RECONSTRUCTION OF VELOCITY FIELD

Dmitri A. Ovsyannikov, Elena D. Kotina,
St. Petersburg State University, St. Petersburg, Russia
Introduction

- Solving of inverse problems of electrodynamics, where by pre-assigned motions (given velocity field) electromagnetic fields were determined, had been investigated in works of G.A. Grinberg, A.R. Lucas, B. Meltzer, V.T. Ovcharov, V.I. Zubov, E.D. Kotina [1-9].

- It should be noted, that the problem of determination of velocity field is a separate task. In particular, the task of determination of velocity field could be considered as the problem of the optimal control theory [10]. In this case it is needed to find the velocity field securing necessary beam dynamics.

- In this paper we suppose that the distribution density of particles in phase space is known. The problem of finding the velocity field is considered as a minimization problem. Similar problem is widely discussed in the literature for image processing based on the so-called optical flow. This approach was also used for the motion correction for radionuclide tomographic studies [11]. In this work the problem of determining the velocity field in solving the problem of charged particle beam formation in a stationary magnetic field is also considered.
\[
\frac{dX}{dt} = Y, \quad (1)
\]
\[
\frac{d(mY)}{dt} = eY \times B. \quad (2)
\]
\[
\dot{X} = \eta(t, X). \quad (2)
\]
\[
B = -\frac{m}{e} \text{rot} \eta + h\eta, \quad (3)
\]
\[
div(h\eta) = 0. \quad (4)
\]
Problem statement

Let us consider that particle dynamics is governed by equations:

\[
\begin{align*}
\dot{x} &= u(t, x, y, z), \\
\dot{y} &= v(t, x, y, z), \\
\dot{z} &= w(t, x, y, z).
\end{align*}
\] (5)

Let \( \rho = \rho(t, x, y, z) \) – be the density of particle distribution, which in general depends on time \( t \) and spatial coordinates \( x, y, z \).

Our task is to restore the field of velocities, which is to find functions \( u, v, w \) using given function \( \rho(t, x, y, z) \).
Let us consider that given (1) the density of particle distribution satisfies the transport equation (general Liouville’s equation)

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v + \frac{\partial \rho}{\partial z} w + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0. \tag{6}
\]

or

\[
\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v + \frac{\partial \rho}{\partial z} w + \rho \cdot div f = 0
\]

where \( f = (u, v, w)^* \)
Let us bring in common designations:

\[
\frac{\partial \rho}{\partial t} = \rho_t, \quad \frac{\partial \rho}{\partial x} = \rho_x, \quad \frac{\partial \rho}{\partial y} = \rho_y, \quad \frac{\partial \rho}{\partial z} = \rho_z
\]

\[
\frac{\partial \rho_t}{\partial x} = \rho_{tx}, \quad \frac{\partial \rho_t}{\partial y} = \rho_{ty}, \quad \frac{\partial \rho_t}{\partial z} = \rho_{tz}
\]

\[
\frac{\partial u}{\partial x} = u_x, \quad \frac{\partial u}{\partial y} = u_y, \quad \frac{\partial u}{\partial z} = u_z
\]

\[
\frac{\partial v}{\partial x} = v_x, \quad \frac{\partial v}{\partial y} = v_y, \quad \frac{\partial v}{\partial z} = v_z
\]

\[
\frac{\partial w}{\partial x} = w_x, \quad \frac{\partial w}{\partial y} = w_y, \quad \frac{\partial w}{\partial z} = w_z
\]

\[
\frac{\partial^2 u}{\partial x \partial x} = u_{xx}, \quad \frac{\partial^2 u}{\partial x \partial y} = u_{xy}, \quad \frac{\partial^2 u}{\partial x \partial z} = u_{xz}, \quad \frac{\partial^2 u}{\partial y \partial z} = u_{yz}
\]

\[
\frac{\partial^2 v}{\partial y \partial y} = v_{yy}, \quad \frac{\partial^2 v}{\partial x \partial y} = v_{xy}, \quad \frac{\partial^2 v}{\partial x \partial z} = v_{xz}, \quad \frac{\partial^2 v}{\partial y \partial z} = v_{yz}
\]

