Nonlinear, Nonscaling CW FFAG
Design and Modelling Using Map Methods

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Outline

- FFAGs
- COSY INFINITY, and what it does
- Computations and treatments of fields in COSY
- FFAG Field Modelling with COSY
- Some FFAG simulation results using COSY
  - Field computations, out-of-plane field expansions
  - High-order tracking's with optimal symplectification
  - Consistency checks
  - More examples

Fixed-Field Alternating Gradient Accelerators

- Concept
  - Fixed (time independent) magnetic fields
  - Alternating field gradient – use of the strong focusing idea
- Key advantages
  - Compact
  - Continuous wave operation
  - Large acceptance
- Various new classes of accelerators
  - Proton drivers for muon colliders and neutrino factories
  - Accelerator driven subcritical reactors
  - Medical applications
- Challenges for conventional simulation codes
  - Due to the complicated field arrangements, beam dynamics simulations are difficult if not impossible.

Transfer Map Method and Differential Algebras

- The transfer map $\mathcal{M}$ is the flow of the system ODE.

$$\vec{z}_f = \mathcal{M}(\vec{z}_i, \vec{\delta}),$$

where $\vec{z}_i$ and $\vec{z}_f$ are the initial and the final condition, $\vec{\delta}$ is system parameters.

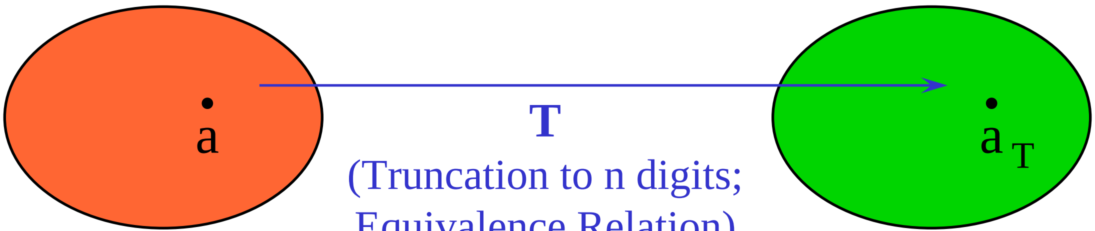
- For a repetitive system, only one cell transfer map has to be computed. Thus, it is much faster than ray tracing codes (i.e. tracing each individual particle through the system).
- The Differential Algebraic method allows a very efficient computation of high order Taylor transfer maps.
- The Normal Form method can be used for analysis of nonlinear behavior.

Differential Algebras (DA)

- it works to arbitrary order, and can keep system parameters in maps.
- very transparent algorithms; effort independent of computation order.

The code **COSY Infinity** has many tools and algorithms necessary.

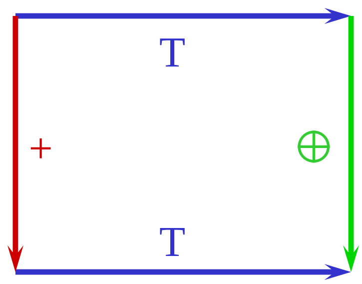
Number Fields and Floating Point Numbers



Real Numbers

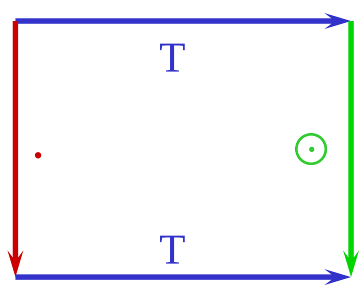
Floating Point "Numbers"

$c = a + b$



$c_T = a_T \oplus b_T$

$c = a \cdot b$



$c_T = a_T \odot b_T$

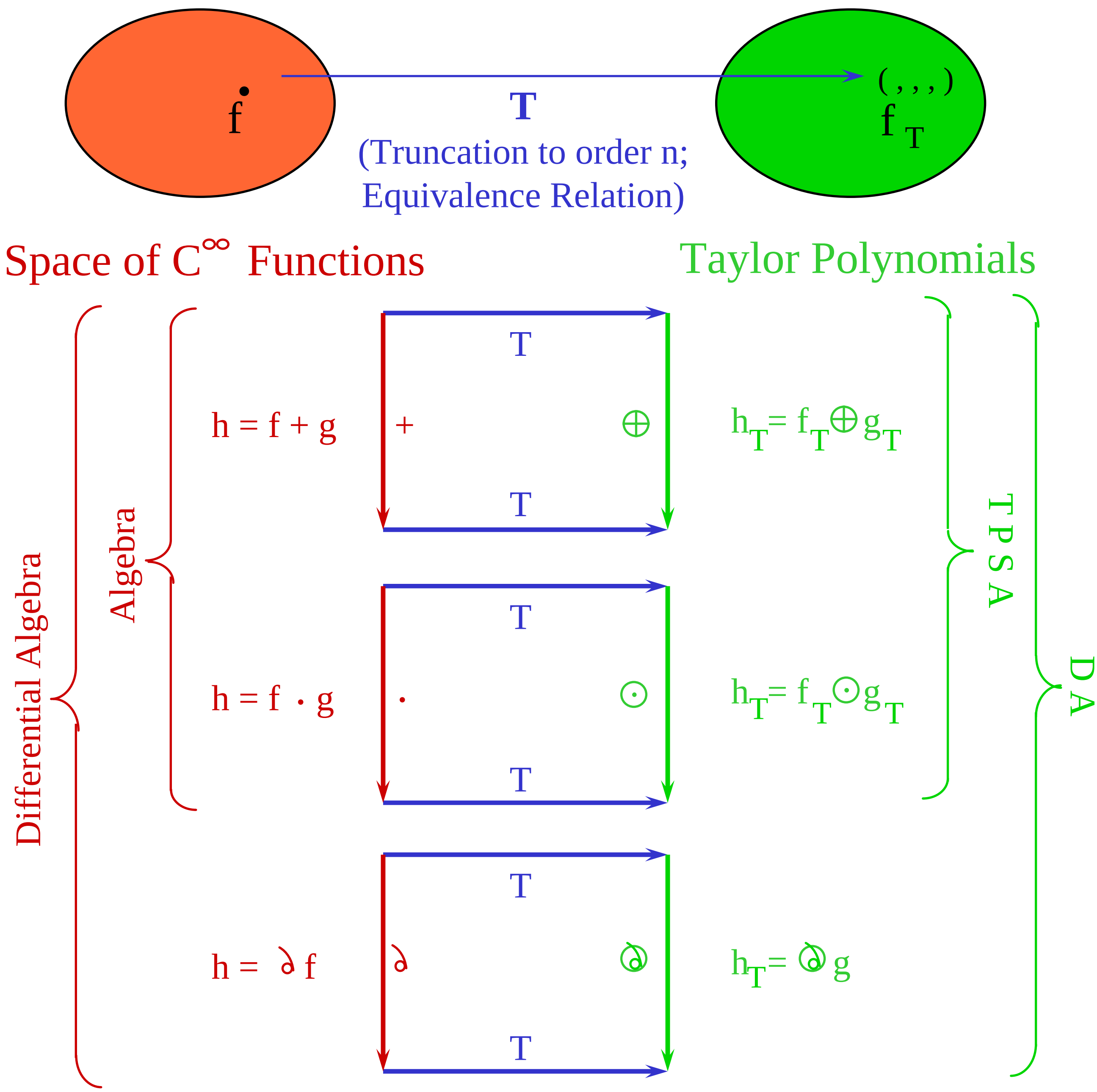
Field
(Also want "exp", "sin" etc: Banach Field)

Diagrams commute "approximately"

Field
("approximately")

T: Extracts Information considered relevant

FUNCTION ALGEBRAS



Differential Algebra
(also want “exp”, “sin” etc: Banach DA)

Diagrams commute exactly

Differential Algebra
(even Banach DA)

T: Extracts Information considered relevant

COSY INFINITY

- Arbitrary order
- Maps depending on parameters (even with mass dependence)
- No approximations in motion or field description
- Large library of elements
- Arbitrary Elements (you specify fields)
- Very flexible input language
- Powerful interactive graphics
- Errors: position, tilt, rotation
- Tracking through maps (with/without symplectification. EXPO)
- Normal Form Methods
- Spin dynamics
- Fast fringe field models using SYSCA approach
- Reference manual (70 pages) and Programming manual (100 pages)
- Currently about 2000 registered users

Field Description in Differential Algebra

There are various DA algorithms to treat the fields of beam optics efficiently. For example, **DA PDE Solver**

- requires to supply only
  - the midplane field for a midplane symmetric element.
  - the on-axis potential for straight elements like solenoids, quadrupoles, and higher multipoles.
- treats arbitrary fields straightforwardly.
  - Magnet (or, Electrostatic) fringe fields:
    The Enge function fall-off model

$$F(s) = \frac{1}{1 + \exp(a_1 + a_2 \cdot (s/D) + ... + a_6 \cdot (s/D)^5)}$$

    where D is the full aperture.
    Or, any arbitrary model including the measured data representation.
  - Solenoid fields including the fringe fields.
  - Measured fields: E.g. Use Gaussian wevelet representation.
  - Etc. etc.

DA Fixed Point PDE Solvers

The **DA fixed point theorem** allows to **solve PDEs iteratively** in **finitely many steps** by rephrasing them in terms of a fixed point problem.

Consider the rather general PDE

$$a_1 \frac{\partial}{\partial x}\left(a_2 \frac{\partial}{\partial x} V\right) + b_1 \frac{\partial}{\partial y}\left(b_2 \frac{\partial}{\partial y} V\right) + c_1 \frac{\partial}{\partial z}\left(c_2 \frac{\partial}{\partial z} V\right) = 0,$$

where a_i, b_i, c_i are functions of x, y, z.

The PDE is re-written in **fixed point form** as

$$V = V|_{y=0} + \int_0^y \frac{1}{b_2}\left(b_2 \frac{\partial V}{\partial y}\right)\Bigg|_{y=0}$$
$$- \int_0^y \frac{1}{b_2} \int_0^y \left(\frac{a_1}{b_1}\frac{\partial}{\partial x}\left(a_2 \frac{\partial V}{\partial x}\right) + \frac{c_1}{b_1}\frac{\partial}{\partial z}\left(c_2 \frac{\partial V}{\partial z}\right)\right) dy dy.$$

Assume the derivatives of V and $\partial V/\partial y$ with respect to x and z are **known in the plane** $y = 0$. Then the right hand side is **contracting** with respect to y (which is necessary for the DA fixed point theorem), and the various orders in y can be **iteratively** calculated by mere iteration.

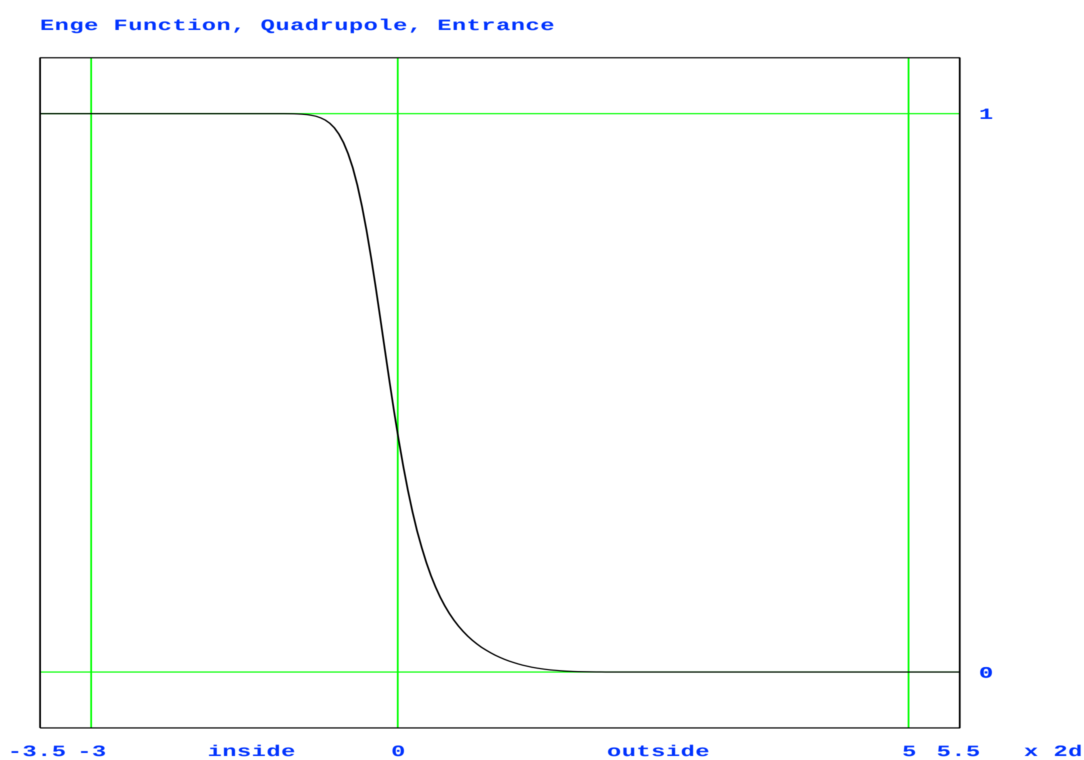
Enge Function for the Fringe Field Fall-off

$$F(s) = \frac{1}{1 + \exp(a_1 + a_2 \cdot (s/D) + ... + a_6 \cdot (s/D)^5)}$$

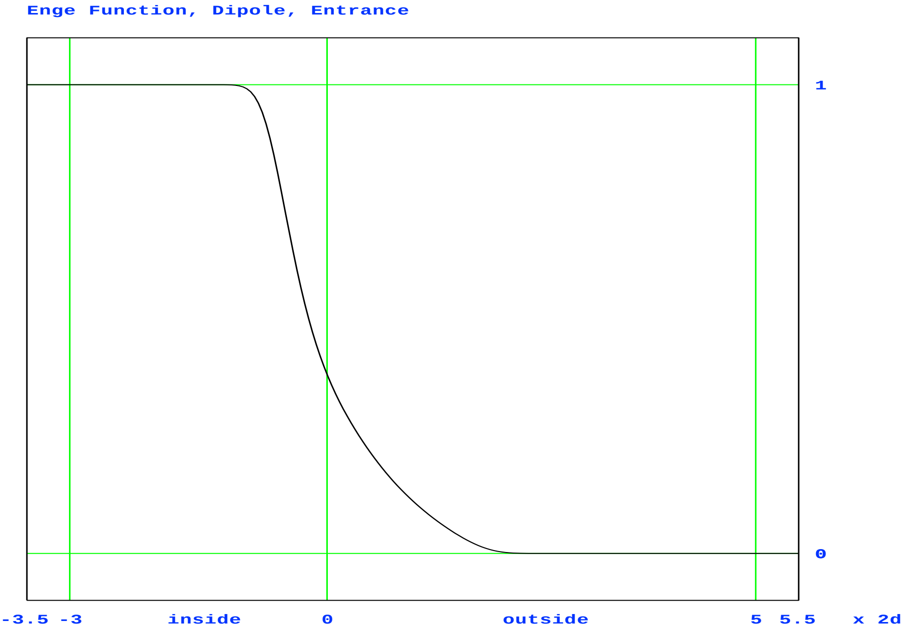
$$D : \text{the full aperture}$$

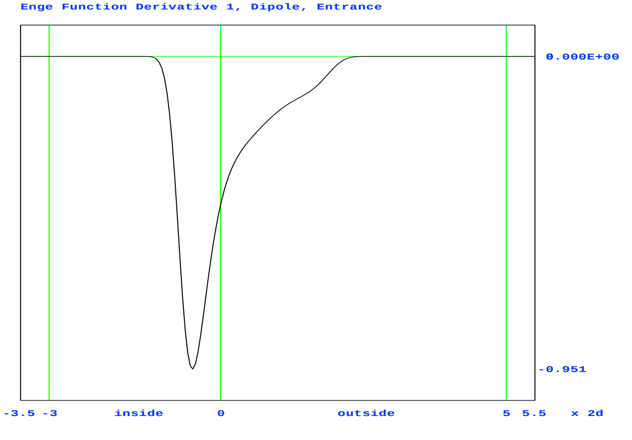
Enge Coefficients of Various Quadrupoles

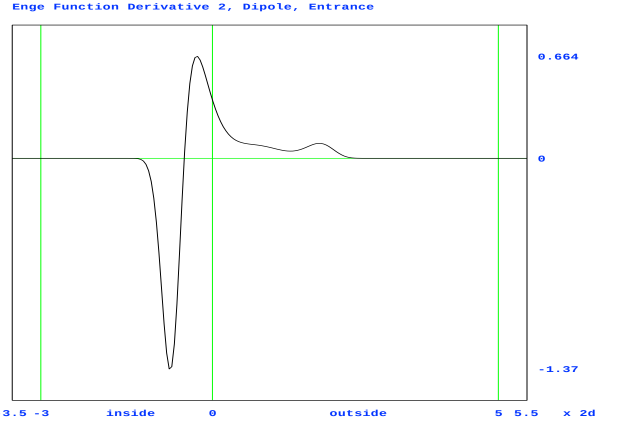
		a_1	a_2	a_3	a_4	a_5	a_6
SLAC-PEP		0.296471	4.533219	−2.270982	1.068627	−0.036391	0.022261
S800 Q II	Entr.	0.0965371	6.63297	−2.718	10.9447	1.64033	0.00
	Exit	0.235452	6.60424	−3.42864	4.38392	−0.573524	0.00
LHC-HGQ lead		−0.939436	3.824163	3.882214	1.776737	0.296383	0.013670



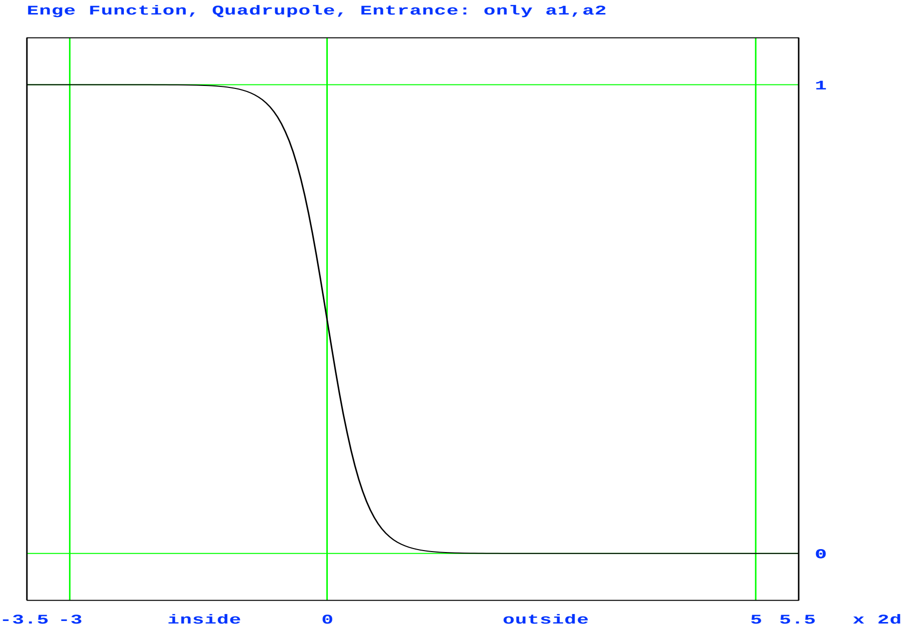
Dipole Enge Function (COSY default)

