

A TWO-DIMENSIONAL FEM CODE FOR IMPEDANCE CALCULATION IN HIGH FREQUENCY DOMAIN *

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Abstract

A new method, using the parabolic equation (PE), for the calculation of both high-frequency impedances of small-angle taper (or collimator) is developed in [1]. One of the most important advantages of the PE approach is that it eliminates the spatial scale of the small wavelength from the problem. As a result, only coarser spatial meshes are needed in calculating the numerical solution of the PE. We developed a new code based on Finite Element Method (FEM) which can handle arbitrary profile of a transition and speed up the calculation by orders of magnitude. As a first step, we completed and benchmarked a two-dimensional code. It can be upgraded to three-dimensional geometry.

INTRODUCTION

Although the computer is powerful nowadays, it is still a challenge to compute the wake field of an ultra short bunch over a long structure, such as a transition from a regular beam pipe to an insertion device. The longitudinal impedance of a small-angle collimator, step-in and step-out transitions, and a pillbox cavity has been calculated in previous work [1] using Mathematica [2] based on parabolic equation. This method can calculate high frequency impedance using relatively coarse mesh.

In this paper we developed a new code based on Finite Element Method (FEM), which can handle arbitrary geometry. As a first step, we completed a two-dimension code, and benchmarked our code with Mathematica code [1] and ECHO2 [3]. Our code shows good agreements with other two codes and it has short execution time and has potential for three-dimensional geometry.

The FEM (sometimes referred to as finite element analysis (FEA)) is a numerical technique for finding approximate solutions of partial differential equations (PDE) as well as of integral equations. It is a powerful approach which can better represent the geometry. The applications of FEM in particle accelerator field is extensive: magnet design, RF cavity design, impedance/wake field calculation, beam dynamics, etc..

The general wave equation is

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = i\omega\mu \mathbf{J} \quad (1)$$

Where $k = \omega/c = 2\pi/\lambda$ is propagation constant and μ is permeability.

We consider an axisymmetric geometry of a beam pipe with a varying radius and a straight axis which is chosen as the z-axis of the cylindrical coordinate system. The beam's field in a beam chamber can be divided into two

parts: field in open space and field due to the image charge and current (so called image field). In circular cylindrical coordinates, the radial image field ε for a single frequency component $\sim \exp(i\omega t)$ satisfies the equations

$$\frac{\partial^2 \varepsilon}{\partial z^2} + \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r \varepsilon}{\partial r} \right) + k^2 \varepsilon = 0, \quad (2)$$

and perfect metal boundary condition at the surface of beam pipe

$$\frac{i}{k} \frac{1}{r} \frac{\partial r \varepsilon}{\partial r} n_r = \left(\varepsilon + \frac{2I_0}{rc} \right) n_z, \quad (3)$$

Here n_r and n_z is the unit normal, I_0 is the beam current. A thin beam is assumed. The longitudinal field can be calculated from the radial image field as

$$E_z = \frac{i}{k} \frac{1}{r} \frac{\partial r \varepsilon}{\partial r}. \quad (4)$$

Finally, the (normalized) longitudinal impedance can be estimated as an integral of the longitudinal field

$$\frac{Z}{Z_0/4\pi} = -\frac{4\pi}{Z_0} \frac{1}{I_0} \int_0^\infty e^{-ikz} E_z(r=0, z) dz = -\frac{ic}{I_0 k} \int_0^{L_0} \frac{1}{r} \frac{\partial r \varepsilon}{\partial r} \Big|_{r=0} dz. \quad (5)$$

To speed up the calculation, the integral in Eq. (5) can be estimated using indirect method [4, 5]:

$$\frac{Z}{Z_0/4\pi} = -2 \sum_n \frac{c_n}{v_n} - \frac{ic}{I_0 k} \int_0^{L_0} \frac{1}{r} \frac{\partial r \varepsilon}{\partial r} \Big|_{r=0} dz \quad (6)$$

$$c_n = \frac{2c}{I_0 J_1(v_n)^2} \int_0^1 r \varepsilon(\rho) J_1(v_n \rho) d\rho, \quad \rho = r/a_0 \quad (7)$$

where v_n are the roots of the first kind of Bessel function of zero-th order; L_0 is the length of the structure and a_0 is the aperture at the exit.

When $\left| \frac{\partial^2 \varepsilon}{\partial z^2} / \frac{\partial \varepsilon}{\partial z} \right| \ll k$, Eq.(2) can be approximately described as

$$\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial r \varepsilon}{\partial r} \right) + 2ik \frac{\partial \varepsilon}{\partial z} = 0 \quad (8)$$

Eq.(8) is parabolic equation used in reference [1]. The FEM can solve either the full equation (2) or parabolic equation (8). The results presented in this paper are from

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the parabolic equation in order to benchmark with Mathematica code [1], which uses PE.

