Spin Dynamics in PTC:

Normal Forms, Invariant Spin Fields, and Resonance Strengths

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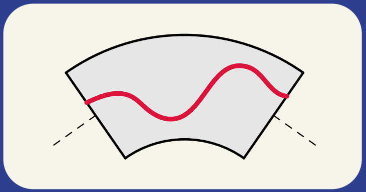
ICAP'2012

PTC's design simplifies modelling of complex accelerator topologies

PTC =

integrator that tracks using local reference frames and local information about fields



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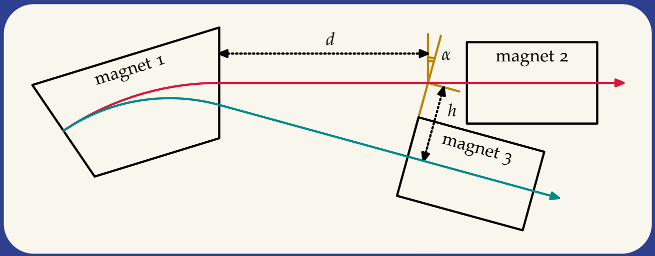
geometric transfoms that relate local reference frames w/rt one another (dynamical Euclidean group)



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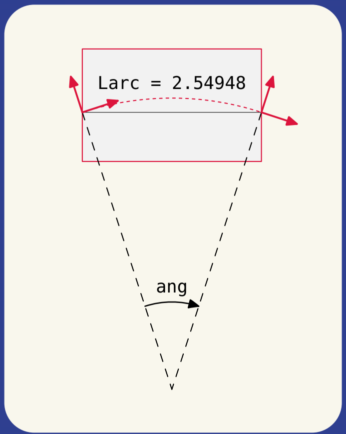
data structures that allow one to model a a recirculator, and also to separate physics and geometry



+ *more*

PTC's map capabilities provide tools for analysis

FPP (*fully polymorphic package*)—derived from LIELIB

PTC + FPP ⇒ maps for arbitrary designs (with errors)

Knobs (parameters in the maps) enable fitting and optimisation

The essential tool is *Normal Form analysis*: $M = A \cdot N \cdot A^{-1}$

$N \Longleftrightarrow$ global invariants, circles—unique

e.g., tunes, chromaticities, ...

$A \Longleftrightarrow$ *s*-dependent global quantities—“gauge” dependent

e.g., lattice functions, ...

A^{-1} converts to normalised coördinates

can “track” the A

NF analysis extends in a natural way to non-linear analyses

anharmonicity, ...

Using a Lie representation simplifies extension to non-linear analysis (or Normal Form in < 100 symbols)

Start with a Lie map: $\mathcal{M} = \mathcal{M}_2\mathcal{M}_3\mathcal{M}_4\cdots = e^{:f_2:}e^{:f_3:}e^{:f_4:}\ldots$

Compute order-k NF: $\mathcal{A}_k\cdots\mathcal{A}_2\,\mathcal{M}\,\mathcal{A}_2^{-1}\cdots\mathcal{A}_k^{-1} = \mathcal{N}_{(k)}\,e^{:\tilde{f}_{k+1}:}\ldots$

Turn the crank: $\mathcal{N}_{(k+1)} = \mathcal{N}_{(k)}\exp(:e^{-:h_2:}g_{k+1}:)\,e^{:\tilde{f}_{k+1}:}\,e^{:-g_{k+1}:}\ldots$

Add exp's (can ignore HOTs): $\tilde{f}_{k+1} - (1 - e^{-:h_2:})g_{k+1} = 0$

Solve for g_{k+1}: $g_{k+1} = \frac{1}{1-e^{-:h_2:}}\tilde{f}_{k+1}$

Keep turning the crank to purify higher-order f_ks.

What can go wrong?

- for integrable motion, *irremovable terms* → $h(J)$.
- with resonance islands present, the phase space topology restricts the algorithm's domain of convergence; but we can still handle a single resonance model (SRM)

PTC applies the same approach to model and analyse spin dynamics ...