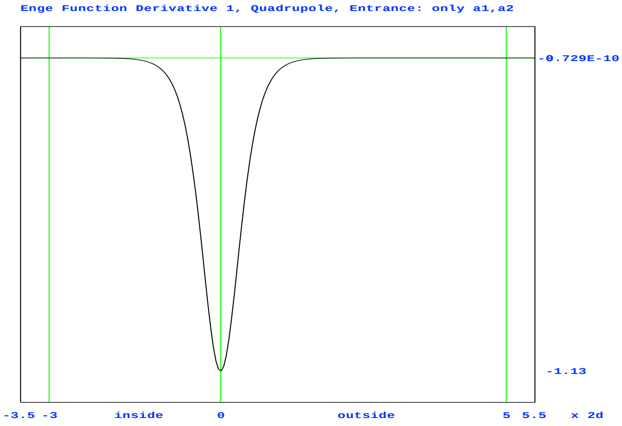


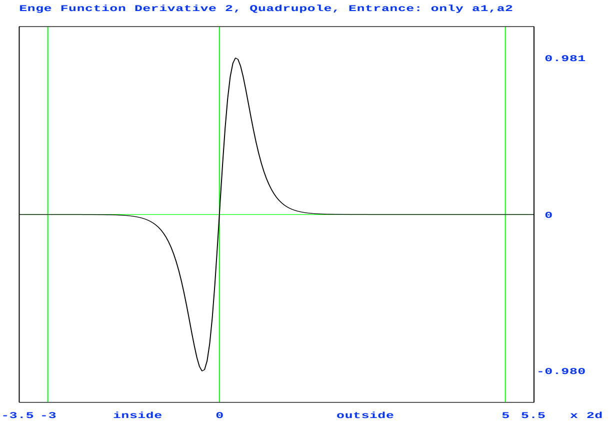




Quadrupole Enge Function (only a_1, a_2)







FFAG Field Modelling with COSY

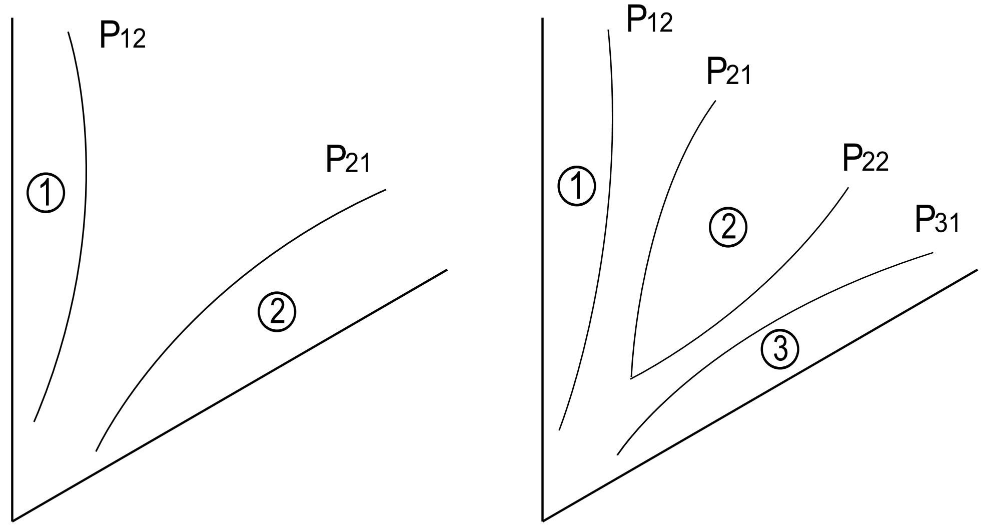
- Sequences of COSY bending magnets
- Generalized FFAG magnets
- Radius-dependent Fourier decomposition
- Gaussian wavelet representation of polar midplane data
- Field representations from 3D data
- Other arrangements

Sequences of COSY Bending Magnets

- Use standard COSY beamline elements
  Various types of combined function bending magnets
  - Tilted and curved entrance edges
  - Various types of fringe fields or measured fields
  - Care:
    - the fields of individual elements do not overlap strongly
    - the field profiles are not too unusual
- The description is relative to
  - A reference orbit and its deflection properties
    - Studying many reference energies becomes tricky

Generalized Nonscaling FFAG Magnets

- Field description is in lab coordinates, applying to all reference orbits
- Superimposed combined-function magnets comprising the FFAG
  - Tilted and curved entrance edges
  - Various types of fringe fields
  - Overlapping fields
- $2n$ sector-shaped cells, pairing one cell and its mirror image
  - Due to the symmetry, closed orbits cross sector lines perpendicularly
- A magnet in each cell assumes a radial field profile $B_{y,i} = B_{0,i} \cdot P_{B,i}(r)$



Radius-Dependent Fourier Decomposition

- Field description is in lab coordinates, applying to all reference orbits
- Describe midplane fields in terms of azimuthal Fourier modes

$$B_y(r,\phi) = a_0 + \sum_{j=1}^{n} a_j(r)\cos\left(j(\phi - \phi_0(r))\right) + \sum_{j=1}^{n} b_j(r)\sin\left(j(\phi - \phi_0(r))\right)$$

  - Lower values of n represent common focusing effects
  - Suitable for scaling FFAGs
- When more radial detail is desired, i.e. having a_{ij} on a Δr grid
  - Have the best fit polynomial P_j to all a_{ij}, and let $\bar{a}_{ij} = a_{ij}/P_j(i\Delta r)$
  - Perform a Gaussian wavelet interpolation

$$a_j(r) = P_j(r) \cdot \sum_i \bar{a}_{ij} \frac{1}{\sqrt{\pi}\sigma} \exp\left(-\frac{(r - i\Delta r)^2}{\sigma^2}\right)$$

Gaussian wavelet representation of polar midplane data

- Field description is in lab coordinates, applying to all reference orbits
- Have a set of midplane field data $B_y^{(i,j)}$ on a regular Δr, $\Delta\phi$ grid.
  - Describe the midplane field by a Gaussian wavelet representation

$$B_y(r,\phi) = \sum_{i,j} G_{\Delta r}(r - r_i) \cdot G_{\Delta\phi}(\phi - \phi_j) \cdot B_y^{(i,j)}$$

  where $G_\sigma(x) = \exp\left(-x^2/\sigma^2\right)/\sigma\sqrt{\pi}$

- Some limitations:
  - The out-of-plane is sensitive to errors in the midplane.field data $B_y^{(i,j)}$
  - First establish the quality of the resulting expansion due to such errors
  - A Fourier representation of the midplane data decreases such trouble
  - Or, use field representations from 3D data

Field Representations from 3D Data

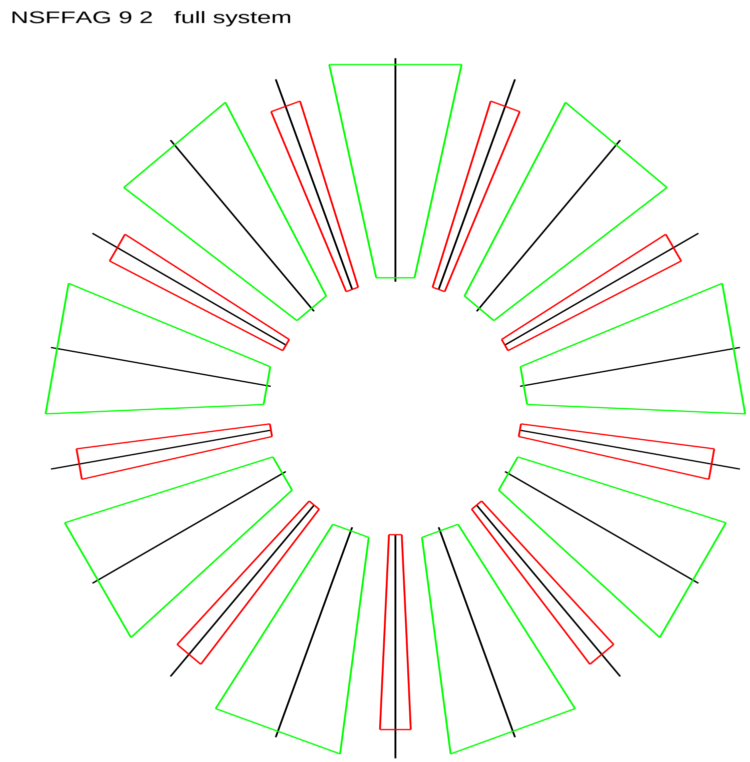
- Field description is in lab coordinates, applying to all reference orbits
- Mathematically, the midplane magnetic fields are sufficient to determine the fields at any point in space based on a power series expansion
- In practice, any such attempt is sensitive to measurement errors
  - Utilize field descriptions that do not rely on the midplane data
  - Rather, utilize surface field data
    - Smoothing out any measurement errors
    - More faithful 3D representations

Other Arrangements

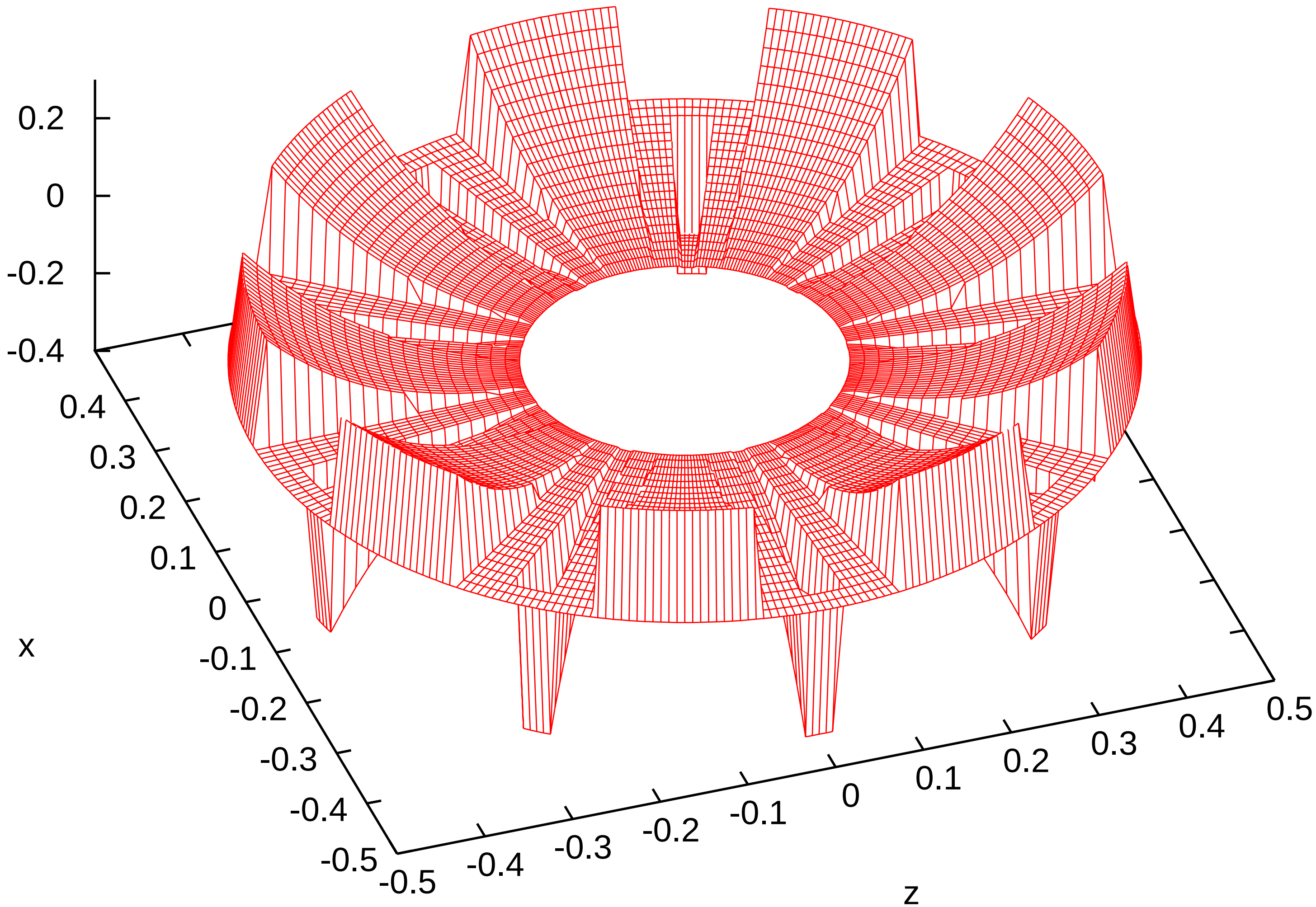
- Describe fields of air coil-dominated magnets
  from the geometry of pieces of the respective coils and the currents
- Perform injection-to-extraction simulations including acceleration elements

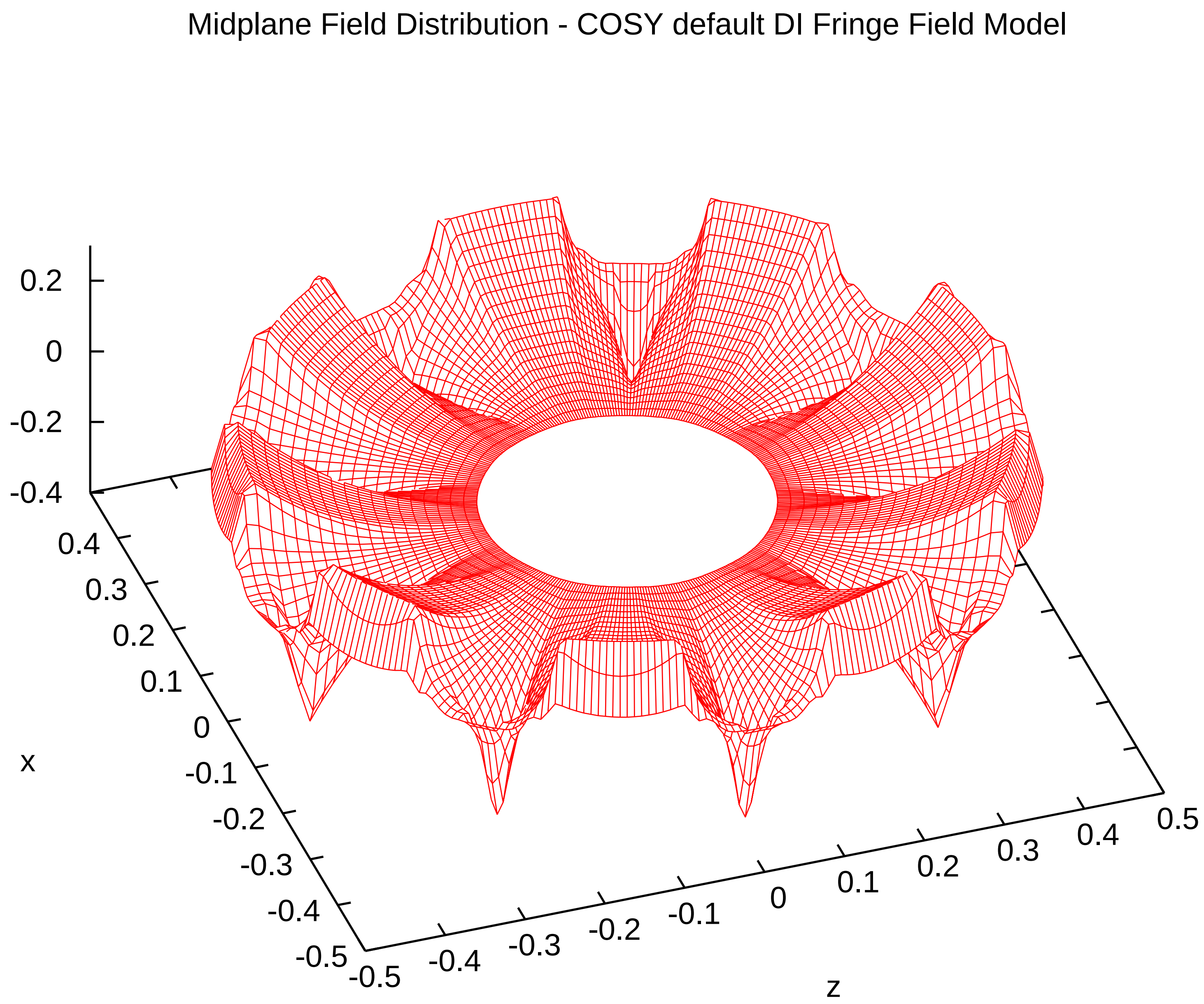
Field Computations, Out-of-Plane Field Expansions

- Example using a nonscaling FFAG model



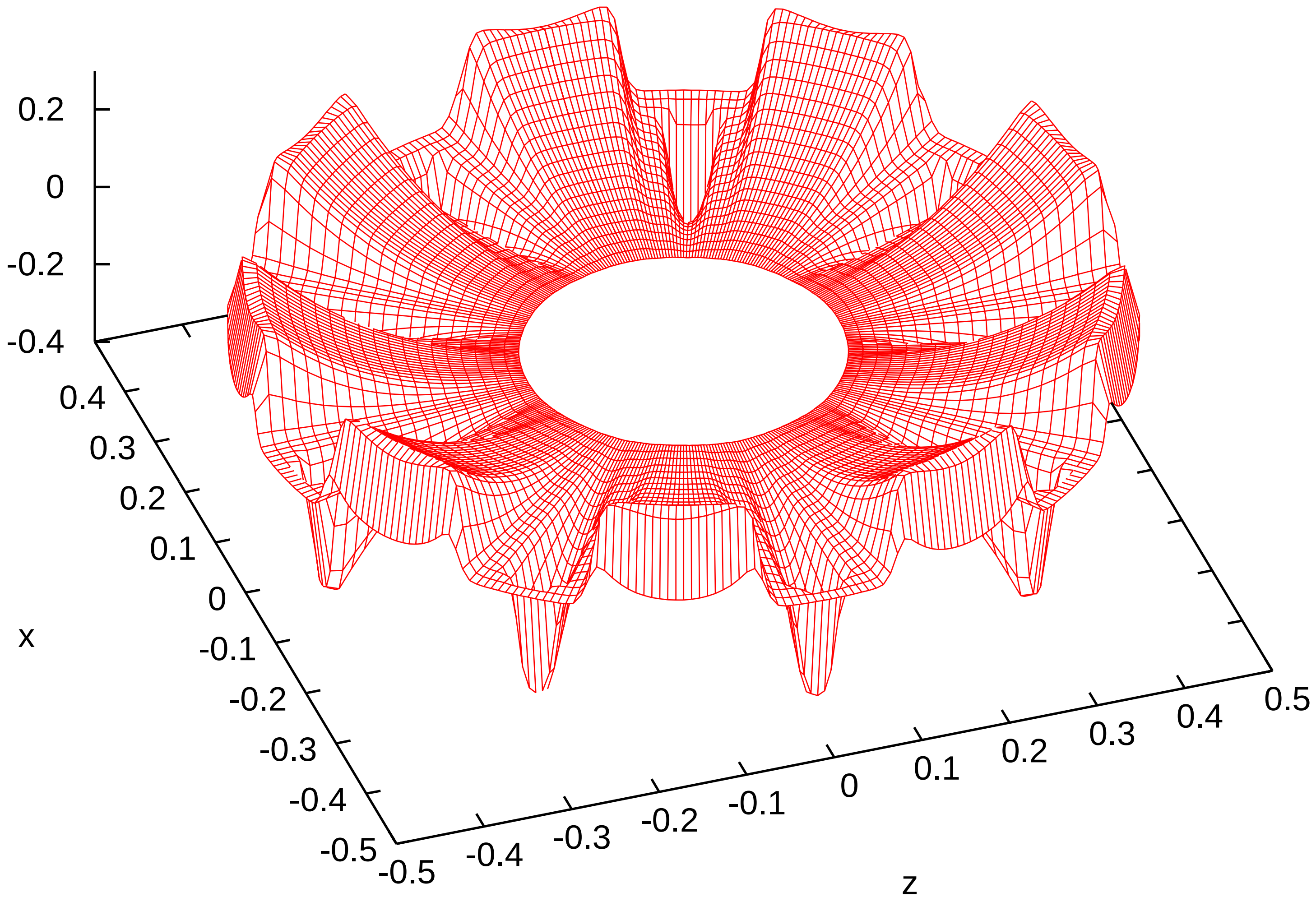
Midplane Field Distribution - Hard Edge Model

