Universität Rostock

Traditio et Innovatio

FAKULTÄT FÜR INFORMATIK UND ELEKTROTECHNIK UNIVERSITÄT ROSTOCK

Arbitrary High-Order Discontinuous Galerkin Method for Solving Electromagnetic Field Problems in Time Domain

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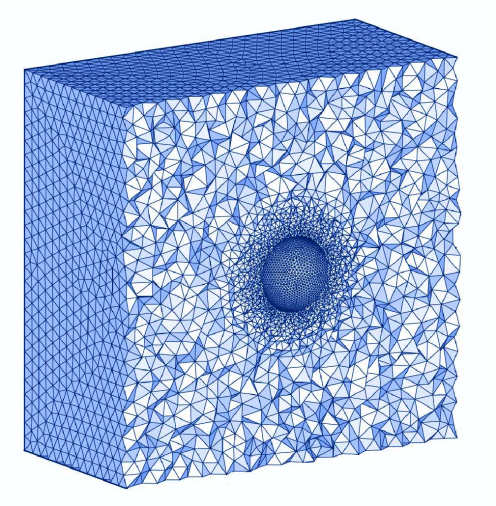
Work supported by Federal Ministry for Research and Education BMBF under contract 05K10HRC

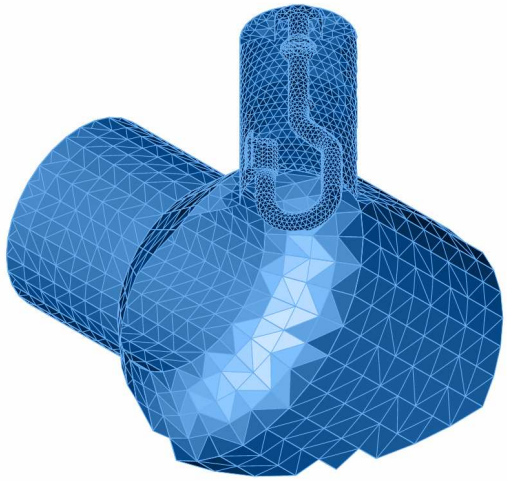
Outline

Introduction

Motivation

- Computation of electromagnetic field distribution as well as scattering parameters can be carried out in time domain
- Fast algorithms $\rightarrow$ implementation on parallel architectures (GPUs)
- Local time stepping (especially for grids with high aspect ratio)

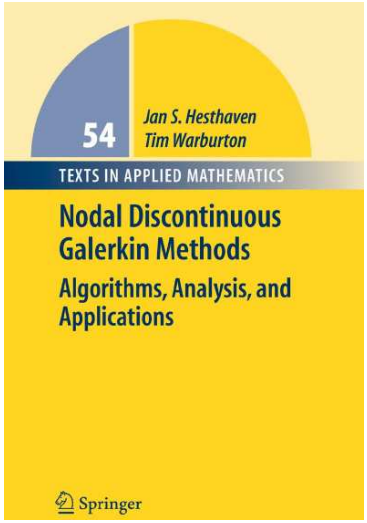




Introduction

Some information

- Implementation of the algorithms within the open source code NUDG++[1]
- Uses Discontinuous Galerkin FEM (DG-FEM) for the spatial discretization of problems
- Well suited for parallel computing architectures (GPUs)
- We are focused on the ADER-DG approach (Arbitrary High-Order Derivative) as a special time integration scheme



[1] Nodal Unstructured Discontinuous Galerkin in C++ (some developers: N. Nunn, N. Gödel, C. R. Bahls, C. Potratz)

References: Hesthaven J. S., Warburton T.: *Nodal Discontinuous Galerkin Methods.* Springer, 2008. - ISBN 978-0-387-72065-4
Dumbser M., Käser M.: *An arbitrary high-order discontinuous Galerkin method for elastic waves on unstructured meshes – II. The three-dimensional isotropic case.*
Taube A., Dumbser M. et. al.: *Arbitrary High-Order Discontinuous Galerkin Schemes for the Magnetohydrodynamic Equations.*

