain non relativistic. The usual parameter defining the strength of a quadrupole is $k = g/(B\rho)$, where $B\rho$ is the rigidity. Recalling that $B\rho e = p_0 = m_p \gamma_p c$, where m_p is the mass of the proton, and $\gamma_p = 1/\sqrt{1-\beta_p^2}$, we find $g = kB\rho = k\gamma_p m_p c/e$. Under these conventions, the equations of motion of an electron become

$$\begin{aligned}
\frac{d^2x}{ds^2} &= -\frac{e}{c^2 m_e \gamma_e} E_x + k\frac{\gamma_p m_p}{\gamma_e m_e}\frac{dz}{ds}x \\
\frac{d^2y}{ds^2} &= -\frac{e}{c^2 m_e \gamma_e} E_y - k\frac{\gamma_p m_p}{\gamma_e m_e}\frac{dz}{ds}y \\
\frac{d^2z}{ds^2} &= -k\frac{\gamma_p m_p}{\gamma_e m_e}\left(\frac{dx}{ds}x - \frac{dy}{ds}y\right).
\end{aligned} \tag{1}$$

The analysis of the motion becomes easier by normalizing the electron coordinates to the transverse beam size σ_r, therefore $x = \sigma_r \tilde{x}$, $y = \sigma_r \tilde{y}$, $z = \sigma_r \tilde{z}$ (note the unusual scaling in z). At the same time we scale the "time" variable s according to $s = \sigma_z \hat{s}$. Therefore the equations of motion read

$$\begin{aligned}
\frac{d^2\tilde{x}}{d\hat{s}^2} &= -\sigma_z^2 \frac{e}{c^2 m_e \gamma_e \sigma_r} E_x + \sigma_z k \sigma_r \frac{\gamma_p m_p}{\gamma_e m_e}\frac{d\tilde{z}}{d\hat{s}}\tilde{x} \\
\frac{d^2\tilde{y}}{d\hat{s}^2} &= -\sigma_z^2 \frac{e}{c^2 m_e \gamma_e \sigma_r} E_y - \sigma_z k \sigma_r \frac{\gamma_p m_p}{\gamma_e m_e}\frac{d\tilde{z}}{d\hat{s}}\tilde{y} \\
\frac{d^2\tilde{z}}{d\hat{s}^2} &= -\sigma_z k \sigma_r \frac{\gamma_p m_p}{\gamma_e m_e}\left(\frac{d\tilde{x}}{d\hat{s}}\tilde{x} - \frac{d\tilde{y}}{d\hat{s}}\tilde{y}\right).
\end{aligned} \tag{2}$$

In these coordinates the "time" variable is normalized to the rms bunch length, and we will consider the range $-3 < \hat{s} < 3$ for a Gaussian bunch profile. Note that the electric field components

$$-\frac{e}{c^2 m_e \gamma_e \sigma_r} E_x, \quad -\frac{e}{c^2 m_e \gamma_e \sigma_r} E_x$$

have already been computed in Ref. [2], where $\omega_e(s)\quad [\mathrm{m}^{-1}] \equiv \sqrt{\lambda(s) r_e}/\sigma_r$, was introduced as the instantaneous linear electron oscillation frequency for an arbitrary longitudinal line particle density $\lambda(s)$, with r_e designating the classical radius of the electron. Using the radial coordinate $\tilde{r} = \sqrt{\tilde{x}^2 + \tilde{y}^2}$, the complete scaled equations of motion for an electron subjected to a proton electric field with longitudinal density $\lambda(\hat{s}) = N_b/(\sqrt{2\pi}\sigma_z)\exp\left(-\hat{s}^2/2\right)$ are