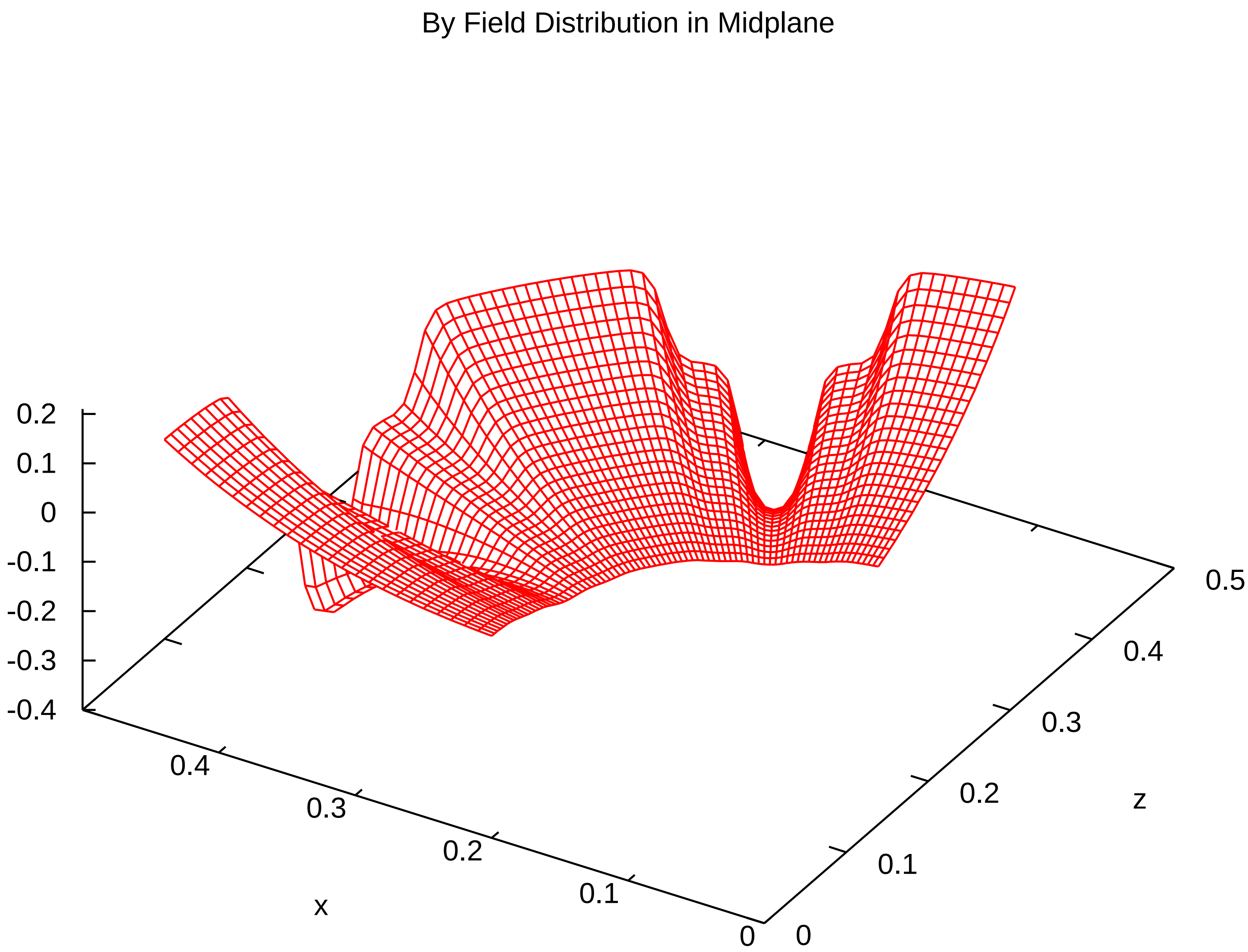




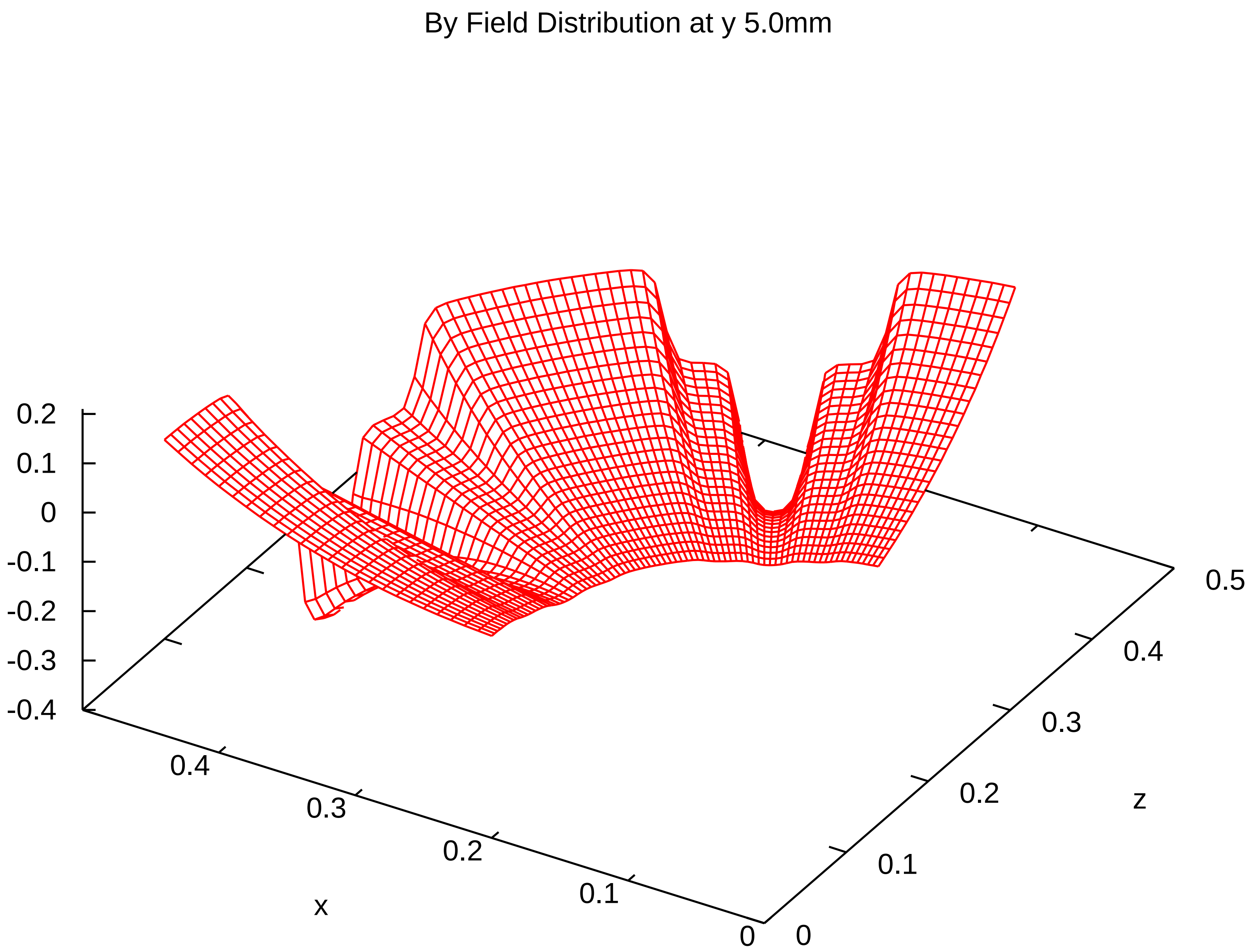
Midplane Field Distribution - COSY default DI Fringe Field Model
0.2
0
-0.2
-0.4
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
x
-0.5
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
0.5
z

Midplane Field Distribution - Permanet Magnet Fringe Field Model

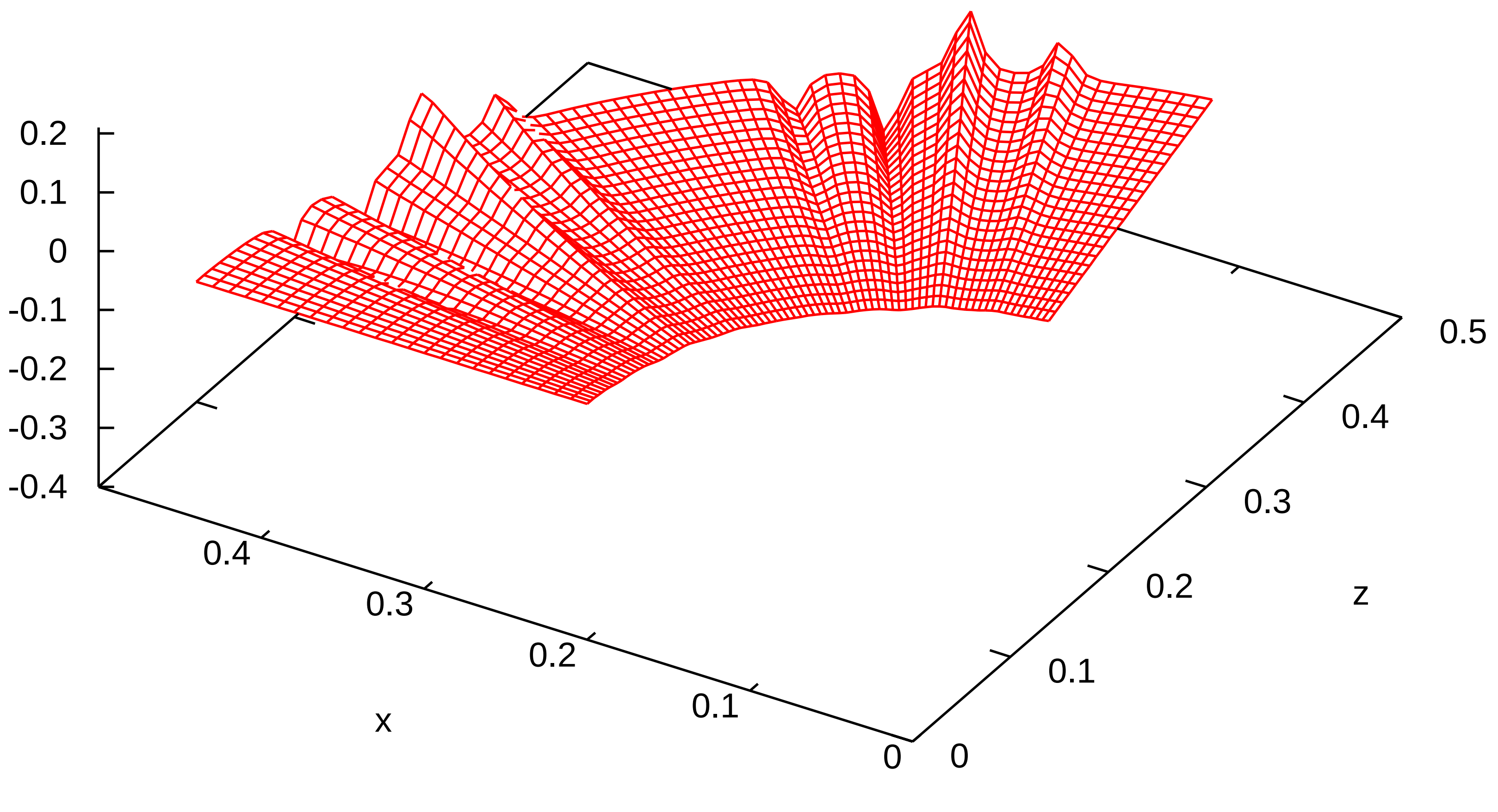


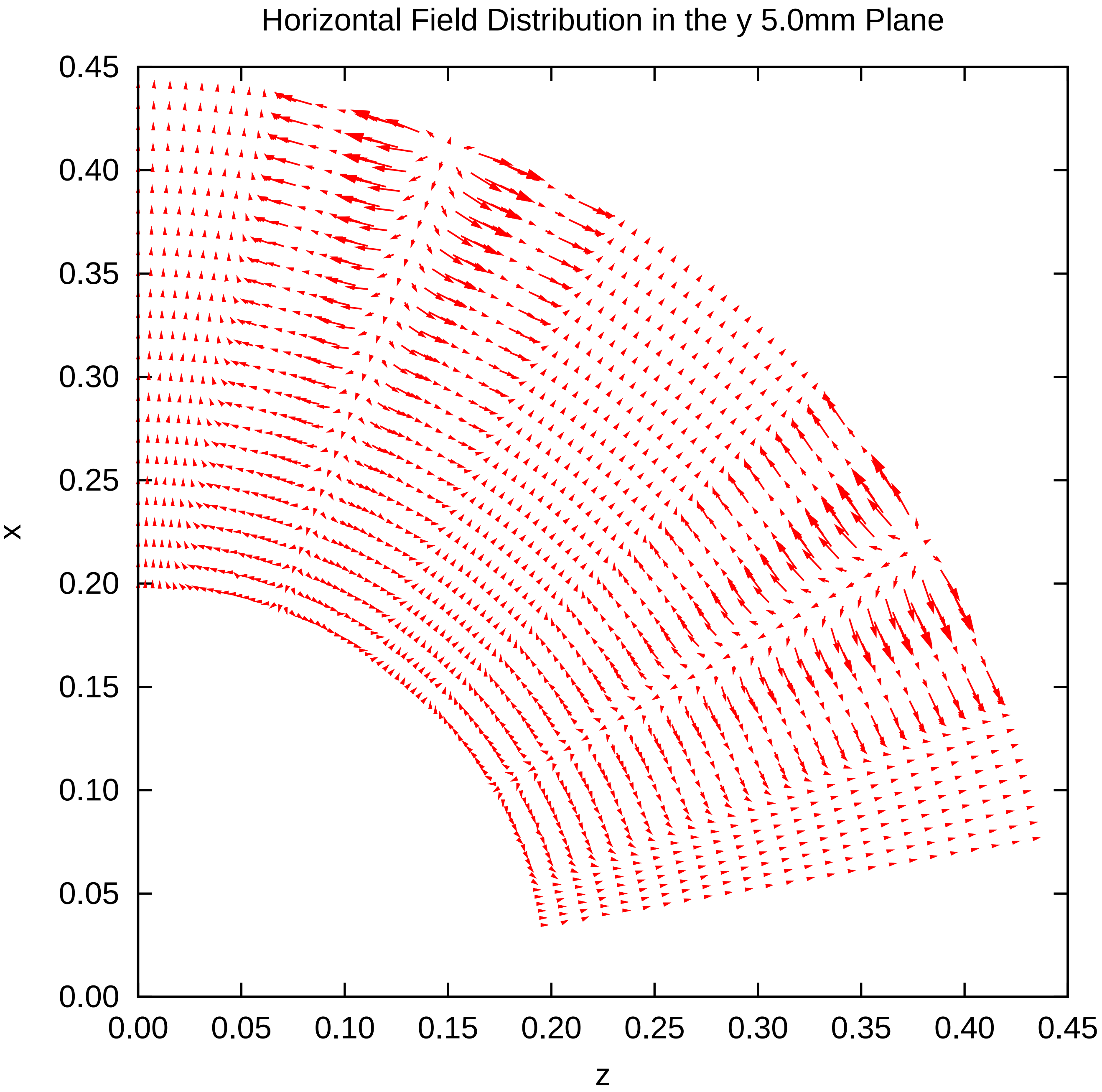


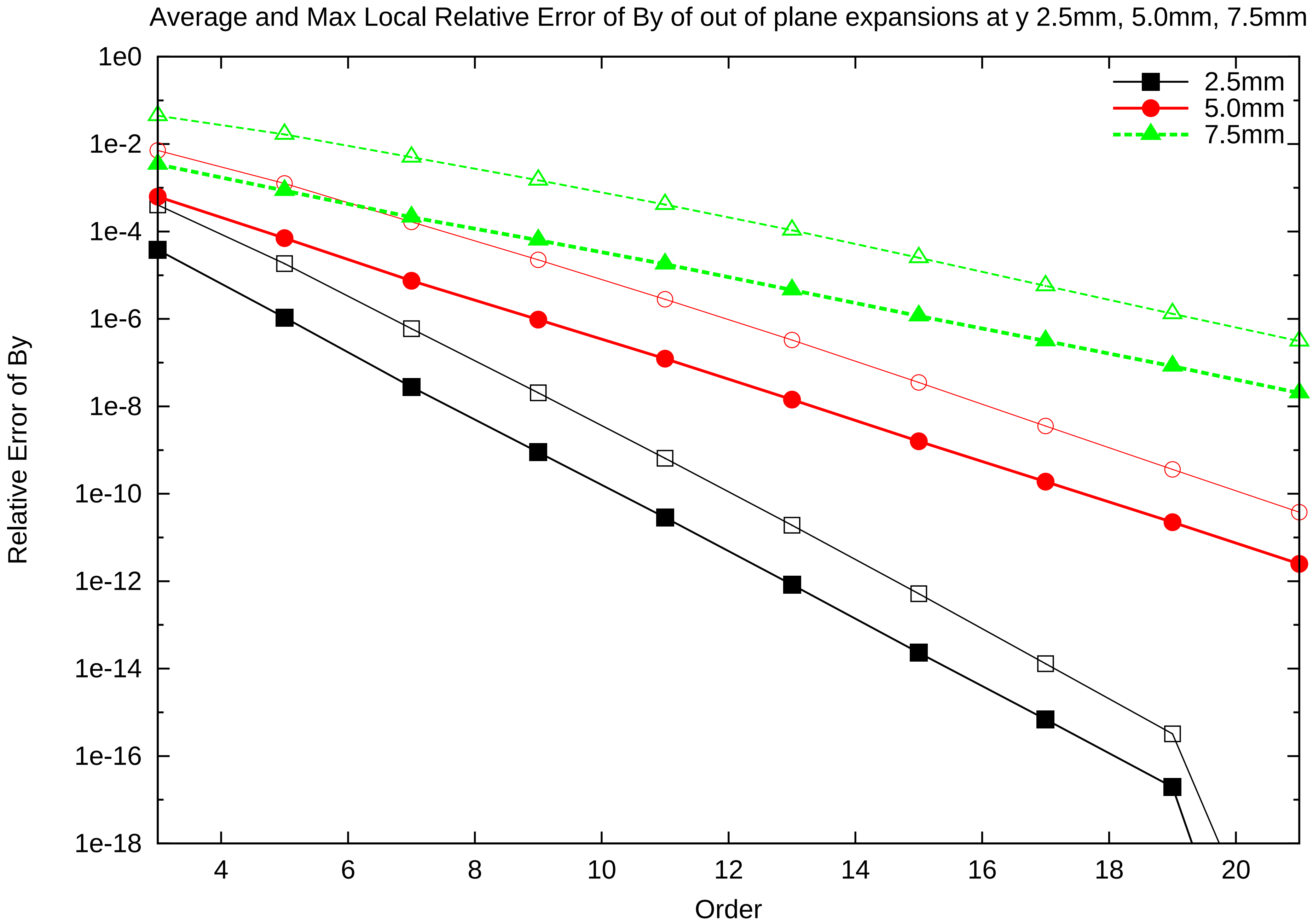
By Field Distribution in Midplane
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0.4
0.3
0.2
0.1
0
x
0
0.1
0.2
0.3
0.4
0.5
z