BENCHMARK WITH MATHEMATIC CODE

A small-angle collimator is used to benchmark with the Mathematica code. The collimator has a geometry given by

$$a(z) = a_0 \left(1 - A \sin^4 \pi \frac{z}{L_{coll}} \right) \quad (9)$$

where a_0 is the maximum aperture of the collimator, L_{coll} is the length of the collimator. A is a factor that determines the minimum aperture and $A=0.5$ is used for this comparison. Figure 1 shows the comparisons of fields given by Mathematica and FEM codes. Dimensionless coordinate r/a_0 and $\zeta(z) = z/ka_0^2$ are used. Figure 2 and 3 show the calculated impedance in low frequency region and high frequency region. Note that the dimensionless parameter l is used in the figure. It is defined as

$$l = L_{coll} / ka_0^2, \quad (10)$$

which is inversely proportional to the frequency. The two codes have very good agreement. At extreme low frequency, both codes converge to the theoretical value [6]

$$\frac{Z^{(1)}}{Z_0/4\pi} = -ik \int dz a'^2 = -\frac{5}{8} \pi^2 A^2 \frac{1}{l}. \quad (11)$$

Figure 2 and 3 show the impedance dependence on l . Therefore, the longitudinal impedance $Z(\omega, L)$ of a structure with length L can be scaled from the impedance of a longitudinal compressed structure $R(\omega, L/\alpha)$ with length L/α [7]

$$Z(\omega, L) = R(\omega/\alpha, L/\alpha). \quad (12)$$

This scaling property can greatly simplify the numerical calculation of the longitudinal impedance of small angle structure.

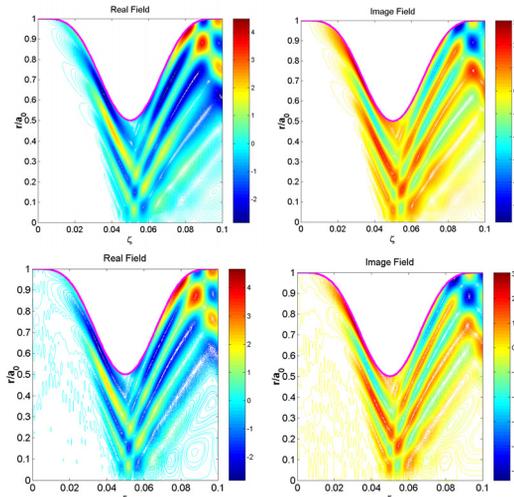


Figure 1: Real (Left) and imaginary (right) field calculated by Mathematica code (top) and FEM code (bottom), the pink line is the boundary of the beam pipe.

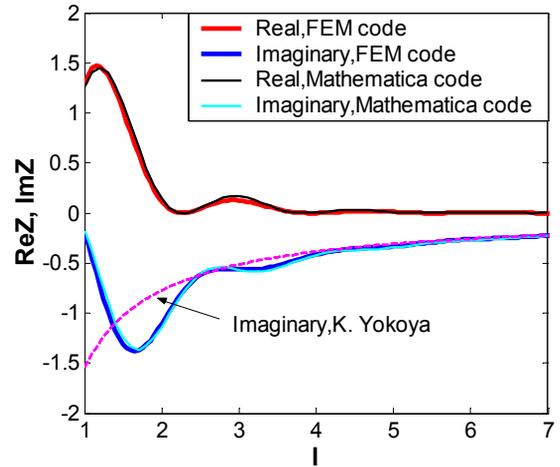


Figure 2: Normalized impedance in low frequency region calculated by FEM and Mathematica codes. The pink dashed-line is the theoretical value by Eq. (11), which was first obtained in [6].

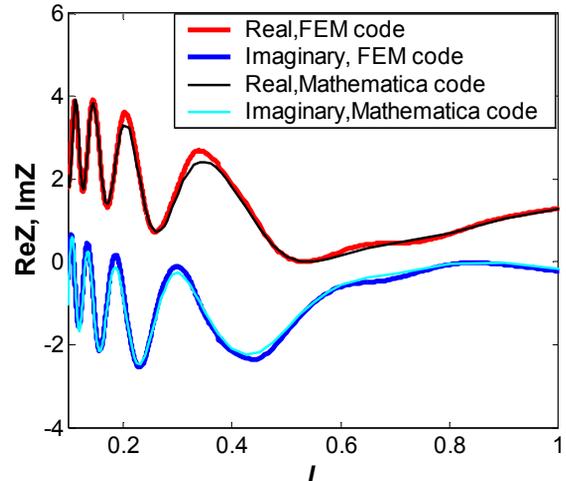


Figure 3: Normalized Impedance in high frequency region calculated by FEM and Mathematica codes.