$$\begin{aligned}
\frac{d^2\tilde{x}}{d\hat{s}^2} &- k\sigma_z\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\frac{d\tilde{z}}{d\hat{s}}\tilde{x} + \sigma_z^2\omega_e^2(\hat{s})\tilde{x} = \\
&- \tilde{x}\sigma_z^2\frac{\omega_e^2(\hat{s})}{\tilde{r}^2}\left[2\left(1 - e^{-\frac{\tilde{r}^2}{2}}\right) - \tilde{r}^2\right] \\
\frac{d^2\tilde{y}}{d\hat{s}^2} &+ k\sigma_z\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\frac{d\tilde{z}}{d\hat{s}}\tilde{y} + \sigma_z^2\omega_e^2(\hat{s})\tilde{y} = \\
&- \tilde{y}\sigma_z^2\frac{\omega_e^2(\hat{s})}{\tilde{r}^2}\left[2\left(1 - e^{-\frac{\tilde{r}^2}{2}}\right) - \tilde{r}^2\right] \\
\frac{d^2\tilde{z}}{d\hat{s}^2} &= -k\sigma_z\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\left(\frac{d\tilde{x}}{d\hat{s}}\tilde{x} - \frac{d\tilde{y}}{d\hat{s}}\tilde{y}\right).
\end{aligned} \tag{3}$$

These equations show that the effect of the quadrupole on the electrons is proportional to the longitudinal scaled velocity $d\tilde{z}/d\hat{s}$ of the electron itself. Note that the last equation can be integrated as

$$\frac{d\tilde{z}}{d\hat{s}} = -\frac{1}{2}k\sigma_z\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\left(\tilde{x}^2 - \tilde{y}^2\right) + C$$

where C is a constant of motion which depends on the initial condition.

FIELD FREE ELECTRON DYNAMICS

Now we discuss whether the motion of the electrons can be thought as being of "field free" type, as it takes place in a drift space, or if it rather resembles an electron pinch under the action of a dipole field. For this purpose consider the coordinates defined as $\phi = \tilde{x} + \tilde{y}$, and $\psi = \tilde{x} - \tilde{y}$. This is similar to a rotation of 45 degree but not divided by $\sqrt{2}$. In these coordinates the electron radial position becomes $\tilde{r}^2 = (\phi^2 + \psi^2)/2$ and the three equations of motion take the form

$$\begin{aligned} \frac{d^2\phi}{d\hat{s}^2} &- k\sigma_z\sigma_r \frac{\gamma_p m_p}{\gamma_e m_e}\frac{d\tilde{z}}{d\hat{s}}\psi + \sigma_z^2\omega_e^2(\hat{s})\phi = \\ &-\phi\sigma_z^2\frac{\omega_e^2(\hat{s})}{\tilde{r}^2}\left[2\left(1-e^{-\frac{\tilde{r}^2}{2}}\right)-\tilde{r}^2\right] \\ \frac{d^2\psi}{d\hat{s}^2} &- k\sigma_z\sigma_r \frac{\gamma_p m_p}{\gamma_e m_e}\frac{d\tilde{z}}{d\hat{s}}\phi + \sigma_z^2\omega_e^2(\hat{s})\psi = \\ &-\psi\sigma_z^2\frac{\omega_e^2(\hat{s})}{\tilde{r}^2}\left[2\left(1-e^{-\frac{\tilde{r}^2}{2}}\right)-\tilde{r}^2\right] \\ \frac{d\tilde{z}}{d\hat{s}} &= -\frac{1}{2}k\sigma_z\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\phi\psi + C \end{aligned} \tag{4}$$

Motion on the diagonal $\tilde{x} + \tilde{y} = 0$. Consider an electron at $\hat{s}_{ini} = -3$, when the bunch enters into the cloud, with initial condition $\phi(\hat{s}_{ini}) = \tilde{z}(\hat{s}_{ini}) = 0$, and at rest, i.e. $d\phi/d\hat{s}(\hat{s}_{ini}) = d\tilde{z}/d\hat{s}(\hat{s}_{ini}) = 0$. Then the third equation of Eqs. 4 yields the constant of motion $C = 0$, and the first equation becomes

$$\begin{aligned} \frac{d^2\phi}{d\hat{s}^2} &+ \frac{1}{2}\left(k\sigma_z\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\right)^2\phi\,\psi^2 + \sigma_z^2\omega_e^2(\hat{s})\phi \\ &= -\phi\sigma_z^2\frac{\omega_e^2(\hat{s})}{\tilde{r}^2}\left[2\left(1-e^{-\frac{\tilde{r}^2}{2}}\right)-\tilde{r}^2\right]. \end{aligned} \tag{5}$$

Then $\phi(\hat{s}) = 0$ is a solution of this equation, i.e. the electron will evolve preserving $\tilde{x} + \tilde{y} = 0$. In fact $\phi(\hat{s}) = 0$ always satisfies Eq. (5) as well as the initial conditions.