Introduction

Maxwell's Equations

- A more compact form of Faraday's and Ampère's law:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \mathcal{A}_1 \frac{\partial \boldsymbol{u}}{\partial x} + \mathcal{A}_2 \frac{\partial \boldsymbol{u}}{\partial y} + \mathcal{A}_3 \frac{\partial \boldsymbol{u}}{\partial z} = 0, \tag{1}$$

where

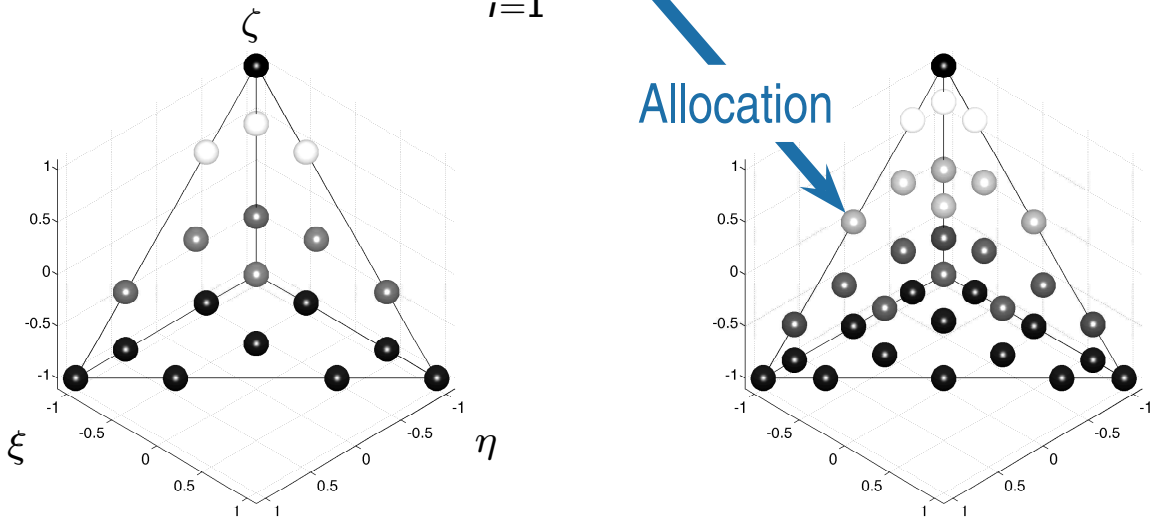
$$\boldsymbol{u}(x, y, z, t) = (H_x, H_y, H_z, E_x, E_y, E_z)^T \tag{2}$$

- To solve this ordinary differential equation, a physical initial condition and boundary conditions are still needed

Spatial Discretization by Discontinuous Galerkin FEM

- Computational domain $\Omega \in \mathbb{R}^3$ partitioned into tetrahedral elements D^k
- Approximate solution $\boldsymbol{u}_h^k$ of (1) inside each tetrahedron D^k is given by:

$$\boldsymbol{u}_h^k = \sum_{i=1}^{N_p} \hat{\boldsymbol{u}}_i^k(t) \cdot \Phi_i^k(x, y, z) \tag{3}$$



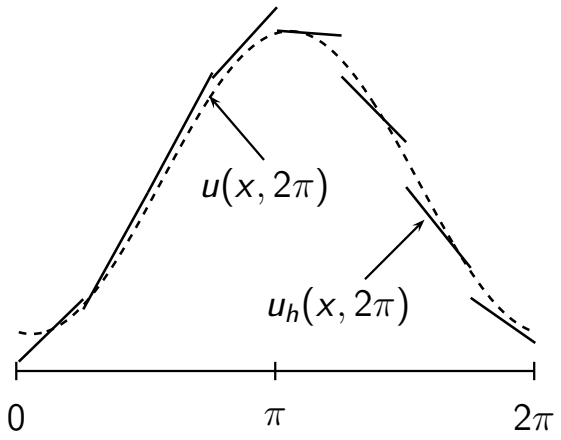
Reference element with the nodes on which the time-dependent degrees of freedom are defined. (left: 3rd order ansatz functions, right: 4th order ansatz functions)

Spatial Discretization by Discontinuous Galerkin FEM

- Multiplication of equation (1) by testfunction $\Phi_j(\boldsymbol{x})$, integration over D^k and finally integration by parts results in:

$$\int_{D^k} \Phi_j \frac{\partial \boldsymbol{u}_h^k}{\partial t} dV + \boxed{\int_{\partial D^k} \Phi_j \boldsymbol{F}_h^k dS} - \int_{D^k} \left(\frac{\partial \Phi_j}{\partial x} \mathcal{A}_1 \boldsymbol{u}_h^k + \dots \right) dV = 0 \qquad (4)$$

- Coupling of adjacent elements by a numerical flux $\boldsymbol{F}_h^k$
- Very suitable for the implementation on massively parallel architectures
- Integral formulation (4) results in a system of linear first order differential equations for every element separately