\[
\frac{\partial^2 w}{\partial x \partial x} = w_{xx}, \quad \frac{\partial^2 w}{\partial x \partial y} = w_{xy}, \quad \frac{\partial^2 w}{\partial x \partial z} = w_{xz}, \quad \frac{\partial^2 w}{\partial y \partial z} = w_{yz}
\]
And introduce the following functional:

$$J(u, v, w) = \int_{M} (\varphi_1 + \alpha^2 \varphi_2) dx dy dz,$$

(7)

where:

$$\varphi_1 = (\rho_t + \rho_x u + \rho_y v + \rho_z w + \rho(u_x + v_y + w_z))^2,$$

(8)

$$\varphi_2 = u_x^2 + u_y^2 + u_z^2 + v_x^2 + v_y^2 + v_z^2 + w_x^2 + w_y^2 + w_z^2.$$

(9)
Euler-Lagrange equations:

\[-\alpha^2 (u_{xx} + u_{yy} + u_{zz}) - \rho \alpha^2 (u_{xx} + v_{xy} + w_{xz}) - \rho (\rho_{xx} u + \rho_{xy} v + \rho_{xz} w) - \rho (\rho_x (2u_x + v_y + w_z) + \rho_y v_x + \rho_z w_x)) = \rho \rho_{tx},\]

\[-\alpha^2 (v_{xx} + v_{yy} + v_{zz}) - \rho \alpha^2 (u_{xy} + v_{yy} + w_{yz}) - \rho (\rho_{xy} u + \rho_{yy} v + \rho_{yz} w) - \rho (\rho_x u_y + \rho_y (u_x + 2v_y + w_z) + \rho_z w_y) = \rho \rho_{ty},\]

\[-\alpha^2 (w_{xx} + w_{yy} + w_{zz}) - \rho \alpha^2 (u_{xz} + v_{yz} + w_{zz}) - \rho (\rho_{xz} u + \rho_{yz} v + \rho_{zz} w) - \rho (\rho_x u_z + \rho_y v_z + \rho_z (u_x + v_y + 2w_z)) = \rho \rho_{tz}.\]
Or in vector form:

\[
-\alpha^2 \begin{pmatrix} u_{xx} + u_{yy} + u_{zz} \\ v_{xx} + v_{yy} + v_{zz} \\ w_{xx} + w_{yy} + w_{zz} \end{pmatrix} - \rho^2 \begin{pmatrix} u_{xx} + v_{xy} + w_{xz} \\ u_{xy} + v_{yy} + w_{yz} \\ u_{xz} + v_{yz} + w_{zz} \end{pmatrix} - \rho \begin{pmatrix} 2 \rho_x \\ \rho_y \\ \rho_z \end{pmatrix} \begin{pmatrix} u_x \\ v_y \\ w_z \end{pmatrix} = \rho \begin{pmatrix} \rho_{tx} \\ \rho_{ty} \\ \rho_{tz} \end{pmatrix}.
\]
For two-dimensional case, when we consider density as
\[
\rho = \rho(t, x, y)
\]
and search for the field of velocities in the form:
\[
\begin{align*}
\dot{x} &= u(t, x, y), \\
\dot{y} &= v(t, x, y),
\end{align*}
\]
formulas (8) and (9) come to:
\[
\begin{align*}
-\alpha^2 (u_{xx} + u_{yy}) - \rho^2 (u_{xx} + v_{xy}) - \rho (\rho_{xx} u + \rho_{xy} v) - \rho (\rho_x (2u_x + v_y) + \rho_y v_x) &= \rho \rho_{tx}, \\
-\alpha^2 (v_{xx} + v_{yy}) - \rho^2 (u_{xy} + v_{yy}) - \rho (\rho_{xy} u + \rho_{yy} v) - \rho (\rho_x u_y + \rho_y (u_x + 2v_y)) &= \rho \rho_{ty}.
\end{align*}
\]
Or in vector form:

\[
\begin{align*}
- \alpha^2 \left( \begin{array}{c}
  u_{xx} + u_{yy} \\
  v_{xx} + v_{yy}
\end{array} \right) & - \rho \left( \begin{array}{cc}
  \rho_{xx} & \rho_{xy} \\
  \rho_{xy} & \rho_{yy}
\end{array} \right) \left( \begin{array}{c}
  u \\
  v
\end{array} \right) & - \rho \left( \begin{array}{ccc}
  2 \rho_x & \rho_x & 0 \\
  \rho_y & 2 \rho_y & \rho_y \\
  0 & \rho_x & \rho_x
\end{array} \right) \left( \begin{array}{c}
  u_x \\
  v_x \\
  u_y
\end{array} \right) \\
\rho \left( \begin{array}{cc}
  \rho_y & 0 \\
  0 & \rho_x
\end{array} \right) \left( \begin{array}{c}
  v_x \\
  u_y
\end{array} \right) & - \rho^2 \left( \begin{array}{c}
  u_{xx} + v_{xy} \\
  u_{xy} + v_{yy}
\end{array} \right) & = \rho \left( \begin{array}{c}
  \rho_{tx} \\
  \rho_{ty}
\end{array} \right).
\end{align*}
\]

(13)
Let $M$ be a set in $R^2$ with boundary smooth enough. Let introduce following operator

$$A(f) = -\alpha^2 \Delta f + \left\{ \begin{array}{l}
\text{div}(\rho f) \rho_{x_1} - \frac{\partial}{\partial x_1} (\rho \text{div}(\rho f)) \\
\text{div}(\rho f) \rho_{x_2} - \frac{\partial}{\partial x_2} (\rho \text{div}(\rho f))
\end{array} \right\}$$

(14)

or

$$A(f) = -\alpha^2 \Delta f - \rho \text{grad div}(\rho f)$$

here

$$f = (f_1, f_2)^T = (u, v)^T, \quad \Delta f = (\Delta u, \Delta v)^T,$$

$$u = u(x_1, x_2), \quad v = v(x_1, x_2).$$
System (10) we can write in the following form:

\[ A(f) = g \quad \text{in} \quad \mathcal{M} \quad \text{(15)} \]

with boundary condition

\[ f = 0 \quad \text{on} \quad \Gamma. \quad \text{(16)} \]

Here

\[ g = \left\{ -\rho_t \rho x_1 + \frac{\partial}{\partial x_1} (\rho \rho_t) \right\} = \rho \text{grad} \rho_t. \]
We should also notice that equation (15) in operational form is valid for vectors $f = (f_1, f_2, \ldots, f_n)$ of arbitrary dimension $n$. At the same time set $M$, should be considered in space $\mathbb{R}^n$. So that the system of equations (10) can also be written in the form (15). It is not hard to demonstrate that operator $A(f)$ is a positively defined one:

$$
(A(f), f) = -\alpha^2 \int_M (\Delta f, f) dx - \int_M (\rho \text{ grad} \text{div}(\rho f), f) dx = \\
\alpha^2 \int_M \varphi_2 dx + \int_M (\text{div}(\rho f))^2 dx > 0, \quad \alpha \neq 0, \quad f \neq 0.
$$
The system of differential equations, defined by the operational equation (15), is a strongly-elliptical system of differential equations [12, 13].

\[ A(f) = L(f) - \rho Hf, \]

\[ L(f) - \rho Hf = \rho \text{ grad} \rho_t, \quad (17) \]
\[ L(f) = - \sum_{\alpha, \beta = 1}^{2} \frac{\partial}{\partial x_{\alpha}} \left[ k_{\alpha \beta} \frac{\partial f}{\partial x_{\beta}} \right], \quad H = \begin{pmatrix} \rho_{x_1 x_1} & \rho_{x_1 x_2} \\ \rho_{x_1 x_2} & \rho_{x_2 x_2} \end{pmatrix}, \]

\[ k_{11} = \begin{pmatrix} \alpha^2 + \rho^2 & 0 \\ 0 & \alpha^2 \end{pmatrix}, \quad k_{12} = \frac{1}{2} \begin{pmatrix} 0 & \rho^2 \\ \rho^2 & 0 \end{pmatrix}, \quad k_{21} = k_{12}, \quad k_{22} = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \alpha^2 + \rho^2 \end{pmatrix}. \]

