Modeling of Ultra-cold and Crystalline Ion Beams

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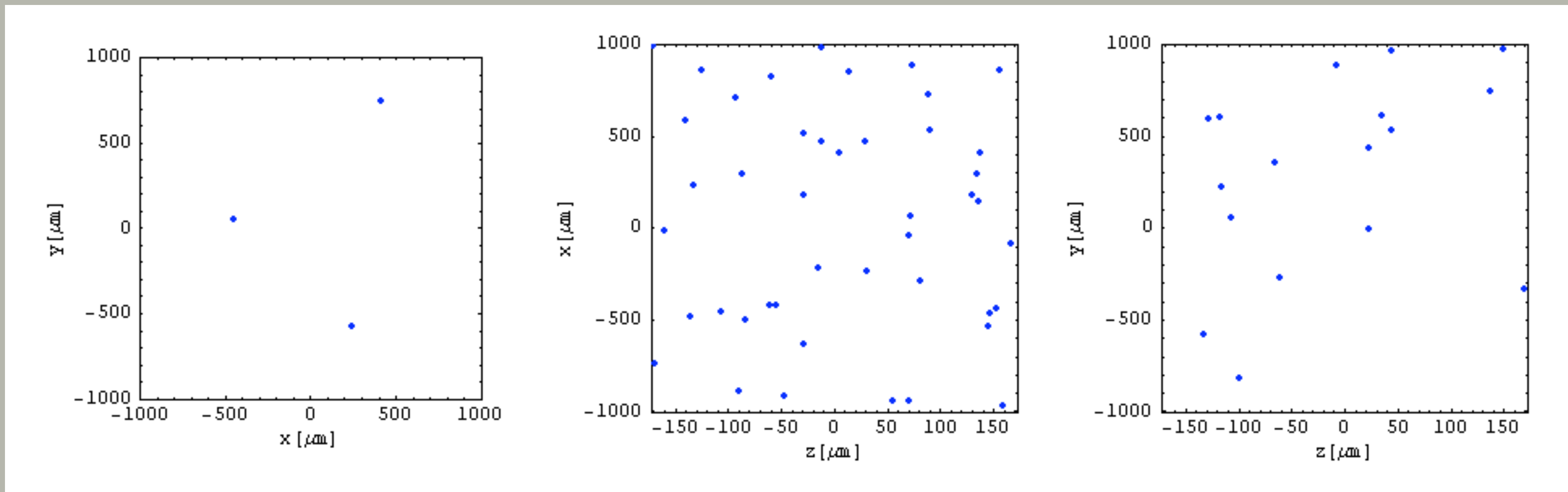
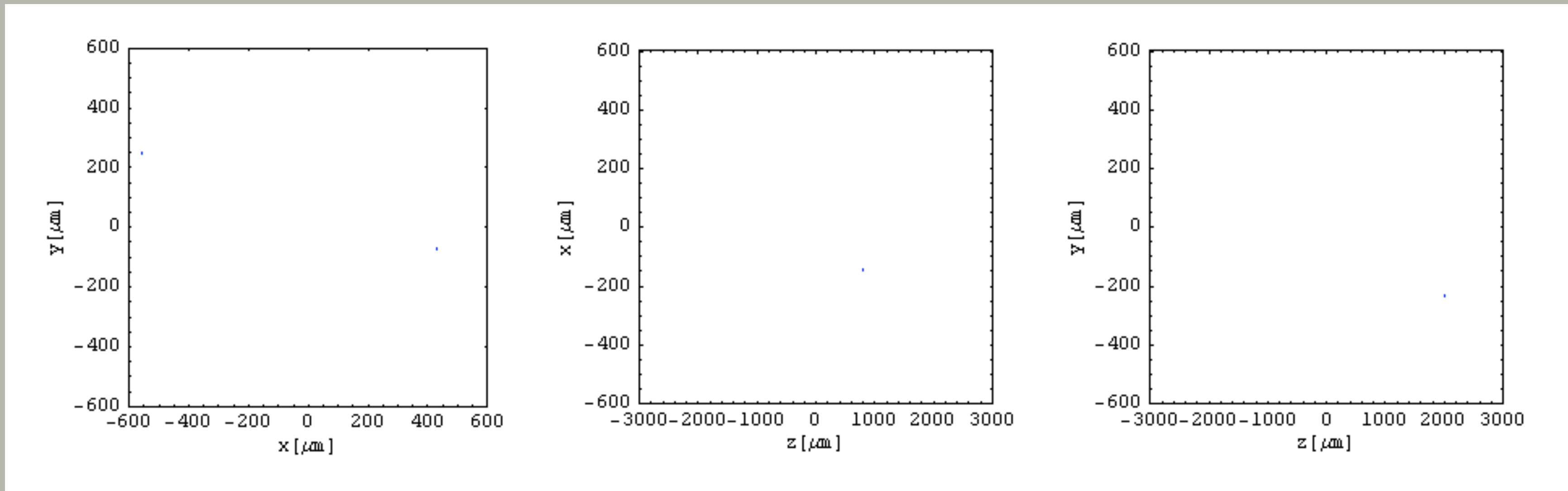
Regular Beam



