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RECONSTRUCTION OF VELOCITY FIELD

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Introduction

- Solving of inverse problems of electrodynamics, where by pre-assigned motions (given velocity field) electromagnetic fields were determined, had been investigated in works of G.A.Grinberg, A.R.Lucas, B.Meltzer, V.T.Ovcharov, V.I.Zubov , E.D. Kotina [1-9].

- It shoud be noted, that the problem of determination of velocity field is a separate task. In particular, the task of determination of velocity field could be considered as the problem of the optimal control theory [10]. In this case it is needed to find the velocity field securing necessary beam dynamics.

- In this paper we suppose that the distribution density of particles in phase space is known. The problem of finding the velocity field is considered as a minimization problem. Similar problem is widely discussed in the literature for image processing based on the so-called optical flow. This approach was also used for the motion correction for radionuclide tomographic studies [11]. In this work the problem of determining the velocity field in solving the problem of charged particle beam formation in a stationary magnetic field is also considered.

$$\frac{dX}{dt} = Y, \qquad (1)$$

$$\frac{d(mY)}{dt} = eY \times B.$$

$$\dot{X} = \eta(t, X). \qquad (2)$$

$$B = -\frac{m}{e} rot\eta + h\eta, \qquad (3)$$

$$div(h\eta) = 0. \qquad (4)$$

Problem statement

Let us consider that particle dynamics is governed by equations:

$$\begin{aligned} \dot{x} &= u(t, x, y, z), \\ \dot{y} &= v(t, x, y, z), \\ \dot{z} &= w(t, x, y, z). \end{aligned} \qquad (5)$$

Let $\rho = \rho(t, x, y, z)$ – be the density of particle distribution, which in general depends on time t and spatial coordinates x, y, z. Our task is to restore the field of velocities, which is to find functions u, v, w using given function $\rho(t, x, y, z)$.

Let us consider that given (1) the density of particle distribution satisfies the transport equation (general Liuville's equation)

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}u+\frac{\partial\rho}{\partial y}v+\frac{\partial\rho}{\partial z}w+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0. \qquad (6)$$

or $$\frac{\partial\rho}{\partial t}+\frac{\partial\rho}{\partial x}u+\frac{\partial\rho}{\partial y}v+\frac{\partial\rho}{\partial z}w+\rho\cdot divf=0$$

where $$f=(u,v,w)^*$$

Let us bring in common designations:

$$\frac{\partial \rho}{\partial t} = \rho_t, \quad \frac{\partial \rho}{\partial x} = \rho_x, \quad \frac{\partial \rho}{\partial y} = \rho_y, \quad \frac{\partial \rho}{\partial z} = \rho_z$$

$$\frac{\partial \rho_t}{\partial x} = \rho_{tx}, \quad \frac{\partial \rho_t}{\partial y} = \rho_{ty}, \quad \frac{\partial \rho_t}{\partial z} = \rho_{tz}$$

$$\frac{\partial u}{\partial x} = u_x, \quad \frac{\partial u}{\partial y} = u_y, \quad \frac{\partial u}{\partial z} = u_z$$

$$\frac{\partial v}{\partial x} = v_x, \quad \frac{\partial v}{\partial y} = v_y, \quad \frac{\partial v}{\partial z} = v_z$$

$$\frac{\partial w}{\partial x} = w_x, \quad \frac{\partial w}{\partial y} = w_y, \quad \frac{\partial w}{\partial z} = w_z$$

$$\frac{\partial^2 u}{\partial x \partial x} = u_{xx}, \quad \frac{\partial^2 u}{\partial x \partial y} = u_{xy}, \quad \frac{\partial^2 u}{\partial x \partial z} = u_{xz}, \quad \frac{\partial^2 u}{\partial y \partial z} = u_{yz}$$

$$\frac{\partial^2 v}{\partial y \partial y} = v_{yy}, \quad \frac{\partial^2 v}{\partial x \partial y} = v_{xy}, \quad \frac{\partial^2 v}{\partial x \partial z} = v_{xz}, \quad \frac{\partial^2 v}{\partial y \partial z} = v_{yz}$$

$$\frac{\partial^2 w}{\partial x \partial x} = w_{xx}, \quad \frac{\partial^2 w}{\partial x \partial y} = w_{xy}, \quad \frac{\partial^2 w}{\partial x \partial z} = w_{xz}, \quad \frac{\partial^2 w}{\partial y \partial z} = w_{yz}$$