Motion on the diagonal $\tilde{x} - \tilde{y} = 0$. The argument is similar. Now we consider an electron starting at $\hat{s}_{ini} = -3$ with initial conditions $\psi(\tilde{s}_{ini}) = \tilde{z}(\tilde{s}_{ini}) = 0$ and $d\psi/d\hat{s}(\tilde{s}_{ini}) = d\tilde{z}/d\hat{s}(\tilde{s}_{ini}) = 0$. Then the third equation yields the constant of motion $C = 0$, and the second equation becomes

$$\begin{aligned} \frac{d^2\psi}{d\hat{s}^2} &+ \frac{1}{2}\left(k\sigma_z\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\right)^2\phi^2\,\psi + \sigma_z^2\omega_e^2(\hat{s})\psi \\ &= -\psi\sigma_z^2\frac{\omega_e^2(\hat{s})}{\tilde{r}^2}\left[2\left(1-e^{-\frac{\tilde{r}^2}{2}}\right)-\tilde{r}^2\right]. \end{aligned} \tag{6}$$

In this case $\psi(\hat{s}) = 0$ is a solution.

The conclusion is that all electrons located on the diagonals and at rest before the bunch passes through the cloud will undergo a field-free pinch dynamics. Note that these electrons will also not posses any longitudinal velocity as $d\tilde{z}/d\hat{s} = 0$ during their entire motion. That means they remain in the transverse plane in which they were located at the beginning of the pinch. One can, therefore, speculate that in a certain region close to the two diagonals the electron motion will be dominated by field-free dynamics. Simulations confirm this behavior.

Motion of Electrons on the Axes

If the electrons on the diagonal do not change their longitudinal position $\tilde{z}$, we should expect that the electrons located along the x or y axis move significantly. Simulations show that the electrons on the two axes (referring to Fig. 1) effectively move along $\tilde{z}$ till (and some beyond) $\tilde{z} \sim 10$. That is they move off their plane of ~ 10 times the beam size. The electrons on the y axis moves positively up to (and some beyond) $\tilde{z} \sim 10$, while the electrons on the x axis move negatively up to (and some beyond) $\tilde{z} \sim -10$.