Example for central flux

Numerical Integration

- Application of explicit time integration schemes beneficial since equation (4) can be given in explicit manner for time derivatives without numerical effort
- Considered problem in compact form:

$$\frac{d\boldsymbol{u}}{dt} = \mathcal{L}(\boldsymbol{u}, t), \qquad \boldsymbol{u}(t_0) = \boldsymbol{u}_0 \tag{5}$$

- Redraft of equation (5) in the integral form:

$$\boldsymbol{u}(t_{n+1}) = \boldsymbol{u}(t_n) + \int_{t_n}^{t_n+\Delta t} \mathcal{L}(\boldsymbol{u}, \tau) d\tau \tag{6}$$

- Key problem lies in the approximation of the integral in equation (6)

Arbitrary High-Order Derivative Time Discretization

Basic idea

- TAYLOR-expansion of the numerical solution in time, which allows for analytical time integration

ADER approach

- Taylor expansion in time

$$\boldsymbol{u}(x,y,z,t) = \sum_{p=0}^{N} \frac{t^p}{p!} \frac{\partial^p}{\partial t^p} \boldsymbol{u}(x,y,z,t)\Big|_{t=t_0} \tag{7}$$

- Replace time derivates by spatial derivates based on problem (1)

$$\frac{\partial^p \boldsymbol{u}}{\partial t^p} = (-1)^p \left(\mathcal{A}_1 \frac{\partial}{\partial x} + \mathcal{A}_2 \frac{\partial}{\partial y} + \mathcal{A}_3 \frac{\partial}{\partial z} \right)^p \boldsymbol{u} \tag{8}$$

Arbitrary high-order derivative time discretization

Impementation - Replace time derivates

- Putting all together with the DG aproximation (3):

$$\boldsymbol{u}_h^k(\xi,\eta,\zeta,t) = \sum_{p=0}^{N} \frac{t^p}{p!}(-1)^p \left(\mathcal{A}_1^* \frac{\partial}{\partial \xi} + \dots\right)^p \hat{\boldsymbol{u}}_i^k(t_0)\Phi_i(\xi,\eta,\zeta) \tag{9}$$

- Projection onto each basis function yields the time evolution of the degrees of freedom $\hat{\boldsymbol{u}}_j^k$ within the time step Δt:

$$\hat{\boldsymbol{u}}_j^k(t)\langle\Phi_n|\Phi_j\rangle = \langle\Phi_n|\sum_{p=0}^{N} \frac{t^p}{p!}(-1)^p \left(\mathcal{A}_1^* \frac{\partial}{\partial \xi} + \dots\right)^p |\Phi_i\rangle \hat{\boldsymbol{u}}_i^k(t_0) \tag{10}$$

Mass matrix $\mathcal{M}_{(n,j)}$

structure like $\langle\Phi_n|\frac{\partial^p}{\partial\xi^\lambda\partial\eta^\mu\partial\zeta^\nu}\Phi_i\rangle$

Arbitrary high-order derivative time discretization

Impementation - Time integration

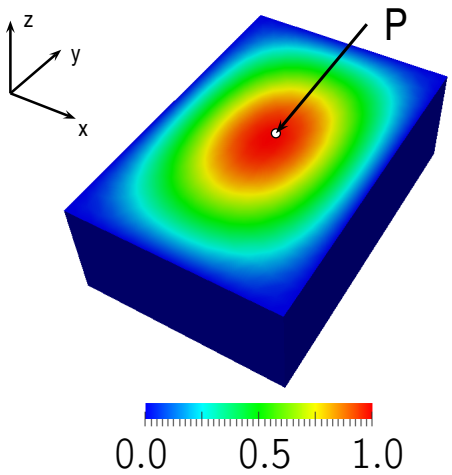
$$\int_{t_0}^{t_0+\Delta t} \hat{\boldsymbol{u}}_j^k(t)dt = \underbrace{\mathcal{M}_{(n,j)}^{-1}\langle\Phi_n|\sum_{p=0}^{N}\frac{\Delta t^{(p+1)}}{(p+1)!}(-1)^p\left(\mathcal{A}_1^*\frac{\partial}{\partial\xi}+\dots\right)^p|\Phi_i\rangle}_{\text{time integration operator}}\hat{\boldsymbol{u}}_i^k(t_0) \quad (11)$$

$$\boldsymbol{u}(t_{n+1}) = \boldsymbol{u}(t_n) + \int_{t_n}^{t_n+\Delta t}\mathcal{L}(\boldsymbol{u},\tau)d\tau \quad (12)$$