Spin-orbit maps: $T(z,\vec{s}) = (M(z),\, S(z)\cdot\vec{s})$

$$T_2 \circ T_1(z,\vec{s}) = (M_2, S_2) \circ (M_1(z), S_1(z)\cdot\vec{s})$$

$$= (M_2 \circ M_1(z),\, S_2(M_1(z))\cdot S_1(z)\cdot\vec{s})$$

- Here spin acts as a spectator: we ignore Stern-Gerlach forces.
- PTC uses $SO(3)$, so spin tracking is rigorously orthogonal.
- The use of FPP means we have available to us knobs for spin.
- What can a spin normal form analysis tell us?
  - –invariants (in the normal form)
    - spin tune: ν_0, as well as ADST
    - resonance strengths, including HO res. strengths
  - –SDGQs (in the normalising map)
    - Invariant Spin Field: $ISF = \mathbf{A}^{-1}\hat{y}$

... including the normal form algorithm

What is a spin normal form? $\mathrm{e}^{\theta_J L_y}$, so rotations about $\hat{y}$.

At order k, this is really $\mathrm{e}^{\theta^{[k]}(J) L_y + \mathbf{\Delta}^{k+1}\cdot\mathbf{L}}$

Here we must solve $\mathbf{\Delta}^{k+1}\cdot\mathbf{L} - \big(1 - \mathrm{e}^{\theta_o\,\mathrm{ad}\,L_y}\mathrm{e}^{-:h_2:}\big)\mathbf{b}^{k+1}\cdot\mathbf{L} = 0$

to obtain $b_y^{k+1} = \dfrac{1}{1 - \mathrm{e}^{-:h_2:}}\Delta_y^{k+1}$ ← avoid orbital resonances

$$\beta^{k+1} = \frac{1}{1 - \mathrm{e}^{\mathrm{i}\,\theta_o}\mathrm{e}^{-:h_2:}}\delta^{k+1}$$ ← avoid spin-orbit resonances

What can go wrong?

- *irremovable terms* in the b_ys lead to θ_J, hence ADST
- equation for β leads to the spin-orbit resonance condition, and to the SRM for spin
- high-energy particles ($G\gamma$ is no longer small)

Single-Resonance Model for spin

PTC: $\mathrm{e}^{\theta_0 \mathsf{L}_y} \mathrm{e}^{\alpha\cdot\Lambda}$ ← eigenfunctions of $\mathrm{ad} L_y$

iterate this map: $$\mathsf{S}^{(2)} = \mathrm{e}^{\theta_0 \mathsf{L}_y} \mathrm{e}^{\alpha(r(z))\cdot\Lambda} \mathrm{e}^{\theta_0 \mathsf{L}_y} \mathrm{e}^{\alpha(z)\cdot\Lambda}$$

$$= \mathrm{e}^{2\theta_0 \mathsf{L}_y} \exp\{\alpha[r(z)] \cdot \mathrm{e}^{\theta_0\,\mathrm{ad}\,\mathsf{L}_y} \Lambda\}\, \mathrm{e}^{\alpha(z)\cdot\Lambda}$$

e.g., small kicks about an axis that rotates $\theta_0 - \phi_y$ per turn.

Can *also* (for this resonance) write this in the form

$$\mathsf{S}^{(2)} = \mathrm{e}^{2\phi_y \mathsf{L}_y} \mathrm{e}^{2\pi\delta\,\mathsf{L}_y} \exp\{\alpha[r(z)] \cdot \mathrm{e}^{-\phi_y\,\mathrm{ad}\,\mathsf{L}_y} \Lambda\}\, \mathrm{e}^{2\pi\delta\,\mathsf{L}_y} \mathrm{e}^{\alpha(z)\cdot\Lambda}$$

$$= \mathrm{e}^{2\phi_y \mathsf{L}_y} \exp\{2[2\pi\delta\,\mathsf{L}_y + |\beta|\sqrt{2J_y}\,\mathsf{L}_z]\}$$

e.g., small kicks about a fixed axis tilted from the vertical.

This latter form is *exactly* the usual SR model found in the literature, from which we identify the resonance strength as

$$\varepsilon = |\beta|/(2\pi)$$

Conclusion—not an ending ...

PTC can model essentially arbitrary accelerator topologies and compute maps for both orbit and spin about even very messy closed orbits.

PTC's normal form analysis yields valuable information about both orbital and spin dynamics in a real machine.

A *spin* normal form analysis yields closed-orbit and amplitude-dependent spin tunes, first- and higher-order resonance strengths, and the invariant spin field. Probably much more.

Where the normal form does not converge, we must return to stroboscopic averaging.