\[ c_1 \sum_{\alpha = 1}^{2} |\xi_{\alpha}|^2 \leq \sum_{\alpha, \beta = 1}^{2} (k_{\alpha \beta} \xi_{\alpha}, \xi_{\beta}) \leq c_2 \sum_{\alpha = 1}^{2} |\xi_{\alpha}|^2, \]

where \( c_1 > 0, \quad c_2 > 0 \) — arbitrary constants,

\( \xi_{\alpha} = (\xi_{\alpha}^1, \xi_{\alpha}^2), \alpha = 1, 2 \) — arbitrary vectors,

\[ |\xi_{\alpha}|^2 = \sum_{s = 1}^{2} (\xi_{\alpha}^s)^2, \quad (k_{\alpha \beta} \xi_{\alpha}, \xi_{\beta}) = \sum_{s, m = 1}^{2} k_{\alpha \beta}^{sm} \xi_{\alpha}^s \xi_{\beta}^m. \]
It is well known that strongly-elliptical systems behave as a single elliptical equation, when we talk about decidability. Let us point out that functional (7) is a quadratic functional, which differs by a constant from the following functional:

\[ J(f) = (A(f), f) - 2(g, f). \]  (18)
Because the operator is positively defined the solution of the equation (15) is also the solution for the task of minimization of the functional (17) [14]. What is more, there exists a unique generalized solution of the system (15) with boundary condition (16) due to the positively defined operator, and under condition of enough smoothness of $\rho$ and the boundary of the set $M$ there also exist a classical equation, due to the embedding theorems [14, 15, 16].
The beam formation in magnetic field

Let us consider a one-dimensional case.

Let us consider an electromagnetic field with axial symmetry, it can be shown that equation of motion of the particles in this case is:

\[ \frac{dr}{dz} = f(r, z). \]  \hspace{1cm} (19)

and:

\[ \frac{\partial \rho(r, z)}{\partial z} + \frac{\partial \rho(r, z)}{\partial r} f(r, z) + \rho(r, z) \frac{\partial f(r, z)}{\partial r} = 0 \] \hspace{1cm} (20)
\[ B_r = \frac{m}{er} \frac{\partial L}{\partial z} + fh \left( \frac{c_1^2 - L^2/r^2}{1 + f^2} \right)^{1/2}, \]

\[ B_z = \frac{-m}{er} \frac{\partial L}{\partial r} + h \left( \frac{c_1^2 - L^2/r^2}{1 + f^2} \right)^{1/2}. \]

\[ L = L(r, z), \quad h = h(r, z) \]

\[
\begin{cases}
-fl \frac{\partial L}{\partial z} + l \frac{\partial L}{\partial r} = d_1, \\
-fq \frac{\partial h}{\partial r} + g \frac{\partial L}{\partial z} - fq \frac{\partial h}{\partial r} + f g \frac{\partial L}{\partial r} = d_2.
\end{cases}
\]

\[ q = c_1^2 - L^2/r^2, \quad g = hL/r^2, \quad l = \frac{m}{e} L/r^2, \quad d_1 = \frac{m}{e} L^2/r^3 - \frac{m}{e} Fq + rg \left( q(1 + f^2) \right)^{1/2}, \]

\[ F = \left( \frac{\partial f}{\partial z} + \frac{\partial f}{\partial r} f \right) / \left( 1 + f^2 \right), \quad d_2 = h \left( c_1^2 f/r + q \left( \frac{\partial f}{\partial r} - fF \right) \right). \]
Let us put the density function as follows:

\[
\rho(r, z) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(r - \bar{r}(z))^2}{2\sigma^2}}.
\] (22)

Let

\[
\bar{r}(z) = r_0 \cos(z_0 z) + a_0
\] (23)

Then

\[
f = r_0 z_0 \sin z - r_0 z_0 \sin(z_0 z) \exp\left(\frac{r^2 - 2(r_0 \cos(z_0 z) + a_0)}{2\sigma^2}\right).
\]
Application for medical imaging

Motion correction of contour

A) B)
Application for medical imaging

Construction of heart left ventricle contour in nuclear medicine
THANK YOU FOR ATTENTION!