If strongly cooled …

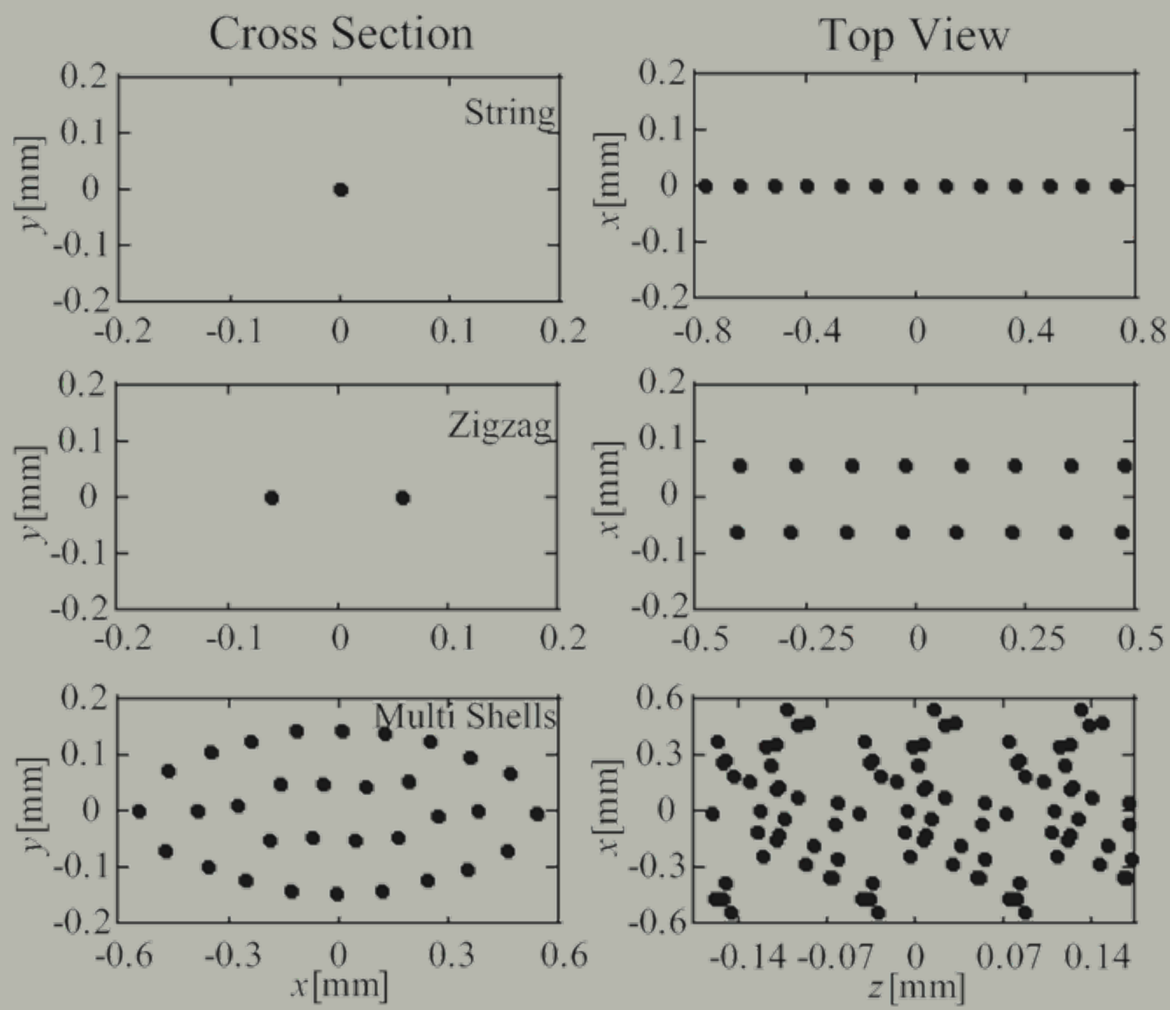


Crystalline Beams

1D crystal
(String)