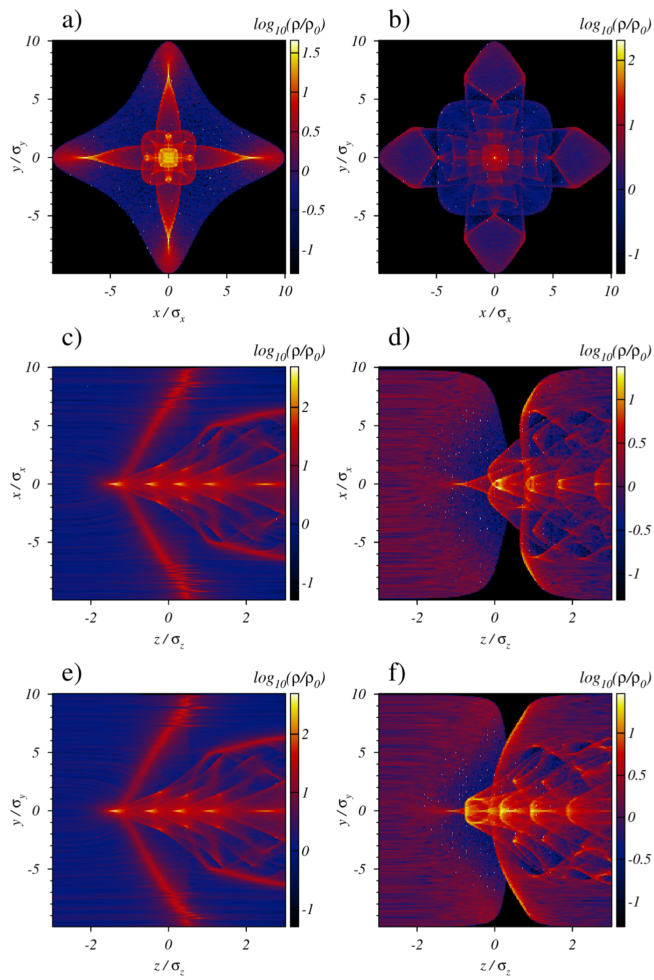

Figure 1: Electron density enhancement in: a) the $x - y$ plane at $z = 0$; b) the $x - y$ plane at $z = 1$; c) the $z - x$ plane at $y = 0$; d) the $z - x$ plane at $y = 2$; e) the $z - y$ plane at $x = 0$; f) the $z - y$ plane at $x = 1$.

STRONG-FIELD APPROXIMATION

Previous electron-pinch simulations took advantage of the strong field approximations. The strong-field approximation is based on the assumption that the motion of an electron is slow along the magnetic field lines (B-lines) so that the orthogonal component of motion to the B-lines makes several cyclotron rotations creating an average effect of cancellation of the transverse motion. The integrated effect is that the motion of the electron follows the B-lines. It also is invoked an adiabaticity and the preservation of an invariant which would be the cyclotron radius. We discuss here the applicability of this approach.

From the equation of motions it is easy to prove that in absence of electric fields the scaled velocity is constant, i.e. $d^2\tilde{v}/d\hat{s}^2 = 0$. The instantaneous cyclotron radius $\tilde{\rho}$ is

$$\frac{1}{\tilde{\rho}}\frac{d}{d\hat{s}}\tilde{R} = \sigma_z|k|\sigma_r\frac{\gamma_p m_p}{\gamma_e m_e}\tilde{r}$$

where $\tilde{r} = \sqrt{\tilde{x}^2+\tilde{y}^2}$, and $\tilde{R} = \sqrt{\tilde{x}^2+\tilde{y}^2+\tilde{z}^2}$ and the normalized cyclotron-oscillation period satisfies

$$\hat{\tau}\tilde{r} = \frac{2\pi}{|k|\sigma_z\sigma_r}\frac{\gamma_e m_e}{\gamma_p m_p}.$$

For the LHC parameters $\sigma_z = 0.114$ m, $k = 0.0097$ m^{-2}, $\sigma_r = 0.88$ mm, $\gamma_e = 1$, $\gamma_p = 450$ we find $\hat{\tau}\tilde{r} = 7.87$. Therefore at small radii $\tilde{r}$ the normalized cyclotron period $\hat{\tau}$ can become larger than the passage time of the bunch through the cloud. Certainly in that case no averaging effect can be invoked to apply the strong field approximation. Fig. 1 shows a pinch of electrons under the effect of an LHC bunch passage in a quadrupole: No strong field approximation is used. The only conditions in which the strong-B approximation can be applied are

1 when $\hat{\tau} \ll 6$ (6 is the scaled time interval for the bunch to go through the e-cloud), or equivalently when $1.33 \ll \tilde{r}$;

2 when the cyclotron radius is small with respect to the variation of the magnetic field, in other words when $\Delta B/B = 2\tilde{\rho}/\tilde{r} \ll 1$ that is

$$\frac{1}{\tilde{r}^2}\frac{d}{d\hat{s}}\tilde{R}\frac{1}{\sigma_z|k|\sigma_r}\frac{\gamma_e m_e}{\gamma_p m_p} \ll 1.$$

For LHC quadrupoles $(d\tilde{R}/d\hat{s})/\tilde{r}^2 \ll 0.8$. Again, this condition will be broken at small radius.

From this discussion we can certainly conclude that when the electrons are inside the beam the strong field approximation breaks down because the instantaneous cyclotron motion has a period longer than the time of passage of the bunch through the cloud. Therefore, it is not expected that the electrons follow the magnetic field lines.