And introduce the following functional:

$$J(u,v,w) = \int_M (\varphi_1 + \alpha^2 \varphi_2)\, dxdydz, \qquad (7)$$

where:

$$\varphi_1 = (\rho_t + \rho_x u + \rho_y v + \rho_z w + \rho(u_x + v_y + w_z))^2, \qquad (8)$$

$$\varphi_2 = u_x^2 + u_y^2 + u_z^2 + v_x^2 + v_y^2 + v_z^2 + w_x^2 + w_y^2 + w_z^2. \qquad (9)$$

Euler-Lagrange equations:

$$-\alpha^2(u_{xx}+u_{yy}+u_{zz})-\rho^2(u_{xx}+v_{xy}+w_{xz})-\rho(\rho_{xx}u+\rho_{xy}v+\rho_{xz}w)-\rho(\rho_x(2u_x+v_y+w_z)+\rho_y v_x+\rho_z w_x))=\rho\rho_{tx},$$

$$-\alpha^2(v_{xx}+v_{yy}+v_{zz})-\rho^2(u_{xy}+v_{yy}+w_{yz})-\rho(\rho_{xy}u+\rho_{yy}v+\rho_{yz}w)-\rho(\rho_x u_y+\rho_y(u_x+2v_y+w_z)+\rho_z w_y)=\rho\rho_{ty},$$

$$-\alpha^2(w_{xx}+w_{yy}+w_{zz})-\rho^2(u_{xz}+v_{yz}+w_{zz})-\rho(\rho_{xz}u+\rho_{yz}v+\rho_{zz}w)-\rho(\rho_x u_z+\rho_y v_z+\rho_z(u_x+v_y+2w_z))=\rho\rho_{tz}.$$

(10)

Or in vector form:

$$-\alpha^2\begin{pmatrix} u_{xx}+u_{yy}+u_{zz} \\ v_{xx}+v_{yy}+v_{zz} \\ w_{xx}+w_{yy}+w_{zz} \end{pmatrix} - \rho^2\begin{pmatrix} u_{xx}+v_{xy}+w_{xz} \\ u_{xy}+v_{yy}+w_{yz} \\ u_{xz}+v_{yz}+w_{zz} \end{pmatrix} - \rho\begin{pmatrix} 2\rho_x & \rho_x & \rho_x \\ \rho_y & 2\rho_y & \rho_y \\ \rho_z & \rho_z & 2\rho_z \end{pmatrix}\begin{pmatrix} u_x \\ v_y \\ w_z \end{pmatrix} -$$

$$\rho\begin{pmatrix} \rho_y & 0 & 0 \\ 0 & \rho_x & 0 \\ 0 & 0 & \rho_x \end{pmatrix}\begin{pmatrix} v_x \\ u_y \\ u_z \end{pmatrix} - \rho\begin{pmatrix} \rho_z & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_y \end{pmatrix}\begin{pmatrix} w_x \\ w_y \\ v_z \end{pmatrix} - \rho\begin{pmatrix} \rho_{xx} & \rho_{xy} & \rho_{xz} \\ \rho_{xy} & \rho_{yy} & \rho_{yz} \\ \rho_{xz} & \rho_{yz} & \rho_{zz} \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \rho\begin{pmatrix} \rho_{tx} \\ \rho_{ty} \\ \rho_{tz} \end{pmatrix}.$$

(11)

For two-dimensional case, when we consider density as

$$\rho = \rho(t, x, y)$$

and search for the field of velocities in the form:

$$\dot{x} = u(t, x, y),$$
$$\dot{y} = v(t, x, y),$$

formulas (8) and (9) come to:

$$\begin{aligned} -\alpha^2(u_{xx} + u_{yy}) - \rho^2(u_{xx} + v_{xy}) - \rho(\rho_{xx}u + \rho_{xy}v) - \rho(\rho_x(2u_x + v_y) + \rho_y v_x)) = \rho\rho_{tx}, \\ -\alpha^2(v_{xx} + v_{yy}) - \rho^2(u_{xy} + v_{yy}) - \rho(\rho_{xy}u + \rho_{yy}v) - \rho(\rho_x u_y + \rho_y(u_x + 2v_y)) = \rho\rho_{ty}. \end{aligned} \quad (12)$$