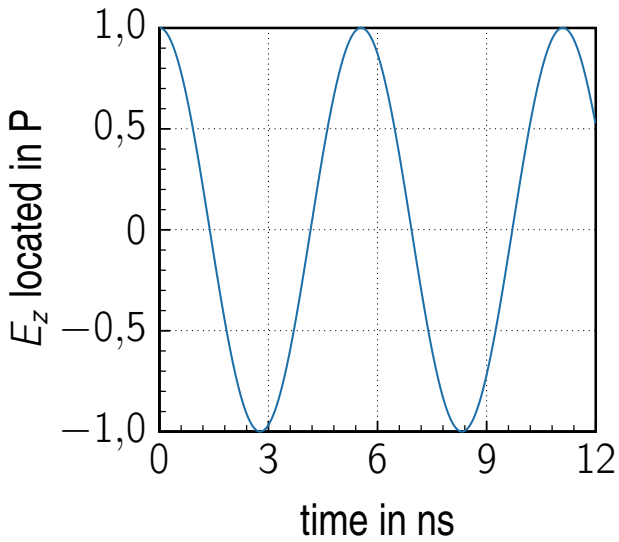
Example: Lossless Rectangular Cavity

Simulation model

- Helmholtz equation for this geometry can be solved analytically
- Size: 1.0 m x 1.5 m x 0.5 m, material: ε_0, μ_0
- Fundamental mode:



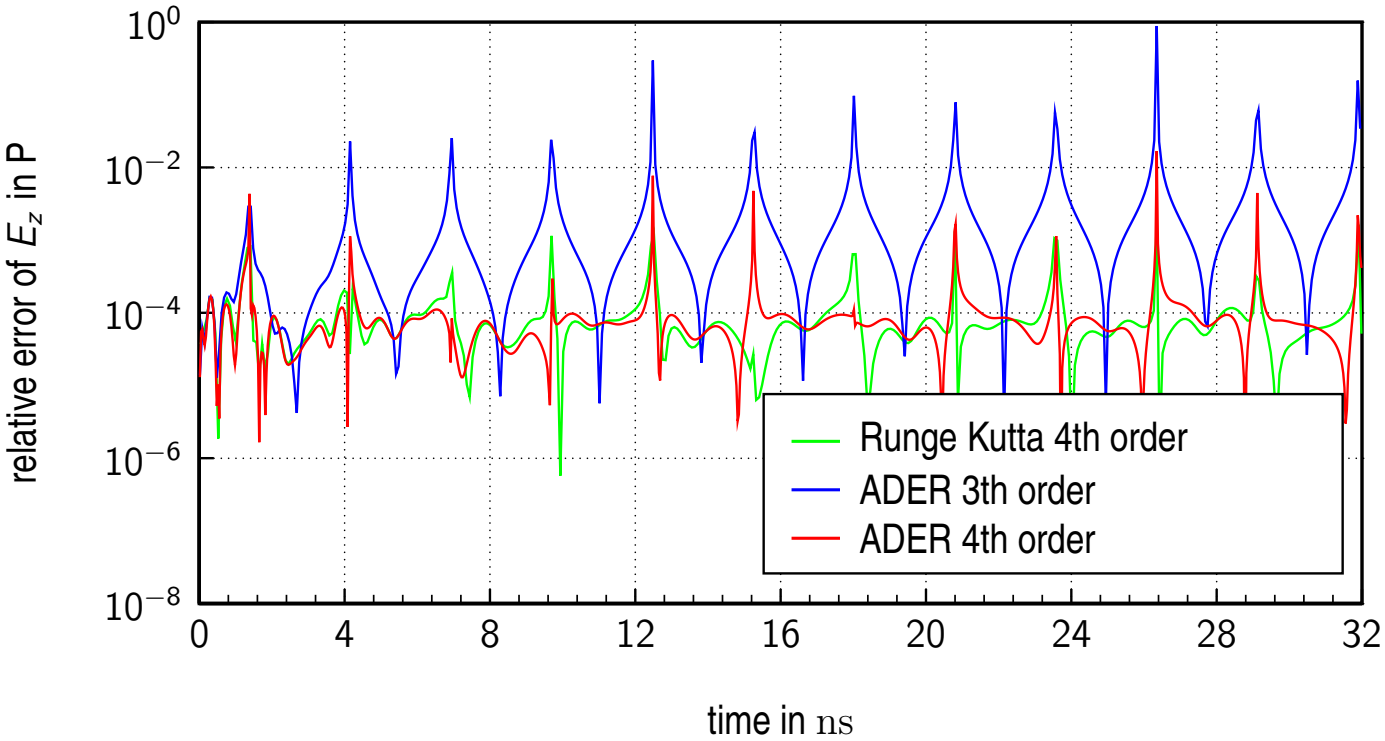
E_z at time $t = 0$



Example: Lossless Rectangular Cavity

Results with 3rd order ansatz functions

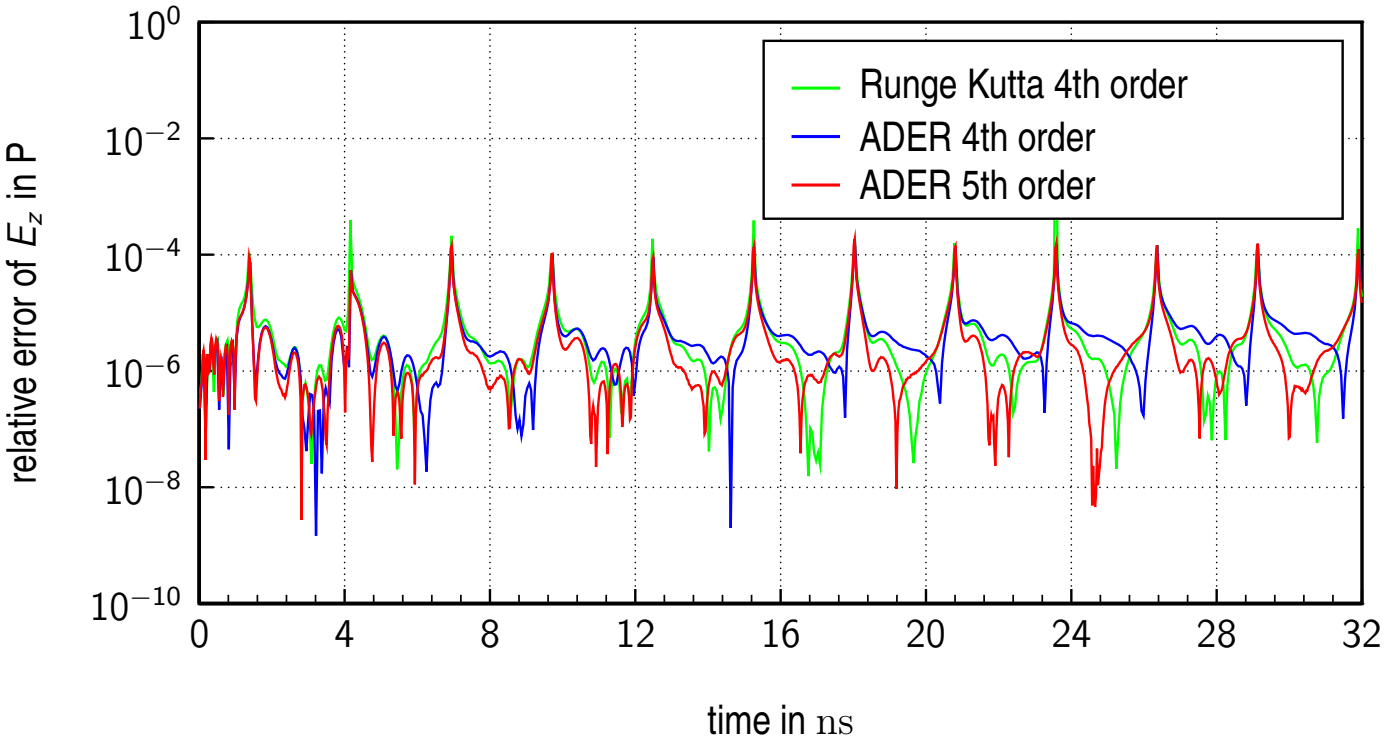
- Comparison between ADER time integration scheme 3rd and 4th order and a 4th order Runge-Kutta scheme



Example: Lossless Rectangular Cavity

Results with 4th order ansatz functions

- Comparison between ADER time integration scheme 3rd and 4th order and a 4th order Runge-Kutta scheme



Summary and Outlook

- By using only one CPU core the 4th order Runge Kutta is $\sim$ 3 times faster than the ADER aproach in the same order
- 3rd order Adams-Bashforth is $\sim$ 1.7 times faster but with a lower convergence order
- For higher order basis functions these ratios reduce $\rightarrow$ especially suitable for high accuracy
- Implementation on parallel architectures probably improves also these ratios related to the ADER aproach
- Time integration for the ADER scheme is relativly expensive in storage and computation but stable for a larger time step
- Next Step: Implementation for GPUs (via Cuda or OpenCL)