By Field Distribution at y 5.0mm
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
0.4
0.3
0.2
0.1
0
x
0
0.1
0.2
0.3
0.4
0.5
z

Bth Field Distribution at y 5.0mm




Horizontal Field Distribution in the y 5.0mm Plane
0.45
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
x
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35
0.40
0.45
z



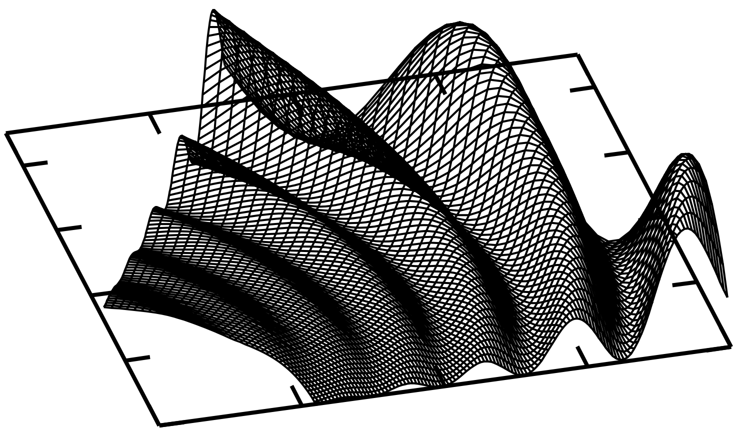
Average and Max Local Relative Error of By of out of plane expansions at y 2.5mm, 5.0mm, 7.5mm
2.5mm
5.0mm
7.5mm
Relative Error of By
Order
1e0
1e-2
1e-4
1e-6
1e-8
1e-10
1e-12
1e-14
1e-16
1e-18
4
6
8
10
12
14
16
18
20

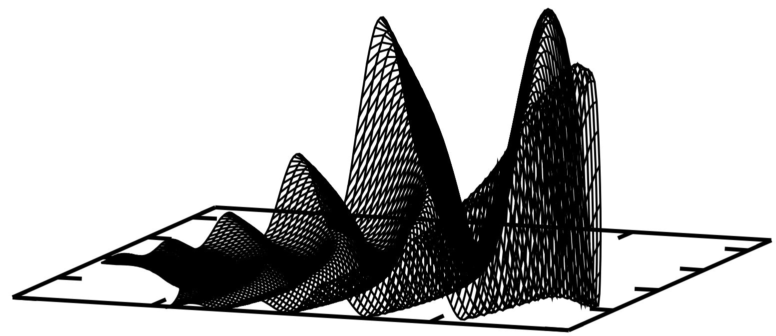
A Flow of Analysis of FFAGs and COSY Tools

- **Closed Orbits.** Determine closed orbits $\overrightarrow{r}_{cl}(s)$ for a set of reference particle energies by optimization.
- **Arbitrary Order Maps.** For each $\overrightarrow{r}_{cl}(s)$, calculate a high-order energy-dependent transfer map $\mathcal{M}(\vec{z}_i, \vec{\delta})$ around it, including high-order effects, such as out-of-plane field expansions and nonlinearities in the Hamiltonian.
- **Linear Properties of Maps.** Determine common linear beam functions including invariant ellipses and tunes near $\overrightarrow{r}_{cl}(s)$.
- **Tracking.** Using $\mathcal{M}(\vec{z}_i, \vec{\delta})$, perform tracking to estimate the dynamic aperture, presence of resonances, etc. There are various methods in COSY preserving the symplecticity, like EXPO with minimal modifications.
- **Acceleration.** Describe the fields including cavities. Study the entire energy range in steps to see the acceleration effects.
- **Amplitude Dependent Tunes and Resonances.** Use COSY tools for nonlinear effects, including the normal form-based computations.
- **Global Parameter Optimization.** Use COSY optimization tools for system parameters, including global optimizations working over a pre-specified search region, differing from conventional local optimizations.

Tracking Study with/without Symplectification

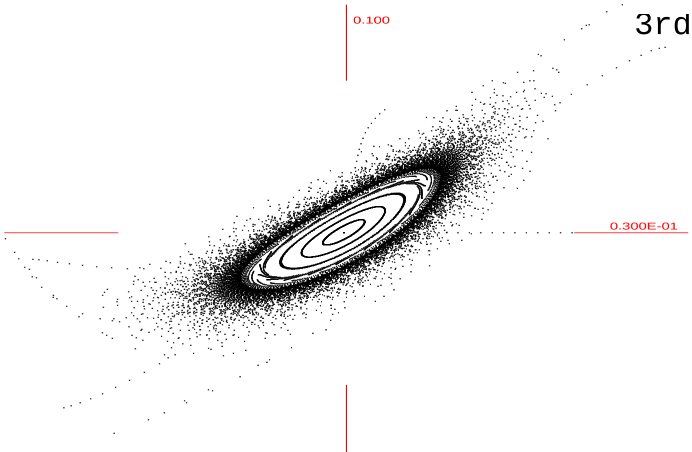
- Example using a scaling FFAG model



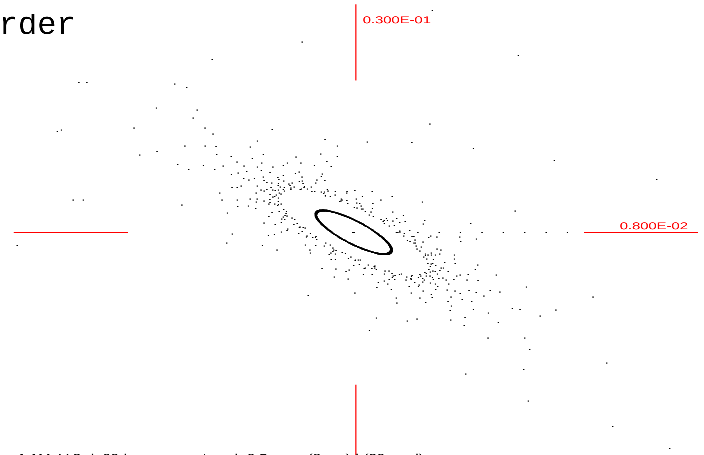


  - 3rd order computations without symplectification
  - 11th order computations without symplectification
  - 11th order computations with EXPO

3rd order

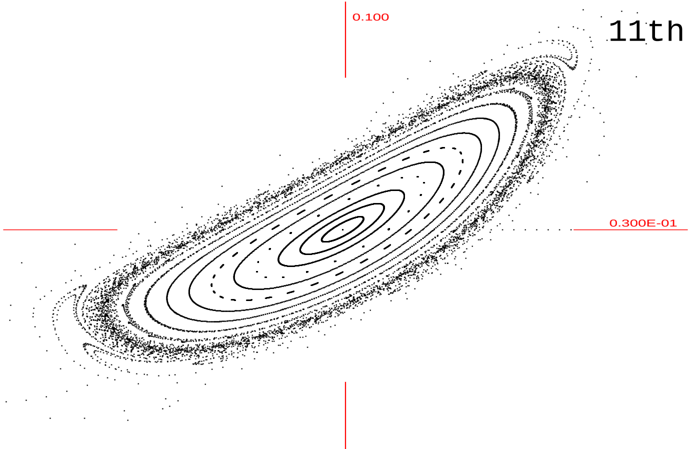


1.1MeV 3rd. 60deg sym. x at each 1mm. x(30mm)-a(100mrad)

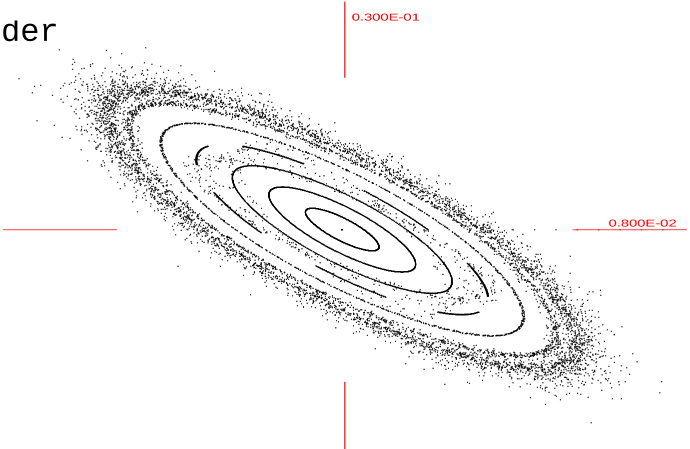


1.1MeV 3rd. 60deg sym. y at each 0.5mm. y(8mm)-b(30mrad)

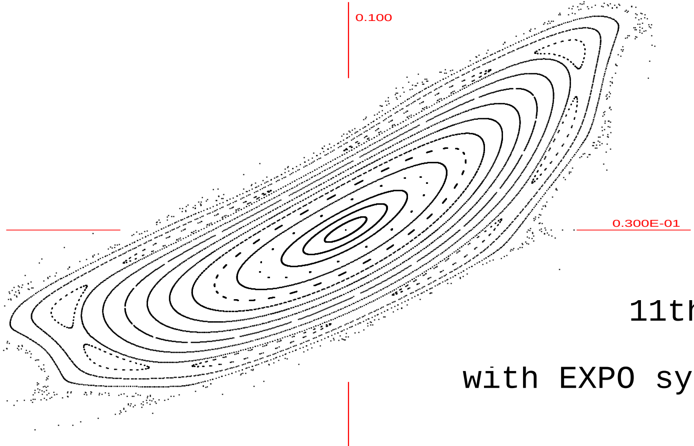
11th order



1.1MeV 11th. 60deg sym. x at each 1mm. x(30mm)-a(100mrad)



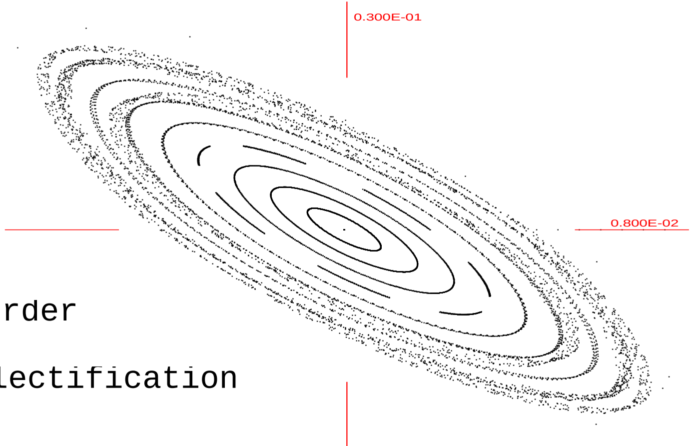
1.1MeV 11th. 60deg sym. y at each 0.5mm. y(8mm)-b(30mrad)



1.1MeV Symplectic 11th. 60deg sym. x at each 1mm. x(30mm)-a(100mrad)

11th order

with EXPO symplectification

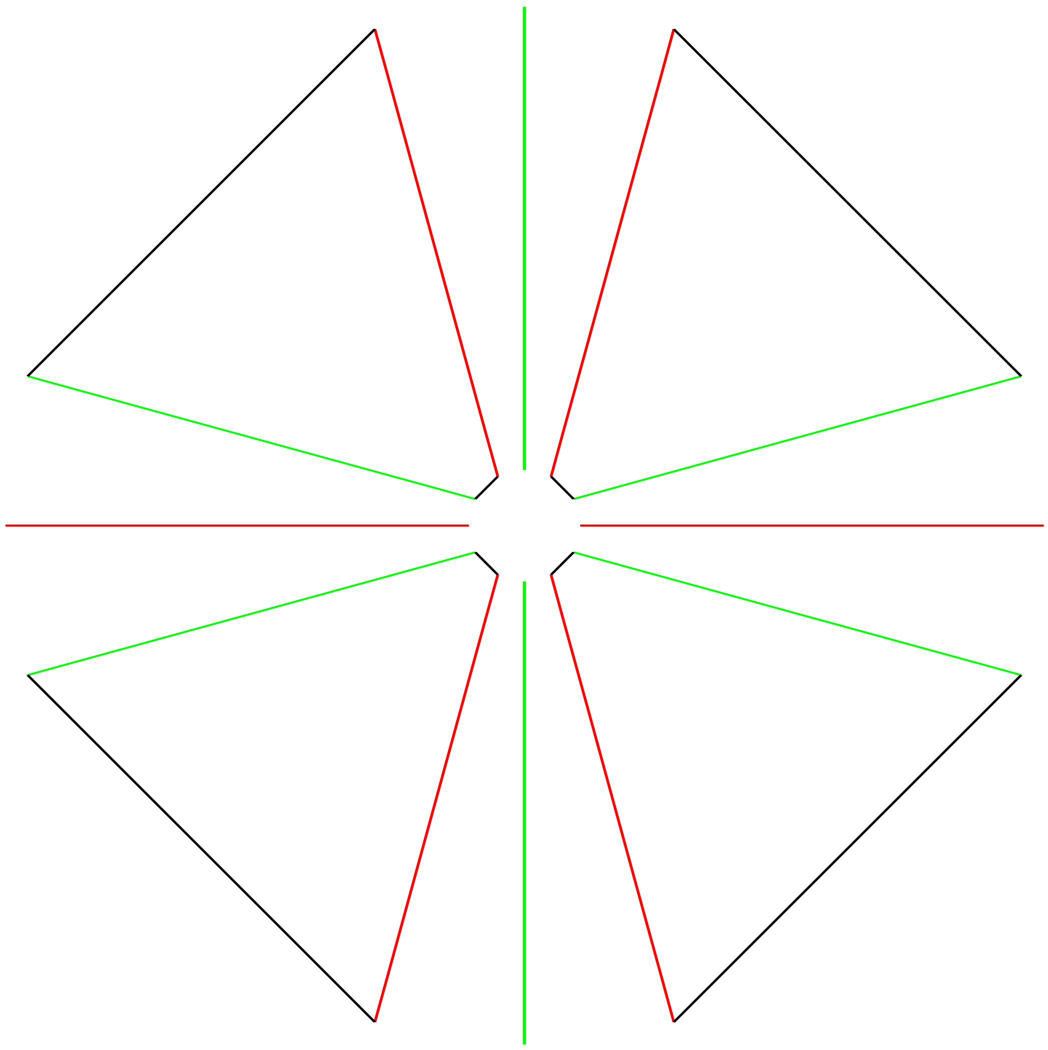


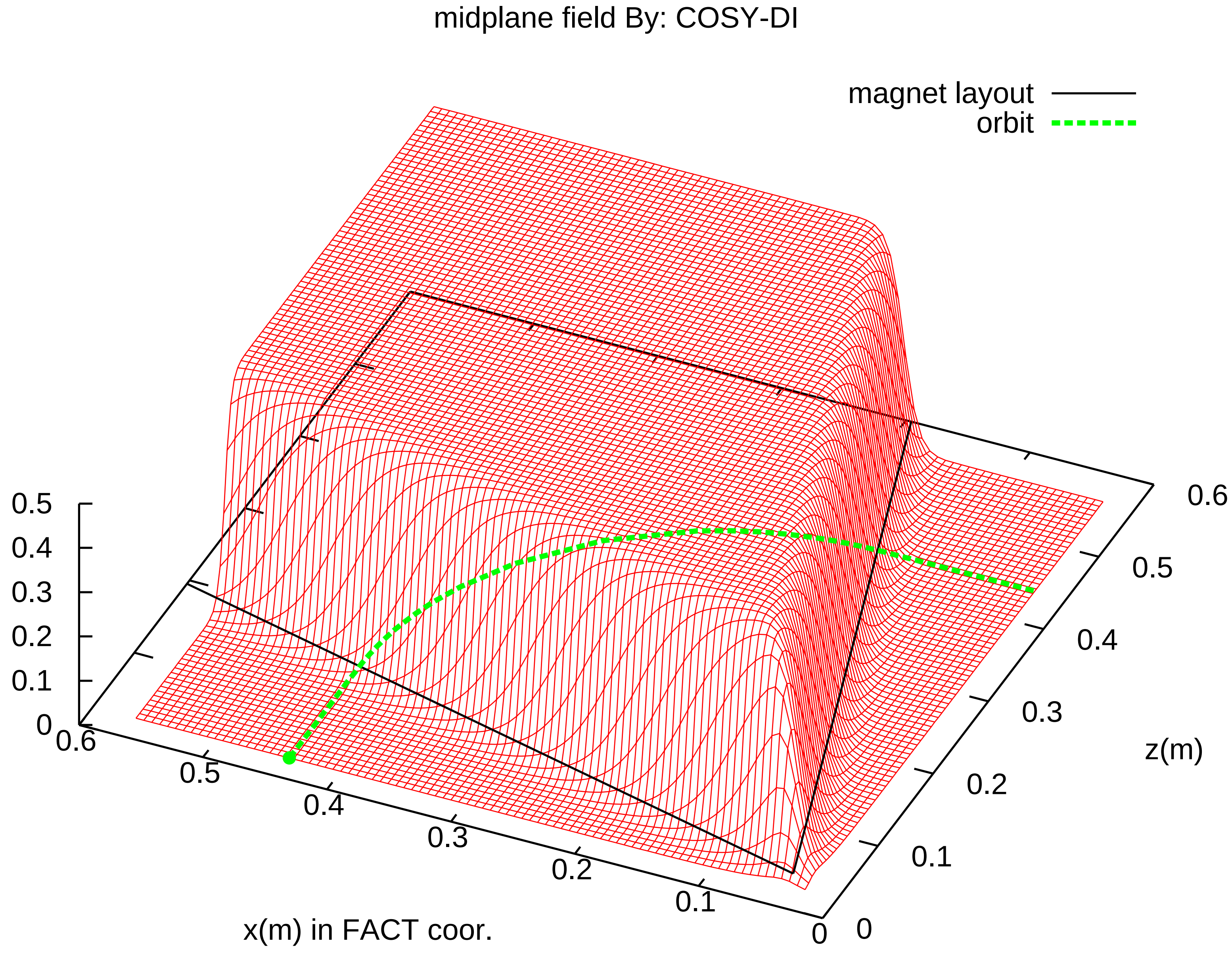
1.1MeV Symplectic 11th. 60deg sym. y at each 0.5mm. y(8mm)-b(30mrad)