Or in vector form:

$$-\alpha^2\begin{pmatrix} u_{xx}+u_{yy} \\ v_{xx}+v_{yy} \end{pmatrix} - \rho\begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{xy} & \rho_{yy} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \rho\begin{pmatrix} 2\rho_x & \rho_x \\ \rho_y & 2\rho_y \end{pmatrix}\begin{pmatrix} u_x \\ v_y \end{pmatrix} -$$

$$\rho\begin{pmatrix} \rho_y & 0 \\ 0 & \rho_x \end{pmatrix}\begin{pmatrix} v_x \\ u_y \end{pmatrix} - \rho^2\begin{pmatrix} u_{xx}+v_{xy} \\ u_{xy}+v_{yy} \end{pmatrix} = \rho\begin{pmatrix} \rho_{tx} \\ \rho_{ty} \end{pmatrix}. \qquad (13)$$

Let $M -$ a set in R^2 with boundary smooth enough. Let introduce following operator

$$A(f) = -\alpha^2 \Delta f + \begin{Bmatrix} div(\rho f)\rho_{x_1} - \dfrac{\partial}{\partial x_1}\left(\rho div(\rho f)\right) \\ div(\rho f)\rho_{x_2} - \dfrac{\partial}{\partial x_2}\left(\rho div(\rho f)\right) \end{Bmatrix} \tag{14}$$

or

$$A(f) = -\alpha^2 \Delta f - \rho \, grad \, div\left(\rho f\right)$$

here

$$f = (f_1, f_2)^T = (u, v)^T, \quad \Delta f = (\Delta u, \Delta v)^T,$$

$$u = u(x_1, x_2), \quad v = v(x_1, x_2).$$

System (10) we can write in the following form:

$$A(f) = g \qquad \text{in} \qquad M \tag{15}$$

with boundary condition

$$f = 0 \qquad \text{on} \qquad \Gamma . \tag{16}$$

Here

$$g = \begin{Bmatrix} -\rho_t \rho_{x_1} + \dfrac{\partial}{\partial x_1}(\rho \rho_t) \\ -\rho_t \rho_{x_2} + \dfrac{\partial}{\partial x_2}(\rho \rho_t) \end{Bmatrix} = \rho \, grad \, \rho_t .$$

We should also notice that equation (15) in operational form is valid for vectors $f = (f_1, f_2, ..., f_n)$ of arbitrary dimension n. At the same time set M, should be considered in space R^n. So that the system of equations (10) can also be written in the form (15). It is not hard to demonstrate that operator $A(f)$ is a positively defined one:

$$\left(A(f), f\right) = -\alpha^2 \int_M \left(\Delta f, f\right) dx - \int_M \left(\rho \,\mathrm{graddiv}(\rho f), f\right) dx =$$

$$\alpha^2 \int_M \varphi_2 dx + \int_M \left(\mathrm{div}(\rho f)\right)^2 dx > 0, \quad \alpha \neq 0, \quad f \neq 0.$$

The system of differential equations, defined by the operational equation (15), is a strongly-elliptical system of differential equations [12, 13].

$$A(f) = L(f) - \rho Hf,$$

$$L(f) - \rho Hf = \rho\, grad \rho_t, \qquad (17)$$

(1)

$$L(f) = -\sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\left[k_{\alpha\beta}\frac{\partial f}{\partial x_\beta}\right], \qquad H = \begin{pmatrix} \rho_{x_1x_1} & \rho_{x_1x_2} \\ \rho_{x_1x_2} & \rho_{x_2x_2} \end{pmatrix},$$

$$k_{11} = \begin{pmatrix} \alpha^2+\rho^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}, \quad k_{12} = \frac{1}{2}\begin{pmatrix} 0 & \rho^2 \\ \rho^2 & 0 \end{pmatrix}, \quad k_{21} = k_{12}, \quad k_{22} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2+\rho^2 \end{pmatrix}.$$

$$c_1 \sum_{\alpha=1}^{2} \left|\xi_\alpha\right|^2 \le \sum_{\alpha,\beta=1}^{2} (k_{\alpha\beta}\xi_\alpha, \xi_\beta) \le c_2 \sum_{\alpha=1}^{2} \left|\xi_\alpha\right|^2,$$

$where \quad c_1 > 0, \quad c_2 > 0 - arbitrary \quad constant,$

$\xi_\alpha = (\xi_\alpha^1, \xi_\alpha^2), \alpha = 1,2 - arbitrary \quad vectors,$

$$\left|\xi_\alpha\right|^2 = \sum_{s=1}^{2} (\xi_\alpha^s)^2, \quad (k_{\alpha\beta}\xi_\alpha, \xi_\beta) = \sum_{s,m=1}^{2} k_{\alpha\beta}^{sm}\xi_\alpha^s\xi_\beta^m.$$

It is well known that strongly-elliptical systems behave as a single elliptical equation, when we talk about decidability. Let us point out that functional (7) is a quadratic functional, which differs by a constant from the following functional:

$$J(f) = \left(A(f), f\right) - 2\left(g, f\right). \qquad (18)$$

Because the operator is positively defined the solution of the equation (15) is also the solution for the task of minimization of the functional (17) [14]. What is more, there exists a unique generalized solution of the system (15) with boundary condition (16) due to the positively defined operator, and under condition of enough smoothness of ρ and the boundary of the set M there also exist a classical equation, due to the embedding theorems [14, 15, 16].