GENERAL MULTIPOLE FIELD

We consider now the pinch of electrons in a general multipole formed by normal and skew components of strengths K_n, J_n. For convenience we define the variable $Z = x+iy$, so that the magnetic field components can be written as

$$V(Z) = B_y + iB_x = B\rho(K_n+iJ_n)\frac{Z^n}{n!}.$$

The equations of motion of an electron then become

$$\begin{array}{l} \frac{d}{dt}\left(m_e\gamma_e\frac{dZ}{dt}\right) = -e(E_x+iE_y) + e\frac{dz}{dt}(B_y - iB_x) \\ \frac{d}{dt}\left(m_e\gamma_e\frac{dz}{dt}\right) = -eRe\left[\frac{dZ}{dt}V(Z)\right] \end{array} \quad (7)$$

Now we note that

$$\frac{dZ}{dt}V(Z) = \frac{d}{dt}B\rho(K_n+iJ_n)\frac{Z^{n+1}}{(n+1)!}.$$

Therefore, the last equation (7) can be written as

$$m_e\gamma_e\frac{dz}{dt} = -eB\rho Re\left[(K_n+iJ_n)\frac{Z^{n+1}}{(n+1)!}\right] + C$$

with C a constant of motion depending on the initial conditions. In summary the equations of motion are

$$\begin{array}{l} \frac{d}{dt}\left(m_e\gamma_e\frac{dZ}{dt}\right) = -e(E_x+iE_y) + e\frac{dz}{dt}(B_y - iB_x) \\ m_e\gamma_e\frac{dz}{dt} = -eB\rho Re\left[(K_n+iJ_n)\frac{Z^{n+1}}{(n+1)!}\right] + C \end{array} \quad (8)$$

Now the fields (E_x, E_y), and (B_x, B_y) can be replaced in the first equation and the same coordinate normalization be applied to 8 as was done in (3).

Which are the electrons subjected to field free pinch under the dynamics of Eqs. 8? For an axisymmetric cross-section of the proton bunch the electric field is always radial as in general $E_x + iE_y = Zf(|Z|)$ (which is not the case if the bunch is not axisymmetric). Therefore the trajectory of the electrons undergoing a field free pinch is radial, of the form $Z = \alpha(t)Z_0$, with $Z_0 = x_0 + iy_0$ denoting the initial condition of the electron and $\alpha(0) = 1$. Of course, in the presence of a multipolar magnetic field, free pinch is possible only where the magnetic field does not play any role in the pinch process. In our model this happen only for those electrons which have $dz/dt = 0$ at any time along a radial trajectory. We will find now which are the allowed trajectories for field free pinch. Lets take an electron at $t = 0$ at rest, i.e. $dz/dt = 0$, then from Eq. 8 we find

$$C = eB\rho Re\left[(K_n+iJ_n)\frac{Z_0^{n+1}}{(n+1)!}\right].$$

The condition of existence of trajectories of the from $Z = \alpha(t)Z_0$ which keep $dz/dt = 0$ are found from the second equation of Eqs. 8, namely

$$(-\alpha(t)+1)Re\left[(K_n+iJ_n)Z_0^{n+1}\right] = 0. \quad (9)$$

Defining $K_n+iJ_n = \overline{K_n}e^{i\phi_n}$, and $Z_0 = R_0e^{i\psi_0}$, the pinch free condition (Eq. 9) is possible if $\cos[\psi_0(n+1)+\phi_n] = 0$, which happens only for the electrons initially located on $2n+1$ lines going thorough the origin and tilted with the angles

$$\psi_0 = \frac{\pi}{2(n+1)} - \frac{\phi_n}{n+1} + \frac{\pi}{n+1}N$$

with $N = 0, ..., 2n+1$. Note that the simultaneous presence of normal and skew component adds an angular shift of $-\phi_n/(n+1)$ to the web of field free pinch lines.

REFERENCES

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[2] G. Franchetti *et al.*, PRSTAB **12**, 124401 (2009).