Consistency Check

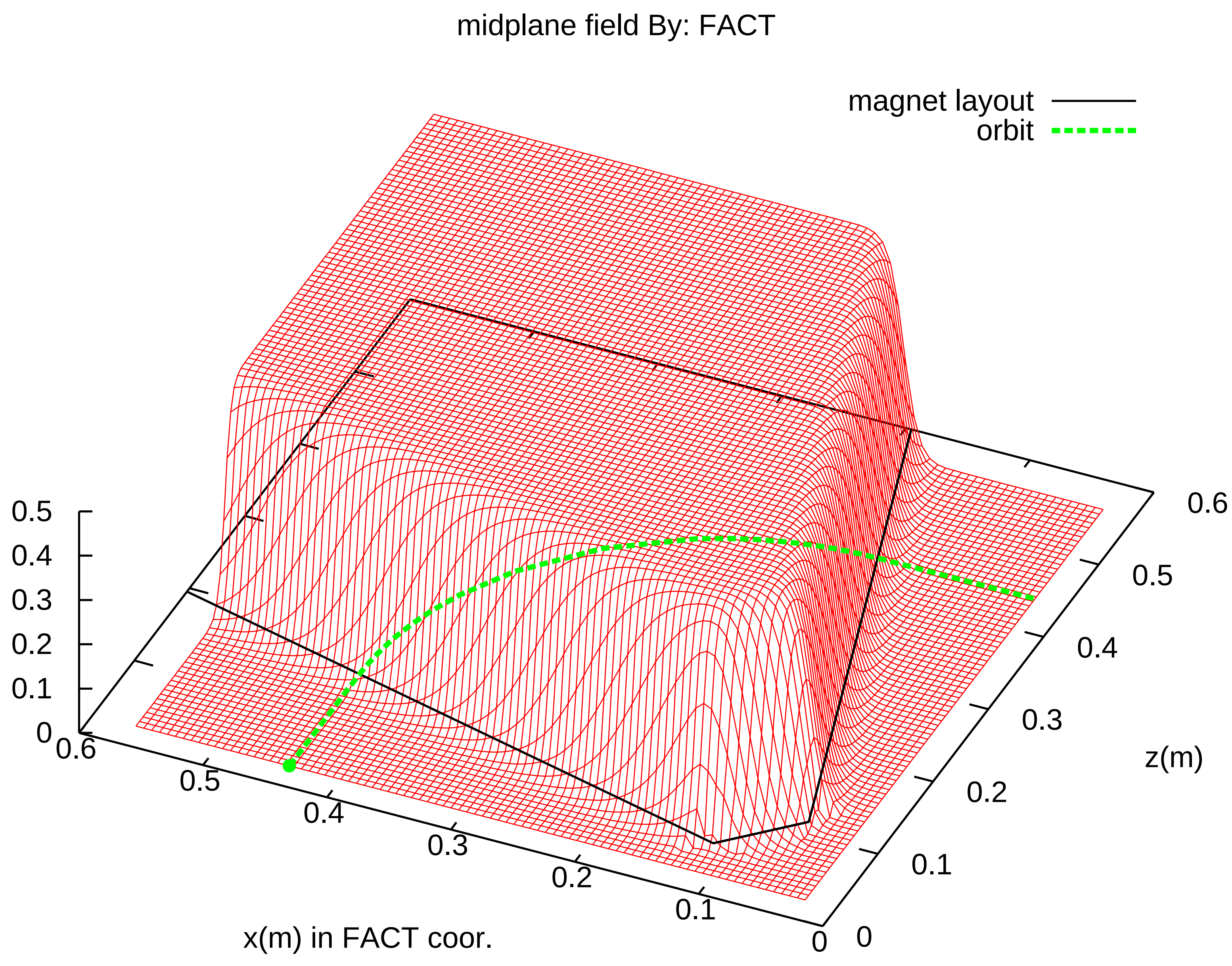
- Example using a cyclotron model
  - Using the method of generalized superimposed FFAG magnets (FACT)
  - Model exactly the same system using a COSY standard DIpole magnet (COSY-DI)
  - The same fringe field fall-off Enge model is applied
  - Compare high-order tracking pictures
    Also compare COSY-DI without any fringe field

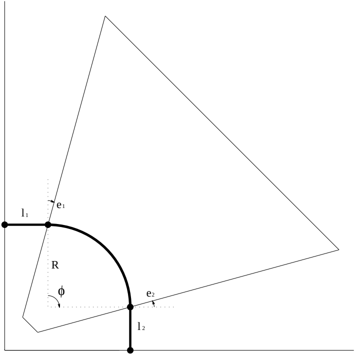
cyclotron 2 3   full system

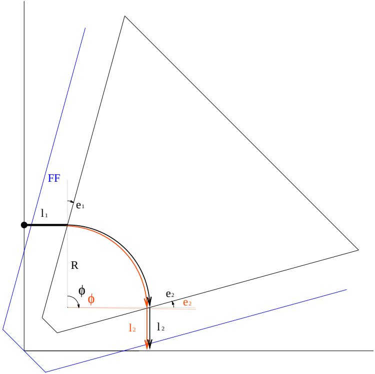


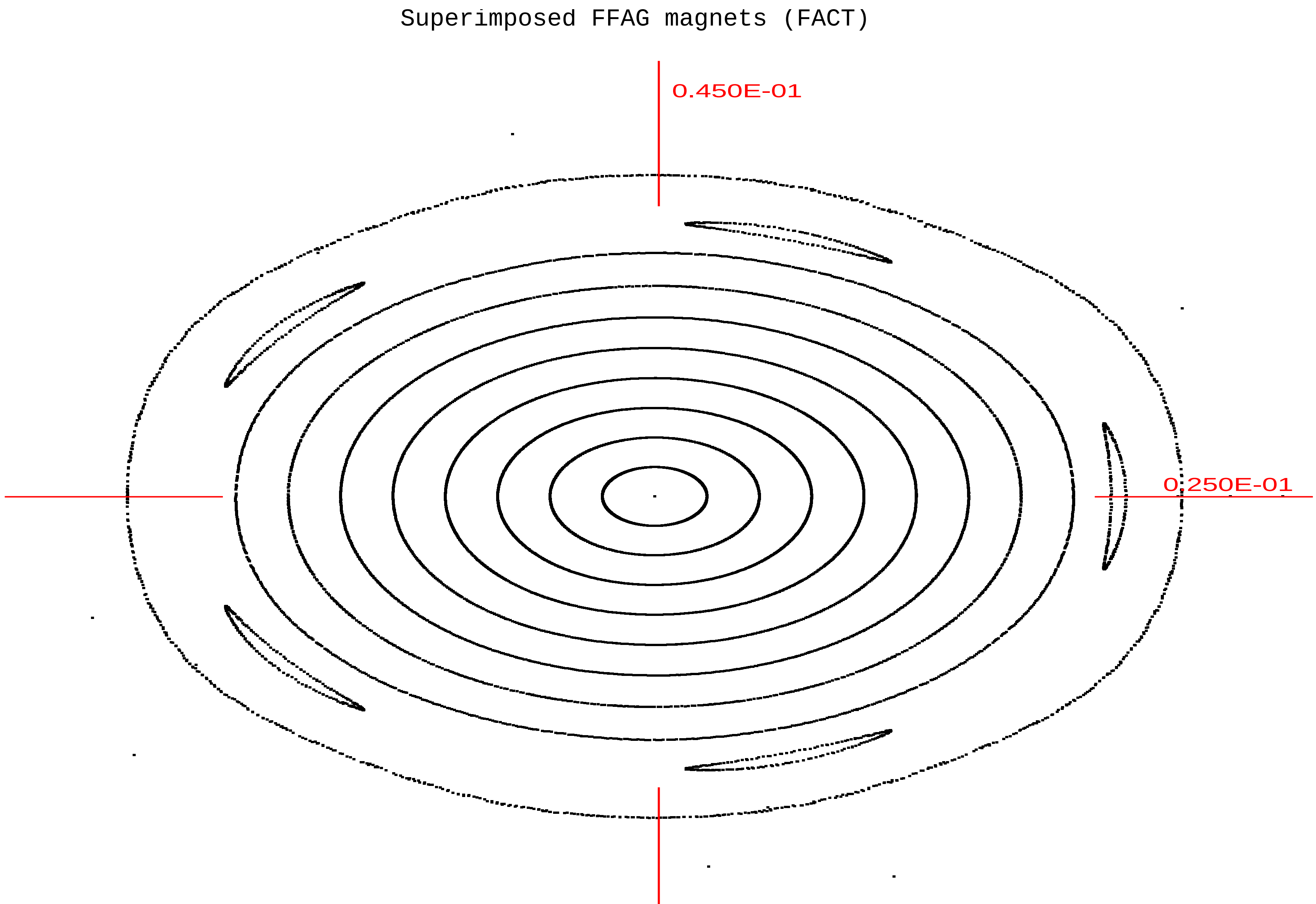


midplane field By: COSY-DI
magnet layout
orbit
0.5
0.4
0.3
0.2
0.1
0
0.6
0.5
0.4
0.3
0.2
0.1
0
x(m) in FACT coor.
0
0.1
0.2
0.3
0.4
0.5
0.6
z(m)


midplane field By: FACT
magnet layout
orbit
0.5
0.4
0.3
0.2
0.1
0
0.6
0.5
0.4
0.3
0.2
0.1
0
x(m) in FACT coor.
0
0.1
0.2
0.3
0.4
0.5
0.6
z(m)

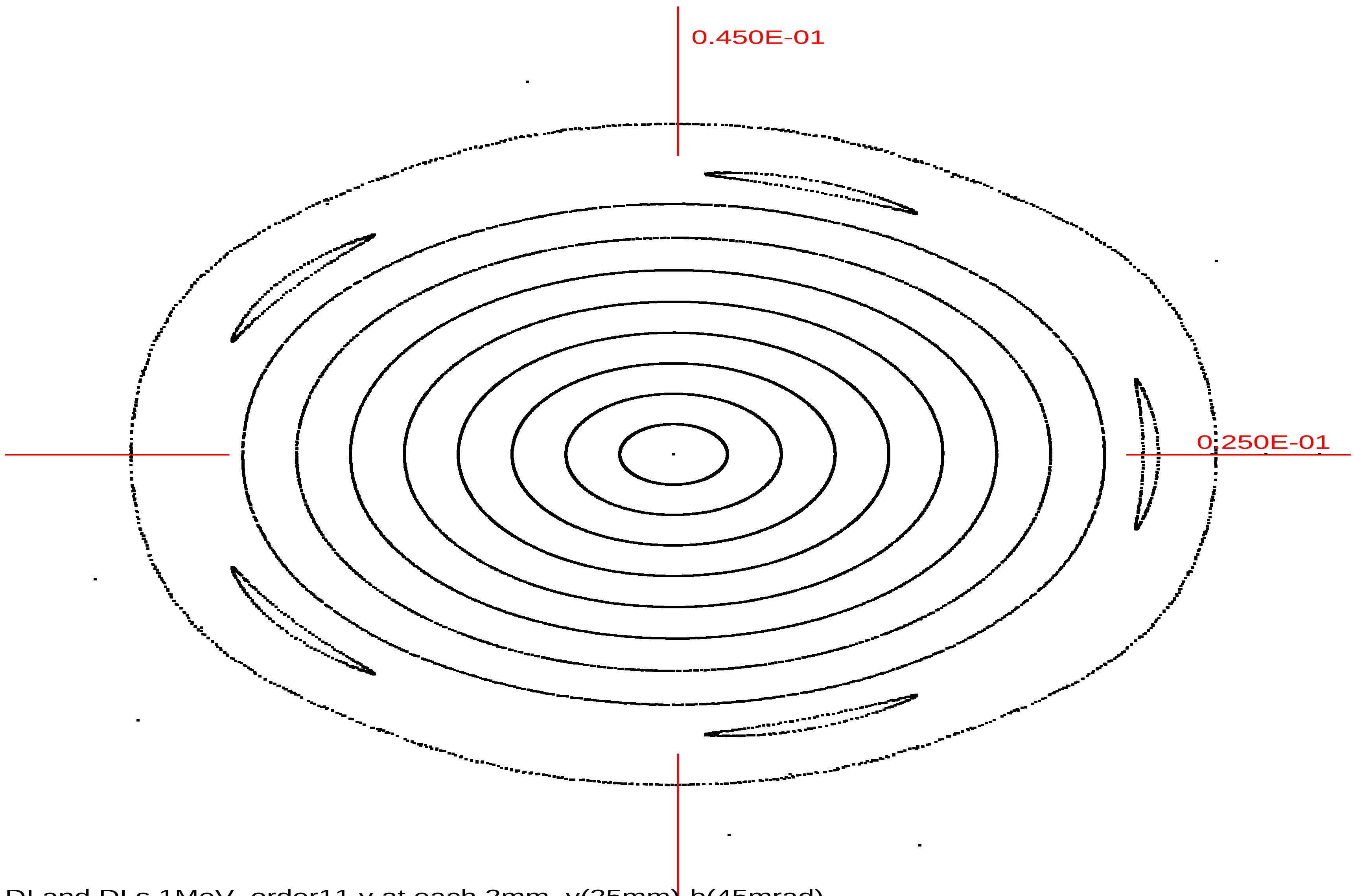

e1
l1
R
ϕ
e2
l2


FF
e₁
l₁
R
ϕ
ϕ
e₂
e₂
l₂
l₂



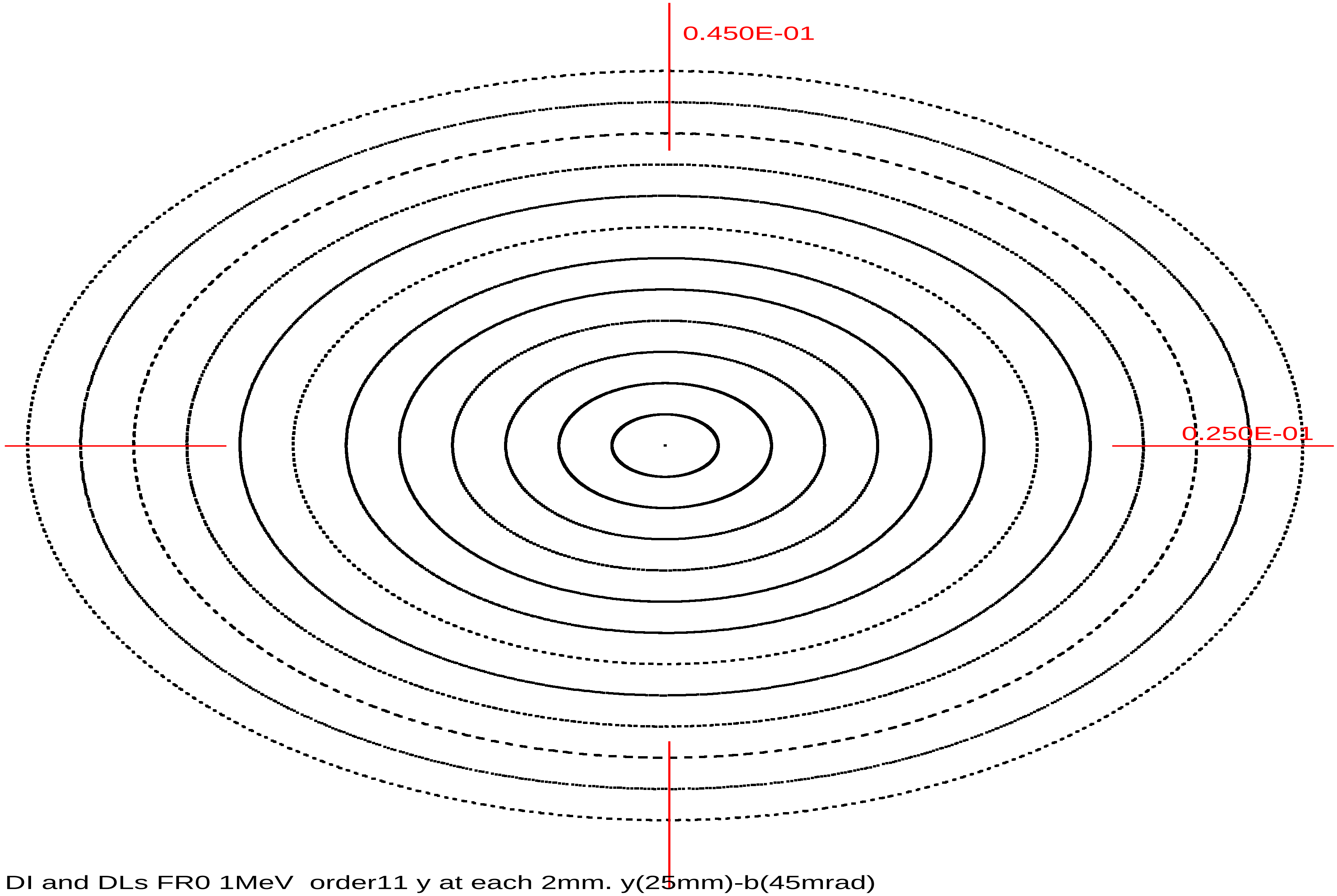
NSFFAG 1MeV order11 y at each 2mm. y(25mm)-b(45mrad)

Modelling exactly the same system by COSY's standard DIpole magnet



DI and DLs 1MeV order11 y at each 2mm. y(25mm)-b(45mrad)

Modelling exactly the same system by COSY's standard DIpole magnet - NO fringe fields