The beam formation in magnetic field

Let us consider a one-dimensional case.

Let us consider an electromagnetic field with axial symmetry, it can be shown that equation of motion of the particles in this case is:

$$\frac{dr}{dz} = f(r,z). \qquad (19)$$

and:

$$\frac{\partial \rho(r,z)}{\partial z} + \frac{\partial \rho(r,z)}{\partial r} f(r,z) + \rho(r,z)\frac{\partial f(r,z)}{\partial r} = 0 \qquad (20)$$

$$B_r = \frac{m}{er}\frac{\partial L}{\partial z} + fh\left(\frac{c_1^2 - L^2/r^2}{1+f^2}\right)^{1/2},$$
$$B_z = -\frac{m}{er}\frac{\partial L}{\partial r} + h\left(\frac{c_1^2 - L^2/r^2}{1+f^2}\right)^{1/2}. \qquad (20)$$

$$L = L(r,z), \;\; h = h(r,z)$$

$$\begin{cases} -fl\dfrac{\partial L}{\partial z} + l\dfrac{\partial L}{\partial r} = d_1, \\ -q\dfrac{\partial h}{\partial z} + g\dfrac{\partial L}{\partial z} - fq\dfrac{\partial h}{\partial r} + fg\dfrac{\partial L}{\partial r} = d_2. \end{cases} \qquad (21)$$

$$q = c_1^2 - L^2/r^2, \; g = hL/r^2, \quad l = \frac{m}{e}L/r^2, \quad d_1 = \frac{m}{e}L^2/r^3 - \frac{m}{e}Fq + rg\left(q(1+f^2)\right)^{1/2},$$

$$F = \left(\frac{\partial f}{\partial z} + \frac{\partial f}{\partial r}f\right)\Big/\left(1+f^2\right), \qquad d_2 = h\left(c_1^2\frac{f}{r} + q\left(\frac{\partial f}{\partial r} - fF\right)\right).$$

Let us put the density function as follows:

$$\rho(r,z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-\bar{r}(z))^2}{2\sigma^2}}. \qquad (22)$$

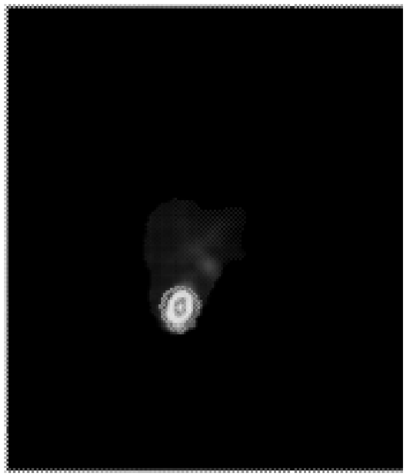
Let

Then

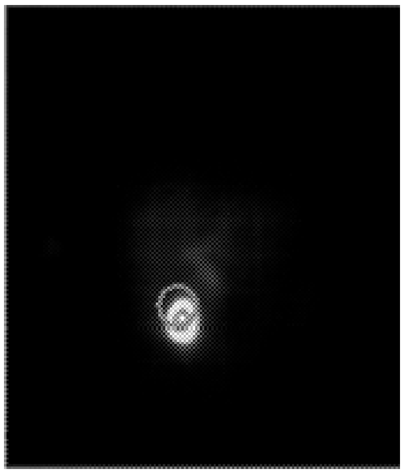
$$\bar{r}(z) = r_0 \cos(z_0 z) + a_0 \qquad (23)$$

$$f = r_0 z_0 \sin z - r_0 z_0 \sin(z_0 z) \exp\left(\frac{r^2 - 2(r_0 \cos(z_0 z) + a_0)}{2\sigma^2} \right).$$