BENCHMARK WITH ECHO2 CODE

ECHO2 is a two-dimensional code which calculates the wake field in time domain and the impedance can be derived from the wake by Fast Fourier Transform (FFT). Figure 4 shows the geometry of two collimators used for the comparisons. Small dimensions are chosen in both examples in order to reduce the CPU time of ECHO2. Meanwhile a small-angle collimator is used since we use parabolic equation. One good example is the transition section from a regular beam chamber to an insertion device. Figure 5 shows comparison of the impedance of the 1st collimator using ECHO2 code and our FEM code. The ECHO2 code used a short bunch length of 0.1mm and mesh size of 0.01mm in order to calculate the impedance with high frequency up to 1000 GHz. The agreement is good up to 1000 GHz. The agreements between the two codes for the second collimator are excellent as shown in Figure 6.

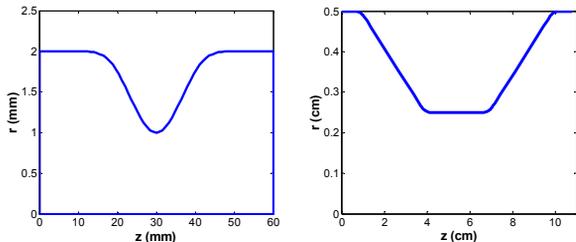


Figure 4: Geometry of collimator one and two used for comparisons

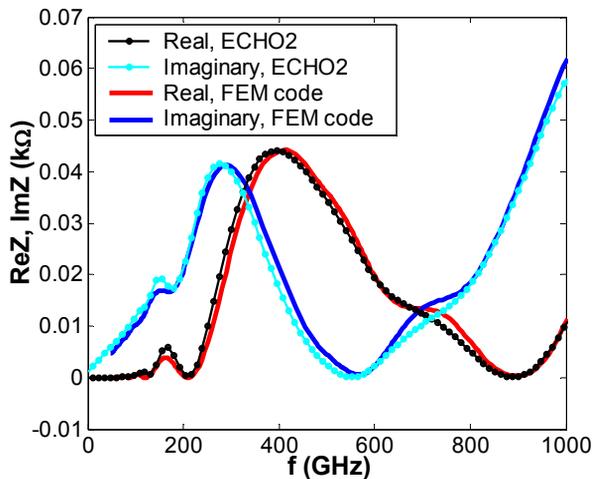


Figure 5: Impedance of the 1st collimator

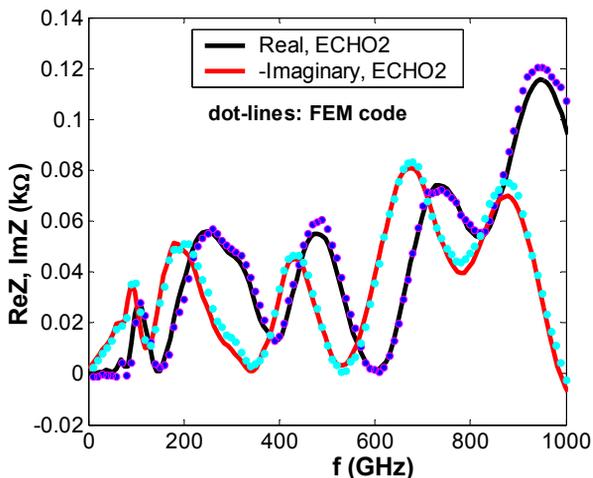


Figure 6: Impedance of the 2nd collimator

DISCUSSION ON NUMERICAL ISSUES

We checked the convergence of the solver. When l is small, such as high frequency field, fine mesh is required because the fields as shown in Figure 1 have fine structure. Both triangular grid and quadrilateral grid are used in the solver. There is a weak dependence on the grid shape as shown in Figure 7. In low frequency domain, the result doesn't vary with grid shape. The small variation of the impedance in high frequency is due to the difference of mesh density. Linear element is used in the above

calculation. Higher order elements can be used to improve the accuracy.

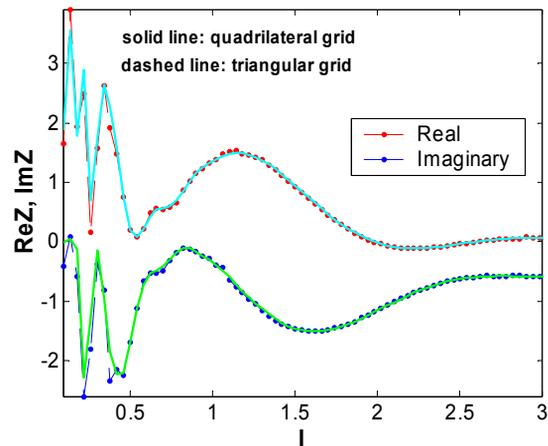


Figure 7: Comparison of triangular and quadrilateral grid

CONCLUSION

A 2D FEM code based on PE approach is developed to directly calculate the high frequency impedance for small angle tapers/collimators. It agrees well with ECHO2. The FEM code can handle arbitrary geometry and speeds up the calculation by orders of magnitude (couple of seconds for one run) comparing with Mathematica code. The impedance is directly calculated in frequency domain. These merits make our code much easy to calculate the wake of ultra short bunch. The same approach can be applied to Three-Dimensional geometry.

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