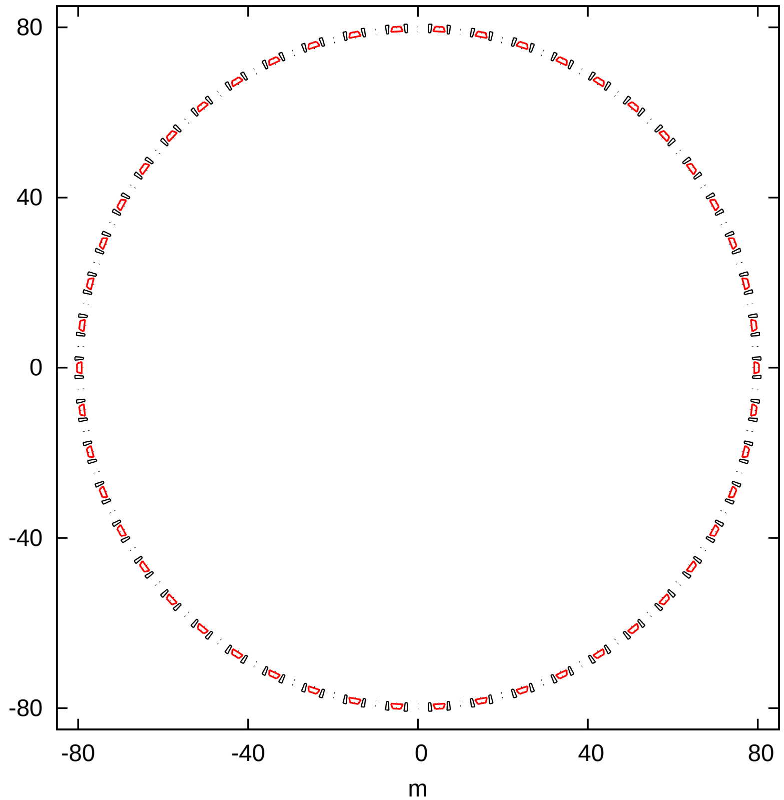


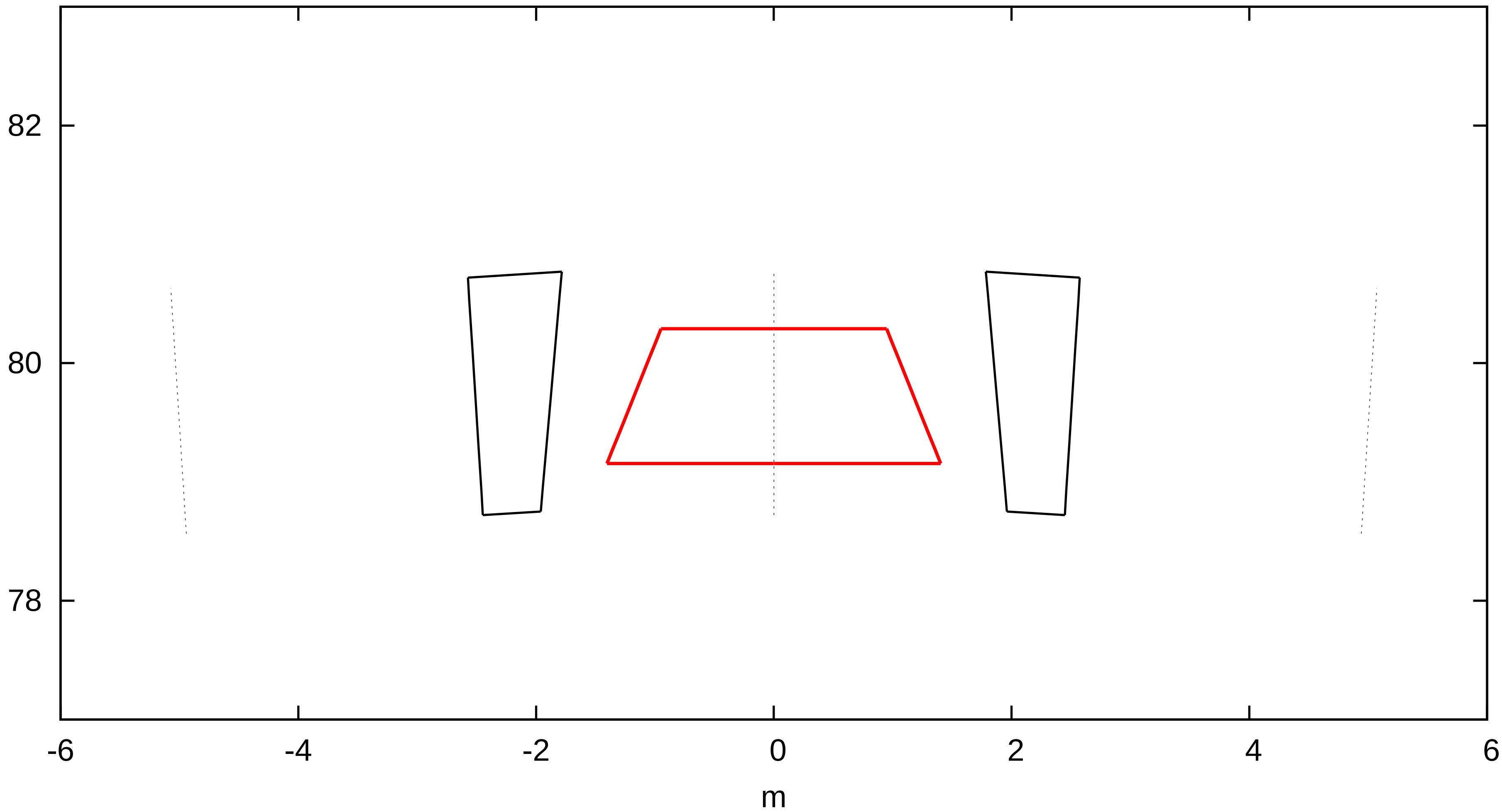
Outlook

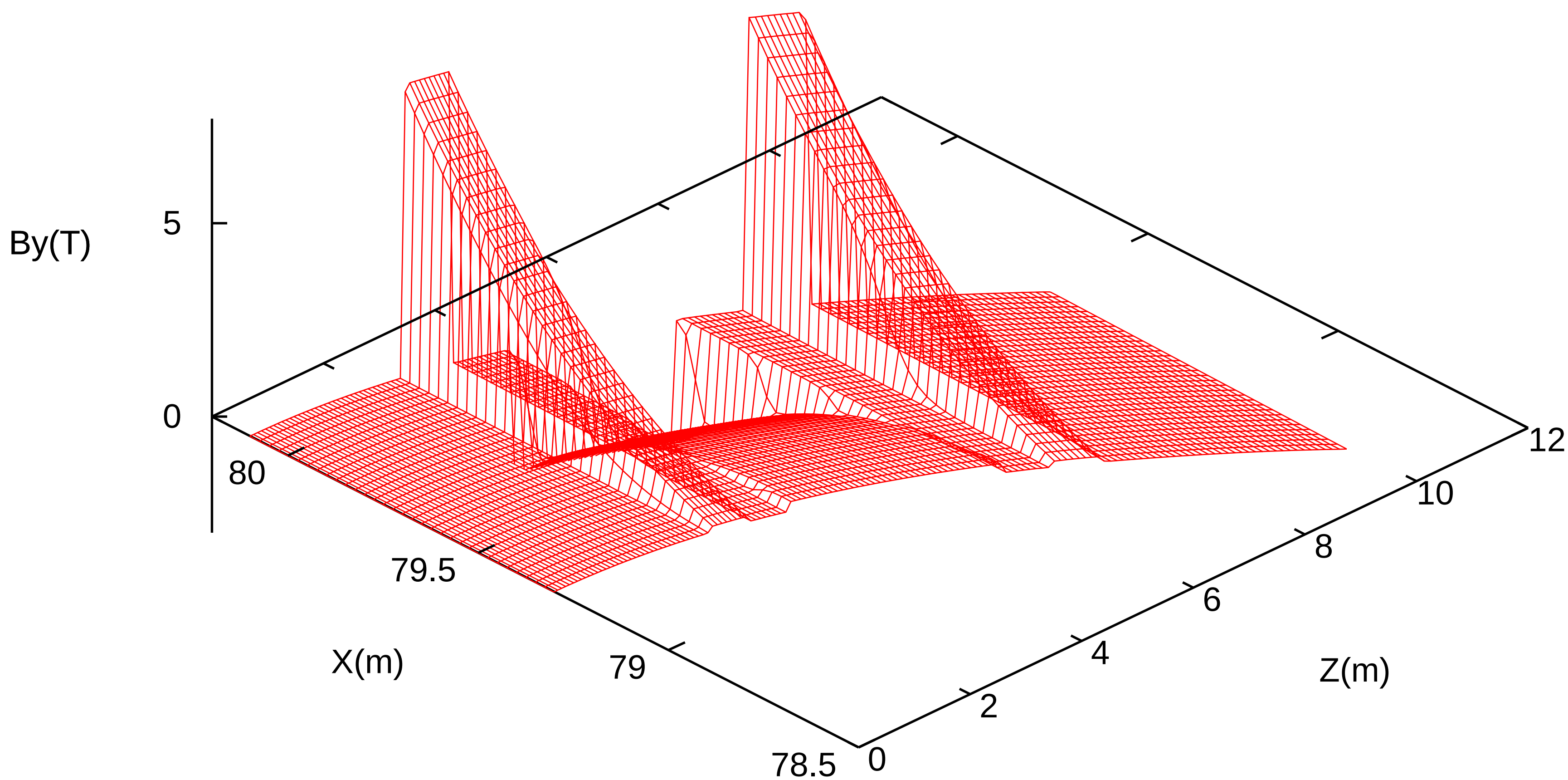
- Consistency checks in inhomogeneous bending magnets

Outlook

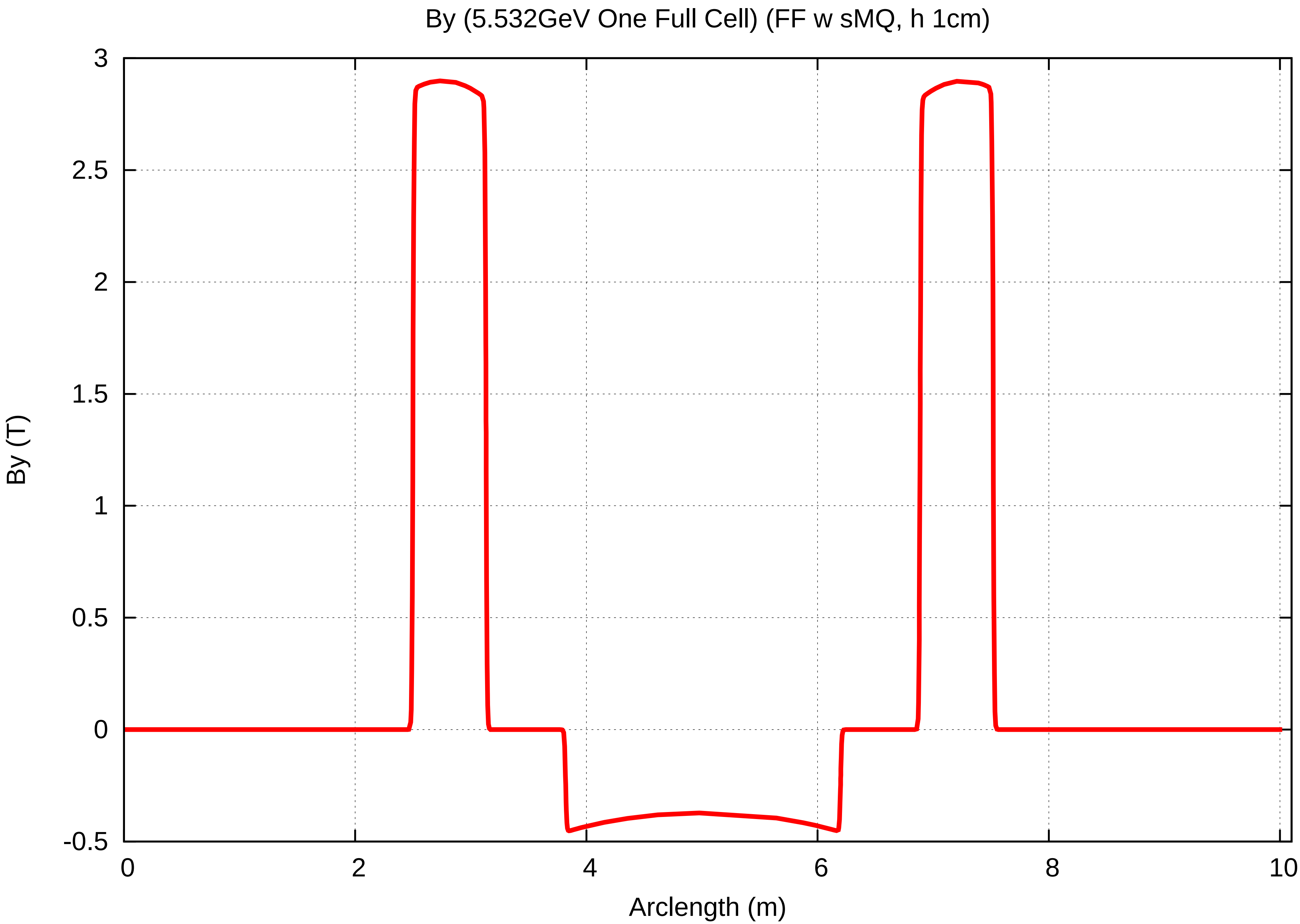
- Consistency checks in inhomogeneous bending magnets
  - How can it be done using conventional simulation codes?
    Example: A nonscaling FFAG design for Project X

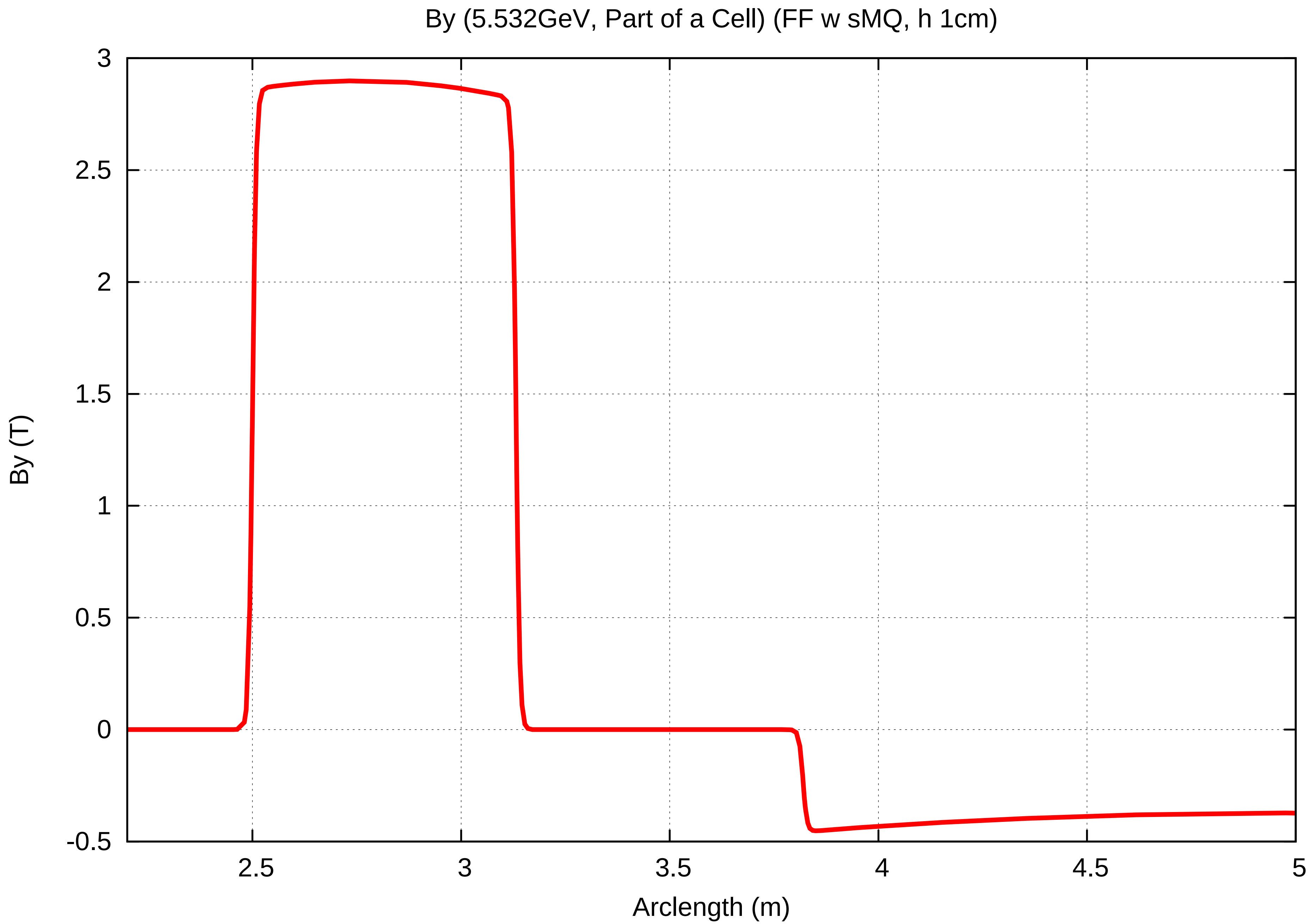



82
80
78
-6
-4
-2
0
2
4
6
m



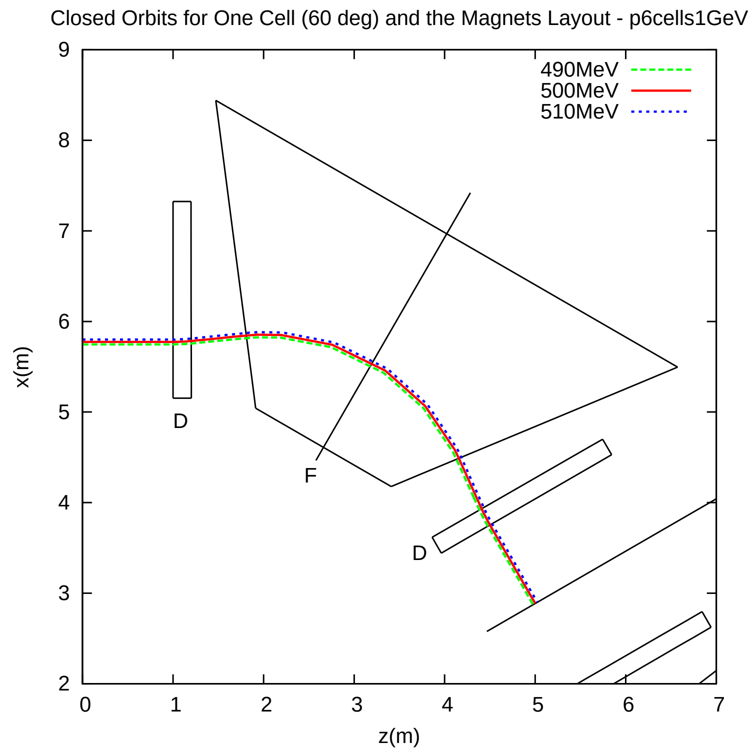
By(T)
5
0
80
79.5
79
78.5
X(m)
0
2
4
6
8
10
12
Z(m)