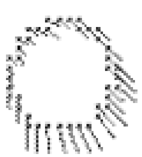
Application for medical imaging



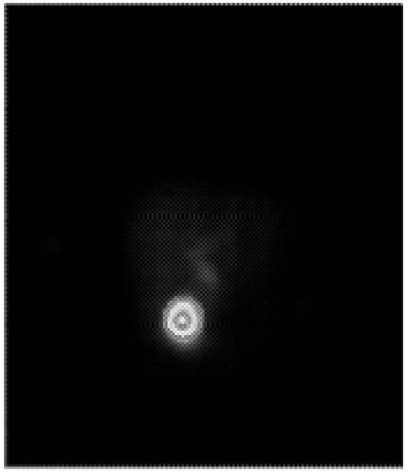
А)



Б)



В)



Г)

Motion correction of contour

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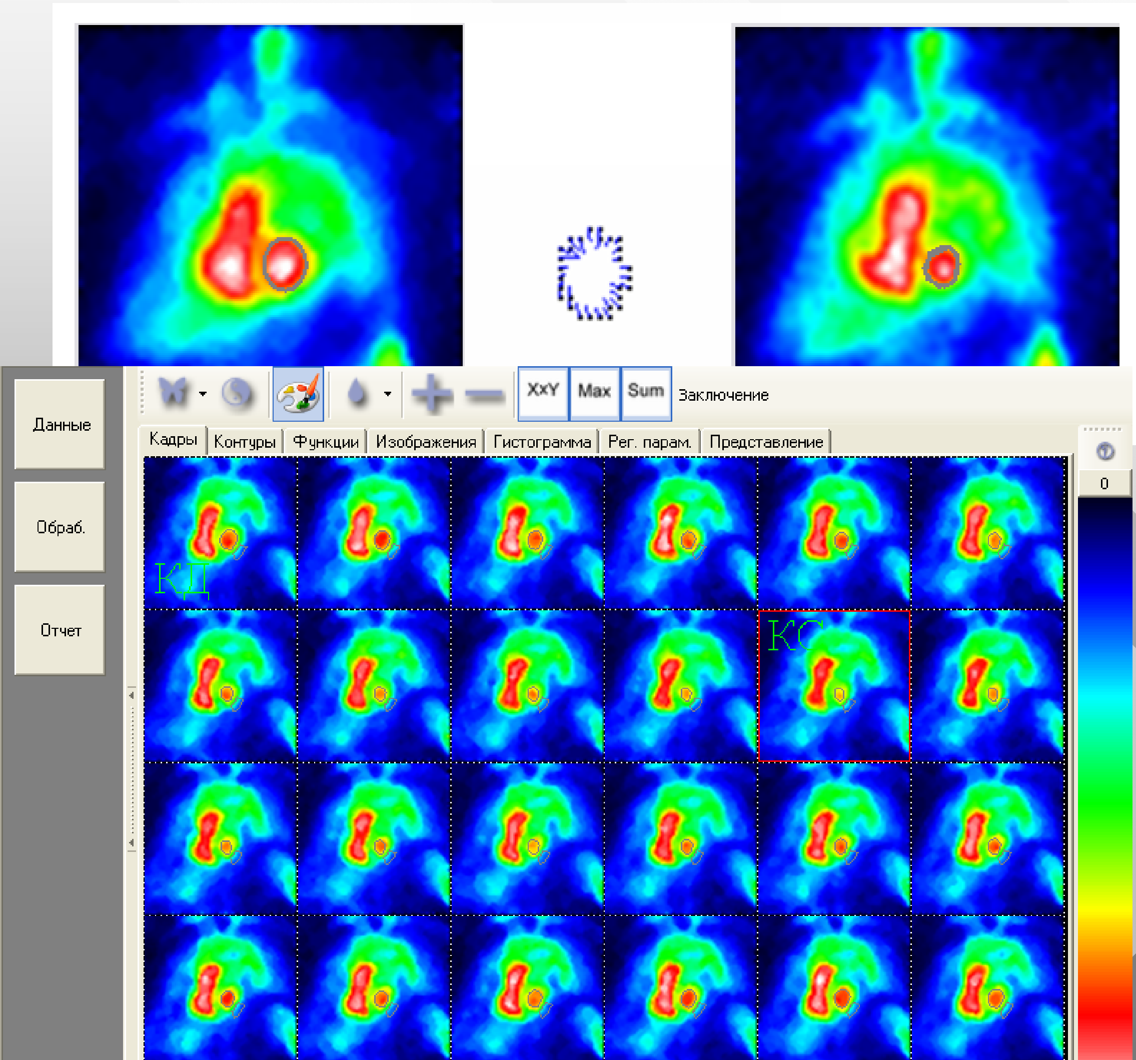
Application for medical imaging



Construction of hart left ventricul contour in nuclear medicine

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THANK YOU FOR ATTENTION!