By (5.532GeV One Full Cell) (FF w sMQ, h 1cm)
3
2.5
2
1.5
1
0.5
0
-0.5
By (T)
0
2
4
6
8
10
Arclength (m)

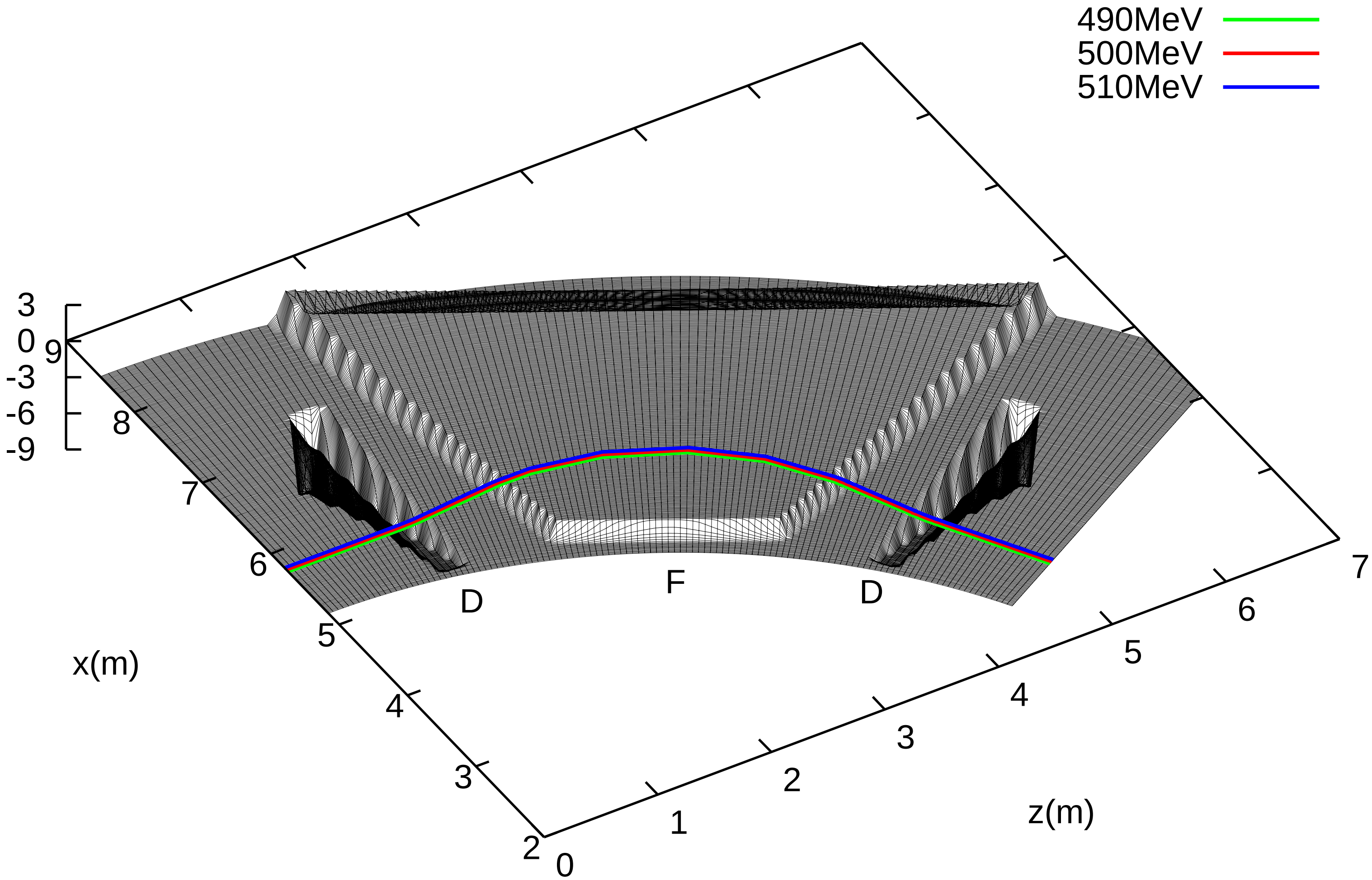

By (5.532GeV, Part of a Cell) (FF w sMQ, h 1cm)
By (T)
Arclength (m)
3
2.5
2
1.5
1
0.5
0
-0.5
2.5
3
3.5
4
4.5
5

Outlook

- Consistency checks in inhomogeneous bending magnets
  - How can it be done using conventional simulation codes?
    Example: A nonscaling FFAG design for Project X
  - How can it be done using codes for field maps?
    Example: A nonscaling 6 cell FFAG design

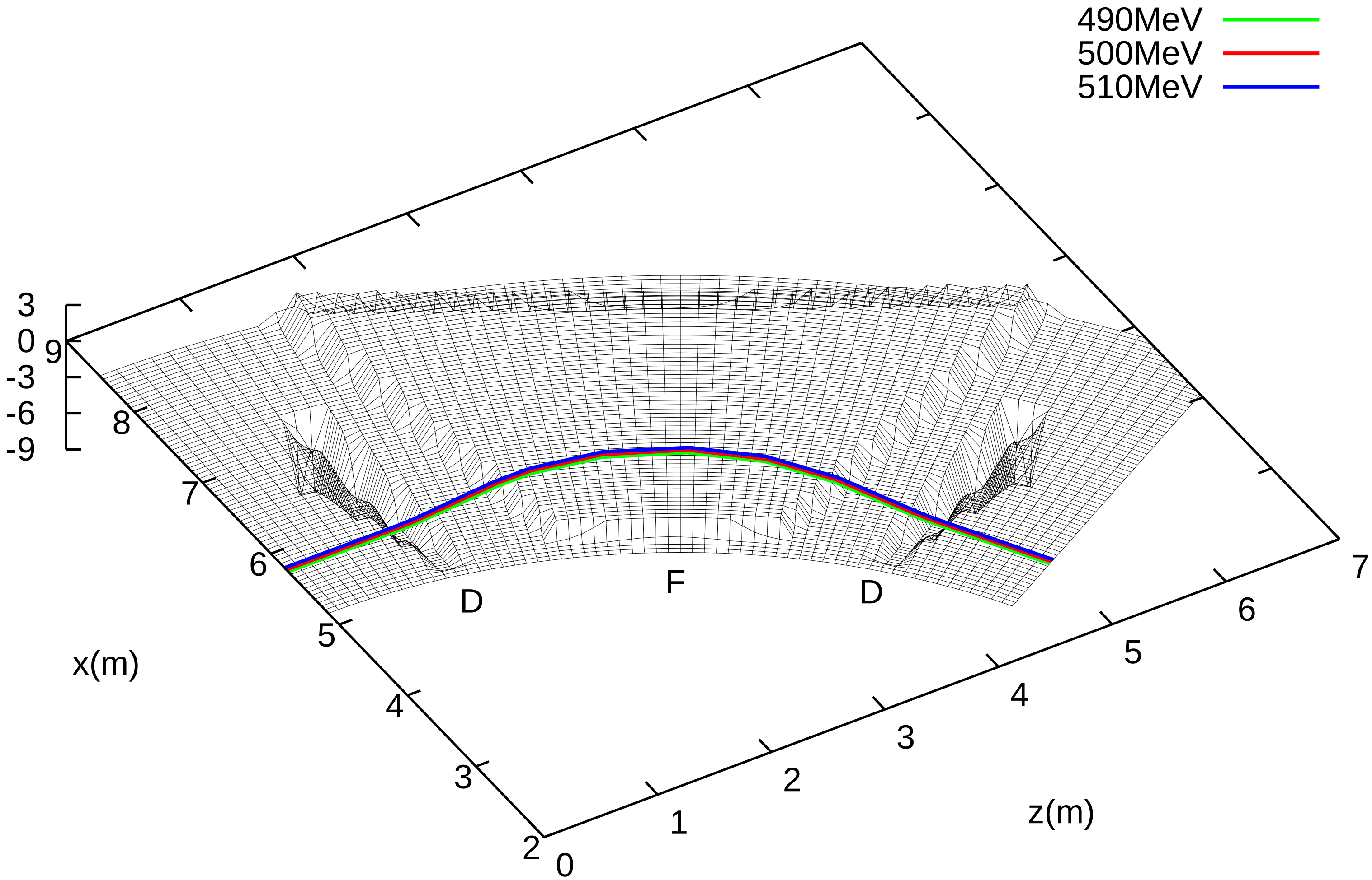


Midplane Field Distribution and Closed Orbits for One Cell (60 deg) - p6cells1GeV



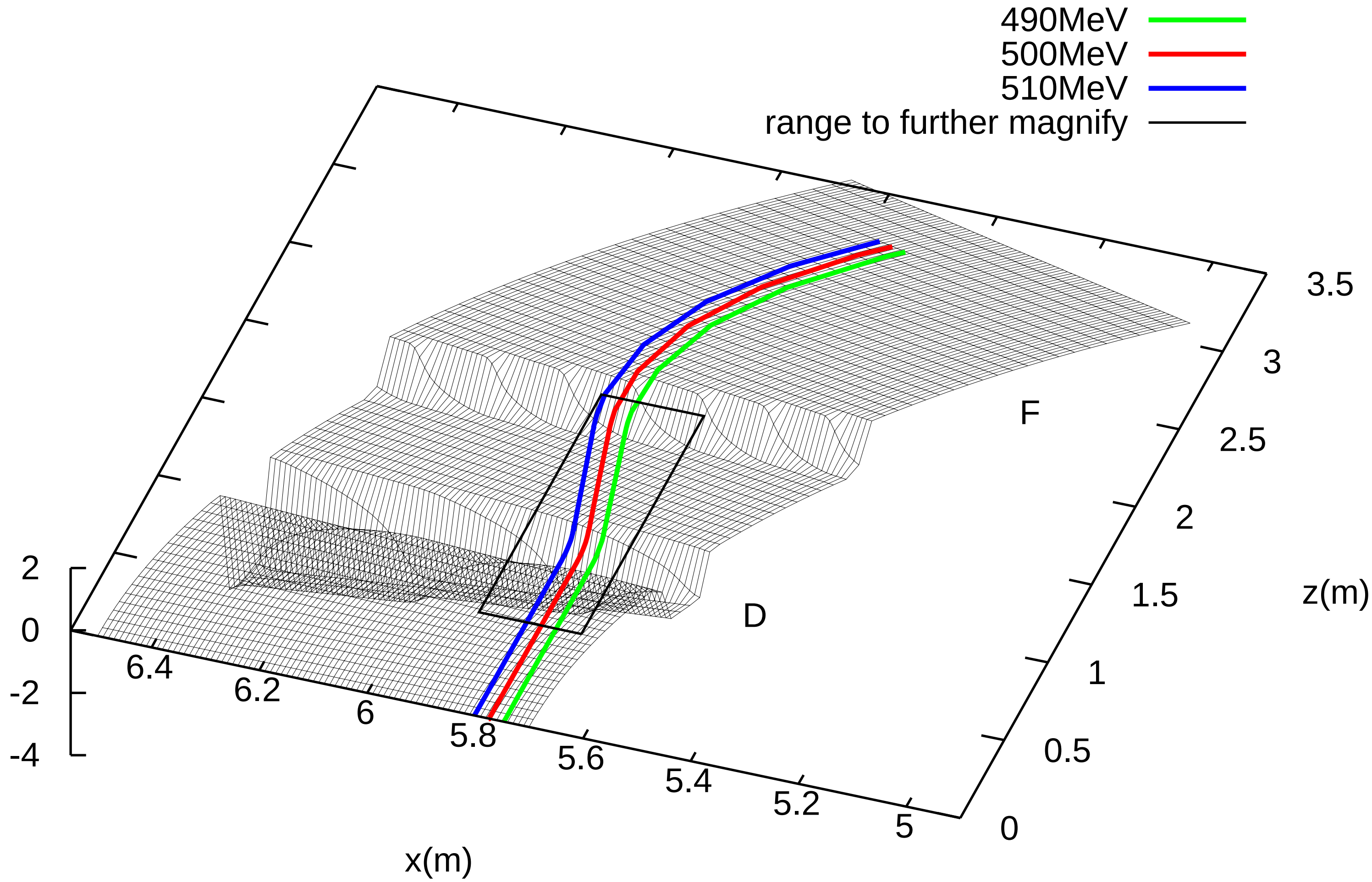
A fine grid (d_R=1cm, d_theta=0.5deg) is used for the field data.

Midplane Field Distribution and Closed Orbits for One Cell (60 deg) - p6cells1GeV



A coarse grid (d_R=5cm, d_theta=1deg) is used for the field data, only for the easy demonstration.

Midplane Field Data and Closed Orbits for a Half Cell (30 deg) Shown in a Limited Range - p6cells1GeV

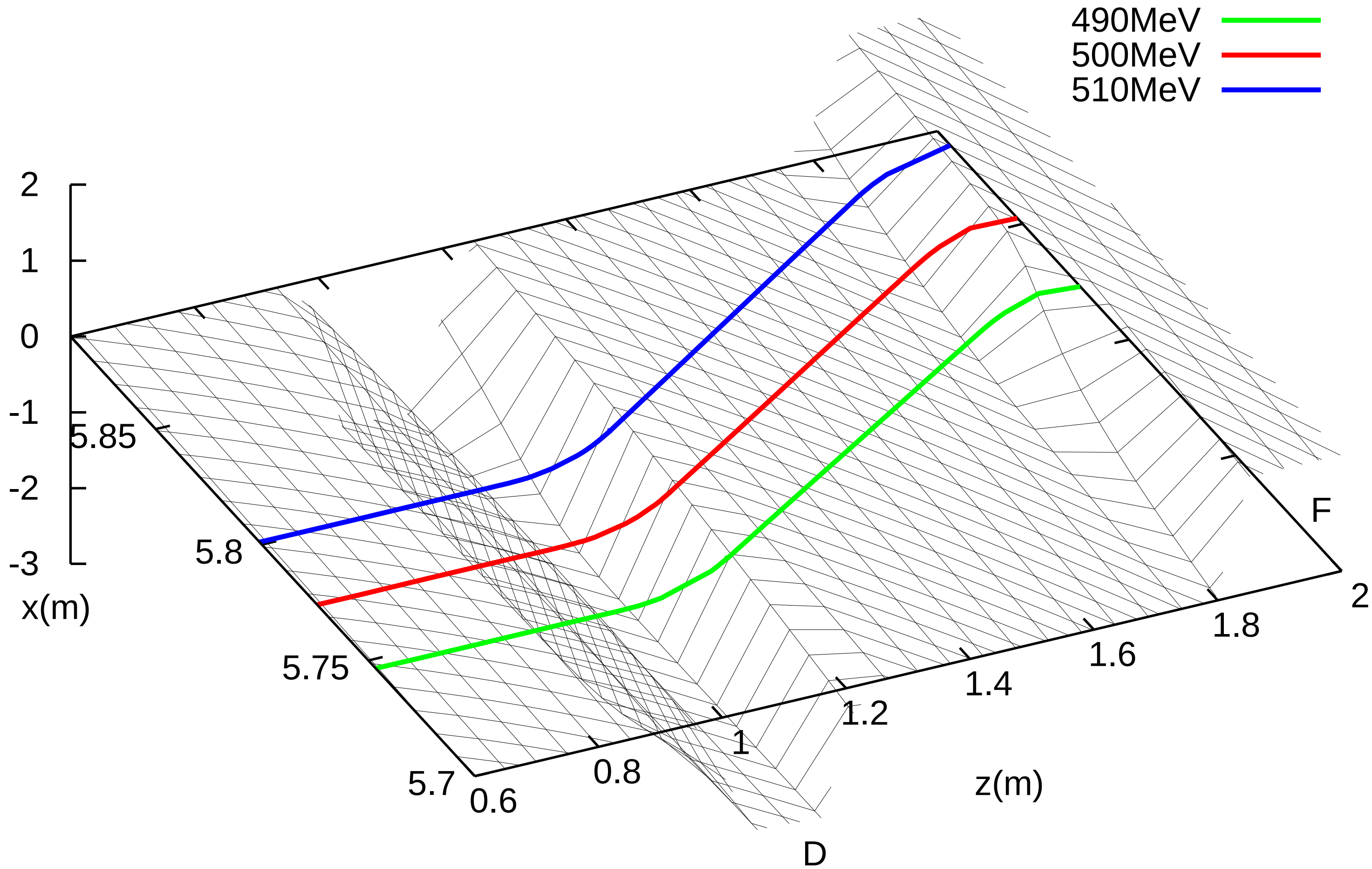


A fine grid (d_R=1cm, d_theta=0.5deg) is used for the field data.

The shown above covers only a limited area of a half cell (30 deg).

The marked rectangular area is shown magnified in the next page.

Midplane Field Data and Closed Orbits in a Magnified Range - p6cells1GeV



A fine grid (d_R=1cm, d_theta=0.5deg) is used for the field data.

The marked rectangular area of the previous page is shown, magnified.

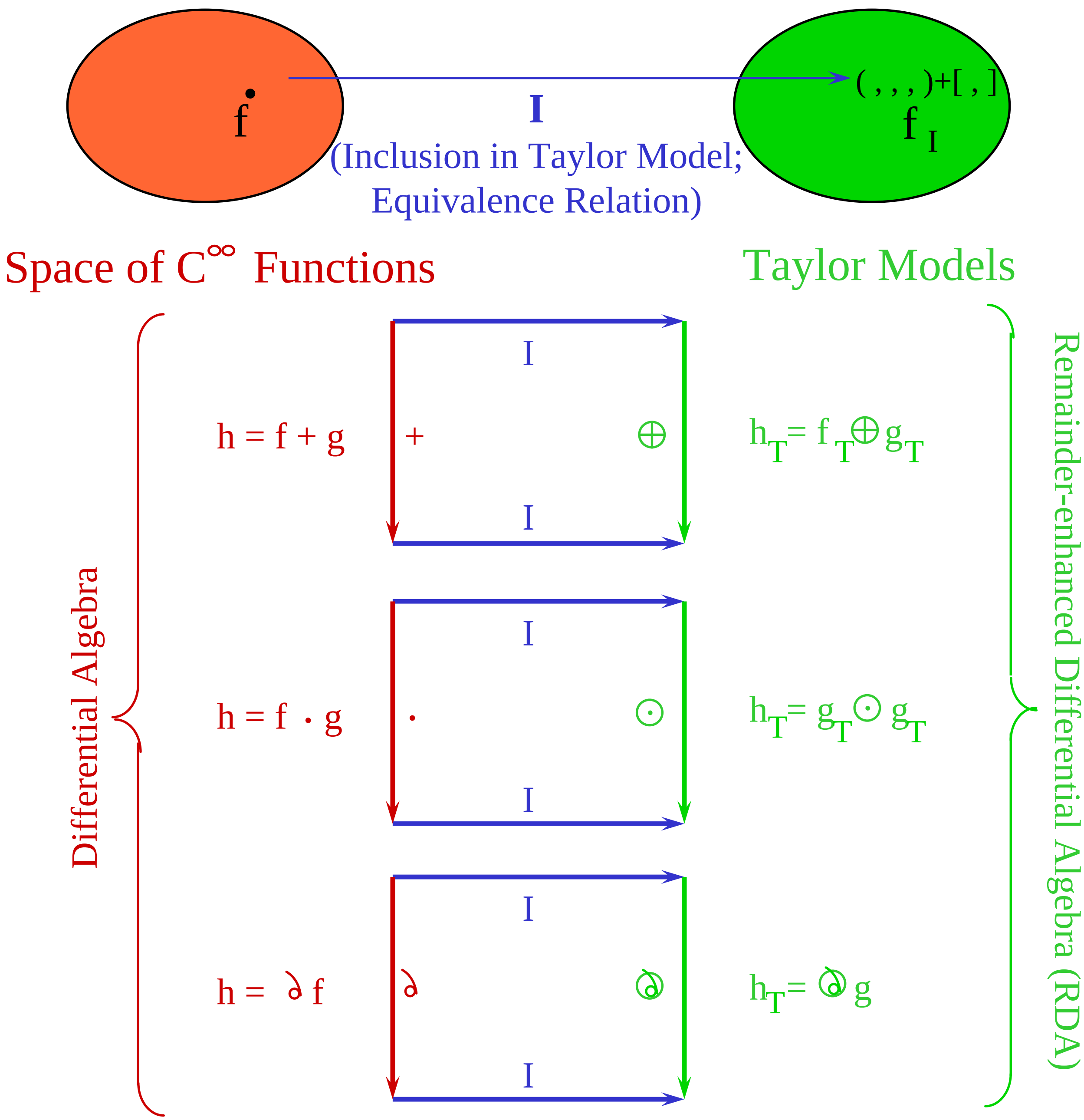
Outlook

- Consistency checks in inhomogeneous bending magnets
  - How can it be done using conventional simulation codes?
    Example: A nonscaling FFAG design for Project X
    $\Longrightarrow$ extremely difficult if not impossible....
  - How can it be done using conventional codes for field maps?
    Example: A nonscaling 6 cell FFAG design
    $\Longrightarrow$ extremely difficult if not impossible....

Outlook

- Consistency checks in inhomogeneous bending magnets
  $\Longrightarrow$ extremely difficult if not impossible....
- Applying automated domain decomposition schemes
  - A selection of a set of reference particle energies can be systematically automated depending on the strength of nonlinearities in the fields. Utilizing the method of Taylor models (Remainder enhanced Differential Algebras) in COSY

Function Algebra Inclusions



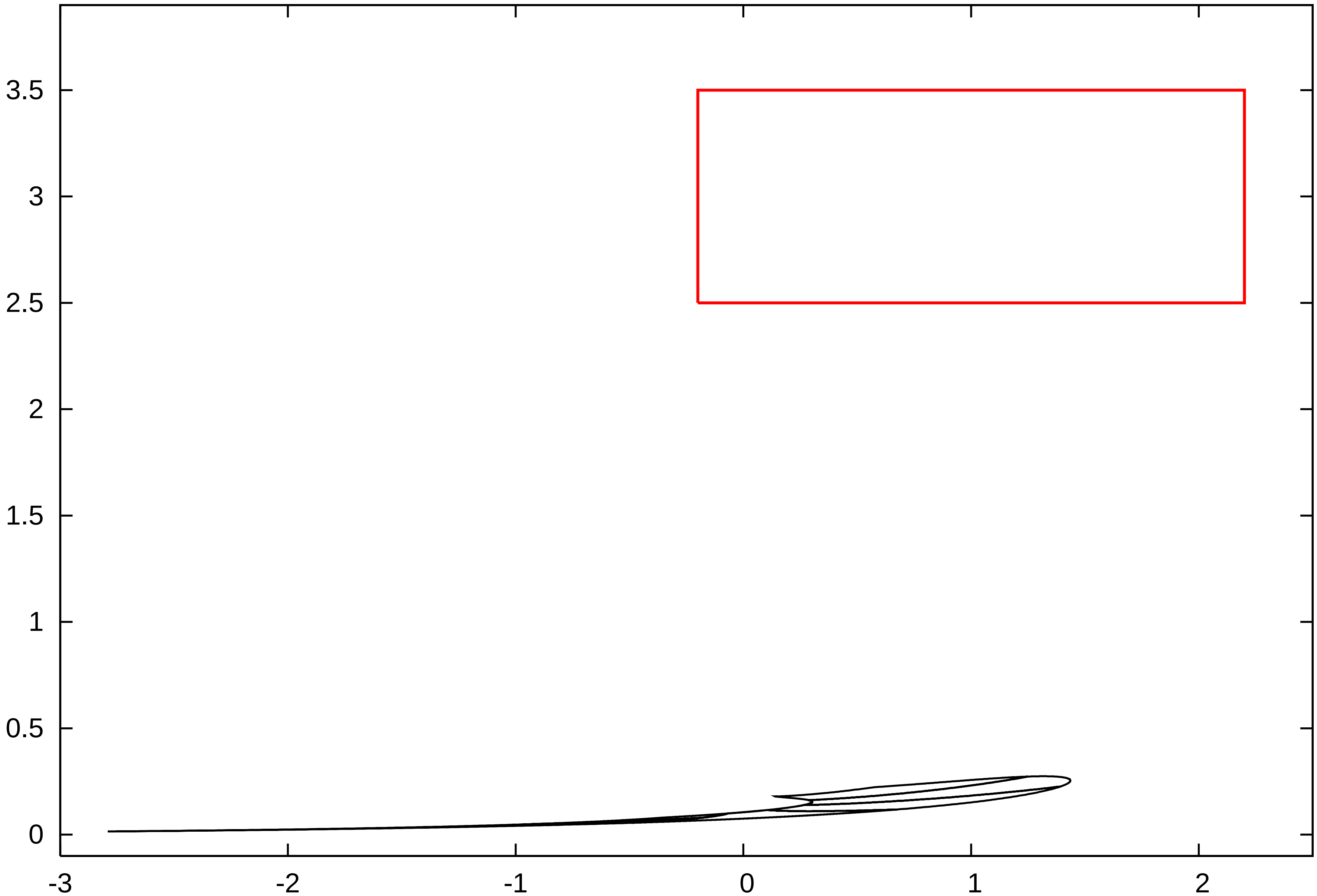
Differential Algebra
(also want “exp”, “sin” etc: Banach DA)

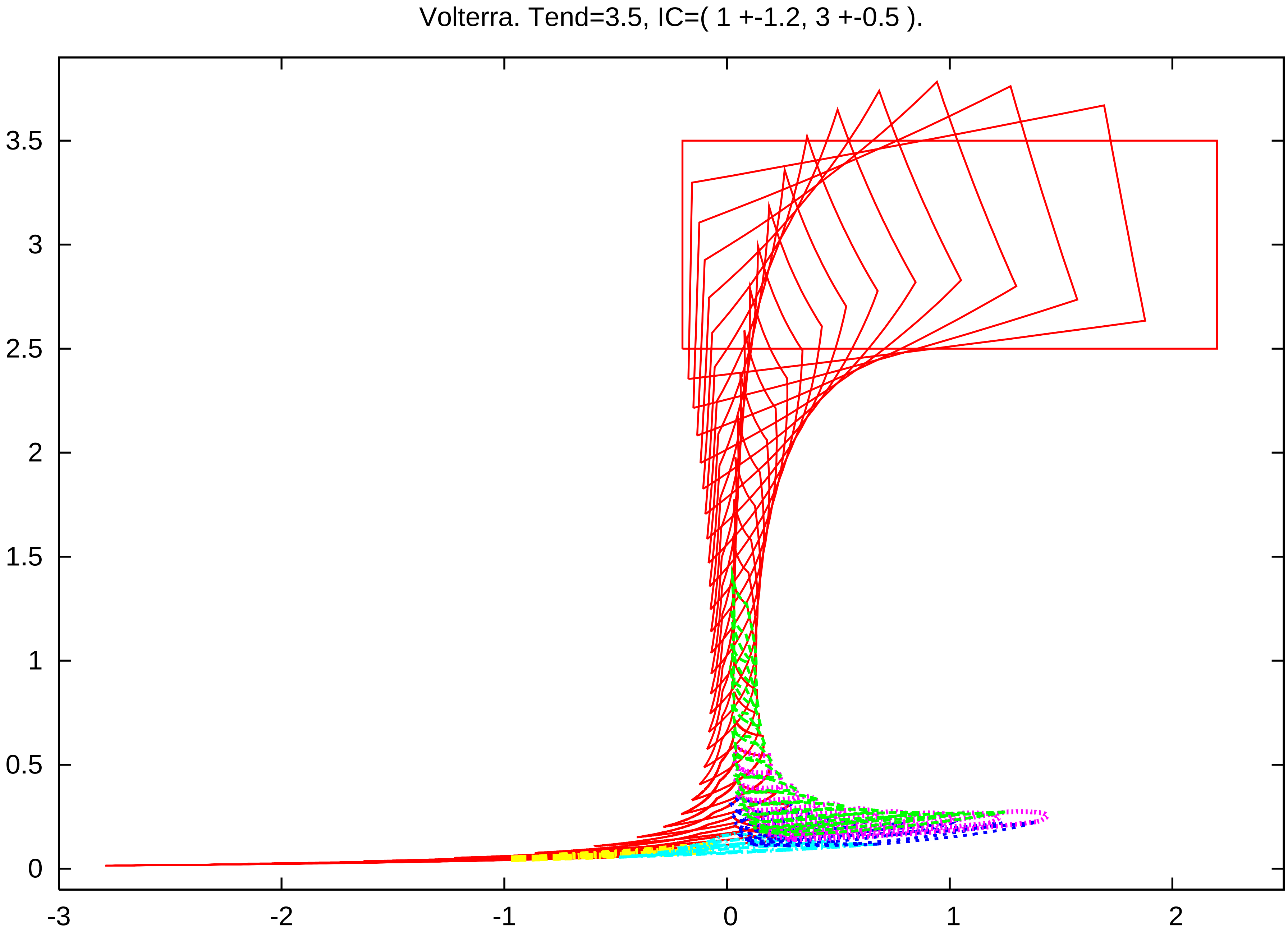
Diagrams commute exactly

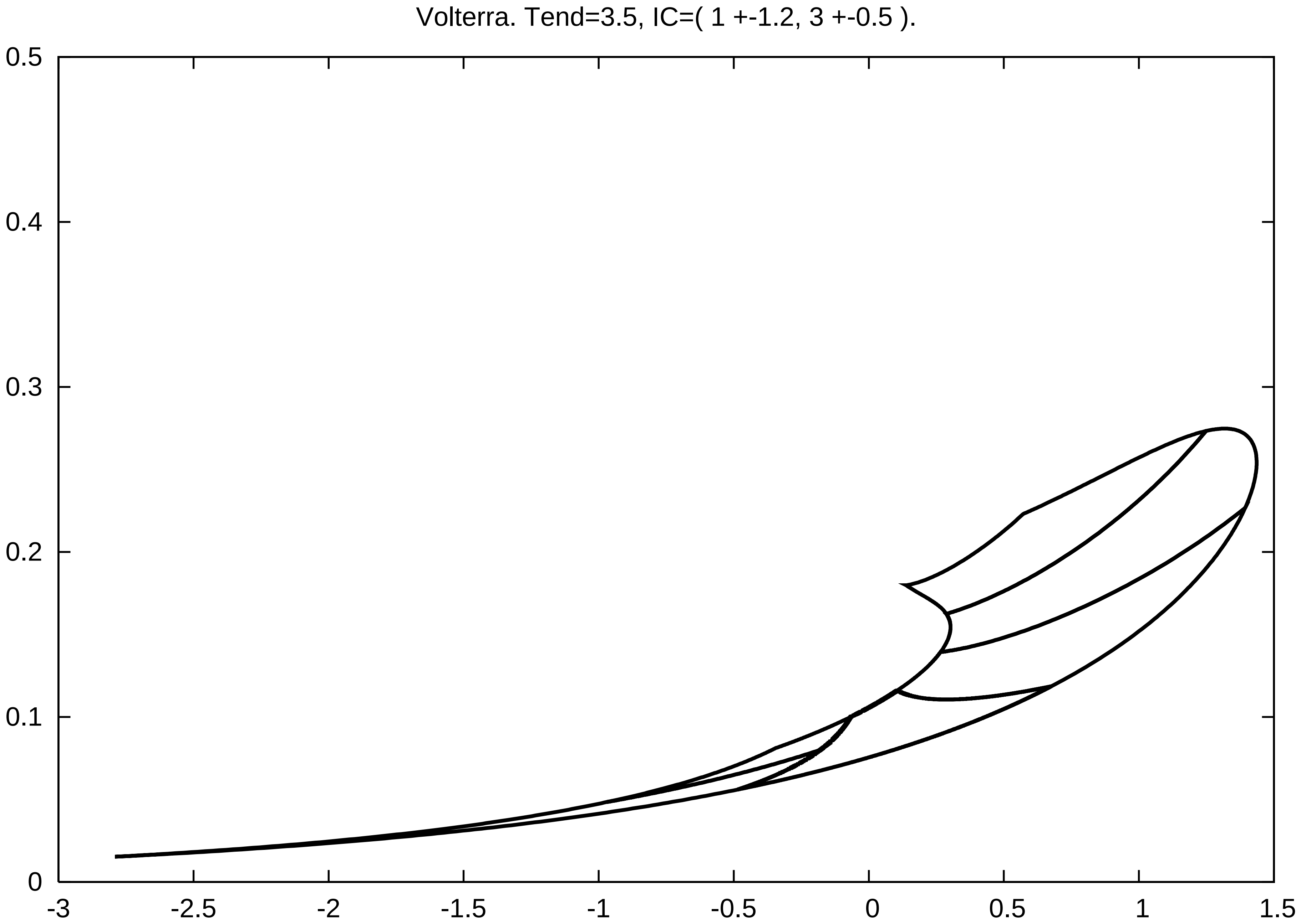
Differential Algebra with Remainder

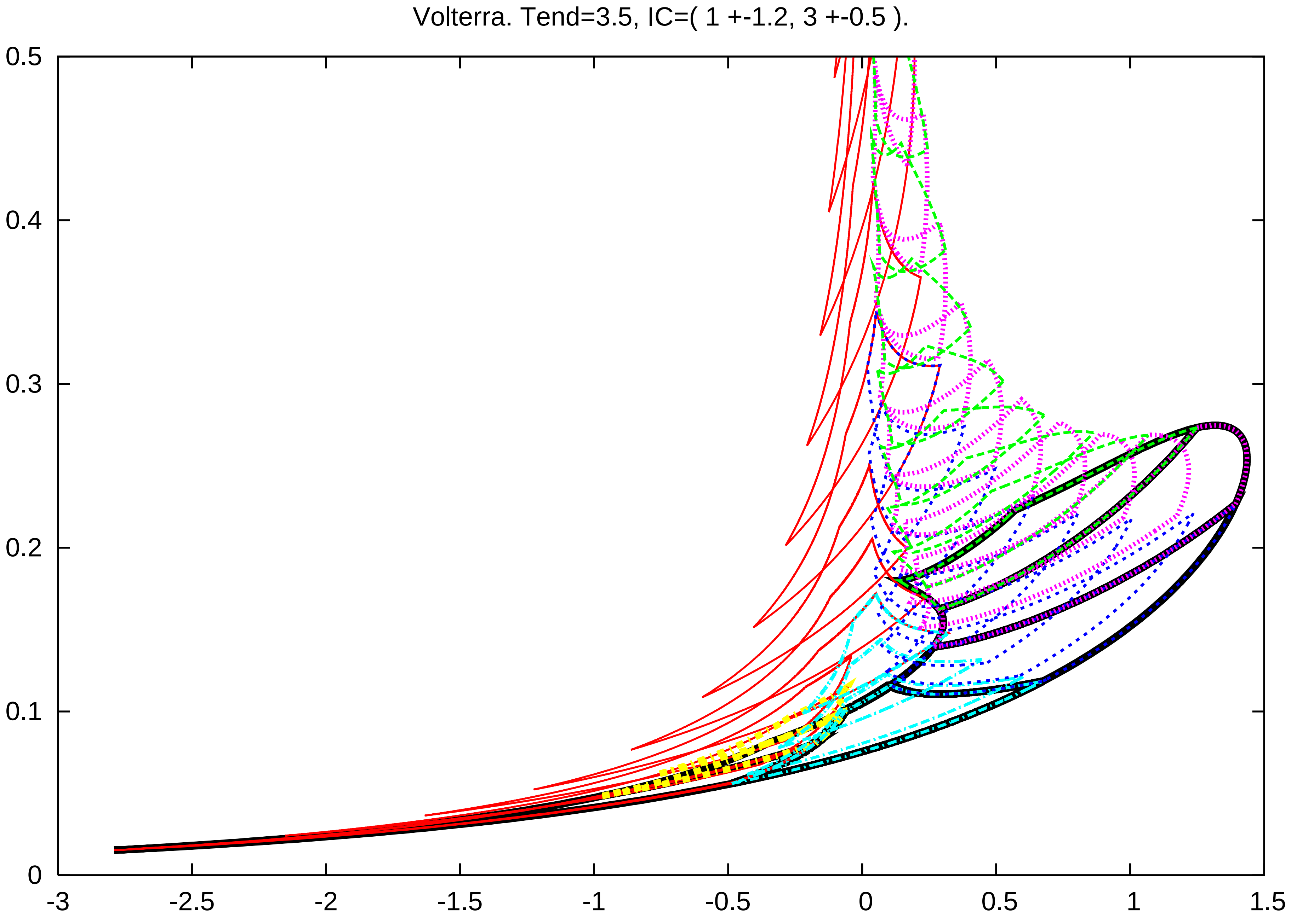
I: Extracts Information considered relevant

Volterra. Tend=3.5, IC=(1 +-1.2, 3 +-0.5).




Volterra. Tend=3.5, IC=(1 +-1.2, 3 +-0.5).
3.5
3
2.5
2
1.5
1
0.5
0
-3
-2
-1
0
1
2


Volterra. Tend=3.5, IC=(1 +-1.2, 3 +-0.5).
0.5
0.4
0.3
0.2
0.1
0
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5


Volterra. Tend=3.5, IC=(1 +-1.2, 3 +-0.5).
0.5
0.4
0.3
0.2
0.1
0
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5

Outlook

- Consistency checks in inhomogeneous bending magnets
  $\Longrightarrow$ extremely difficult if not impossible....
- Applying automated domain decomposition schemes
  - A selection of a set of reference particle energies can be systematically automated depending on the strength of nonlinearities in the fields. Utilizing the method of Taylor models (Remainder enhanced Differential Algebras) in COSY
- Inclusion of space charge effects
  - FMM (Fast Multipole Method) and MLFMA (Multiple Level Fast Multipole Algorithm) are implemented in Differential Algebras in COSY. By He Zhang and Martin Berz