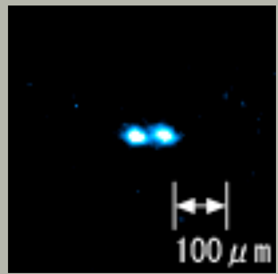
2D crystal
(Zigzag)

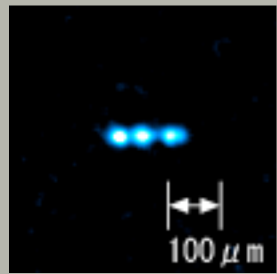
3D crystal
(Shell)

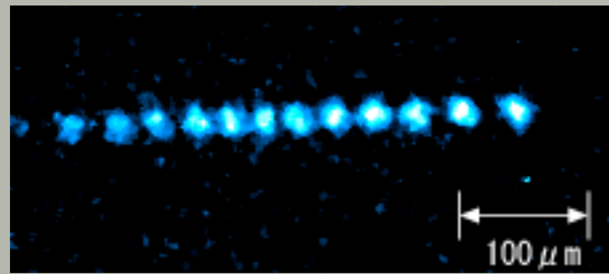


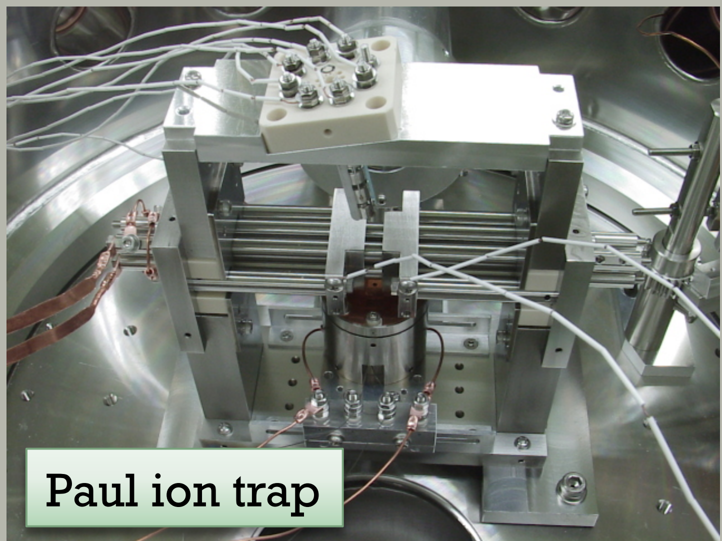
Coulomb Crystals in an Ion Trap

String crystals ($^{40}Ca^+$)

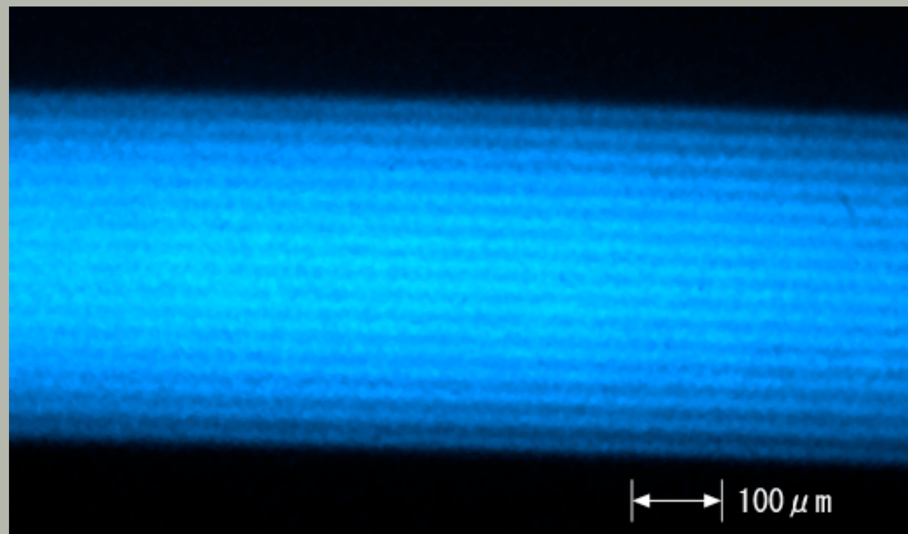








Multi-shell crystal



Beam-frame Hamiltonian

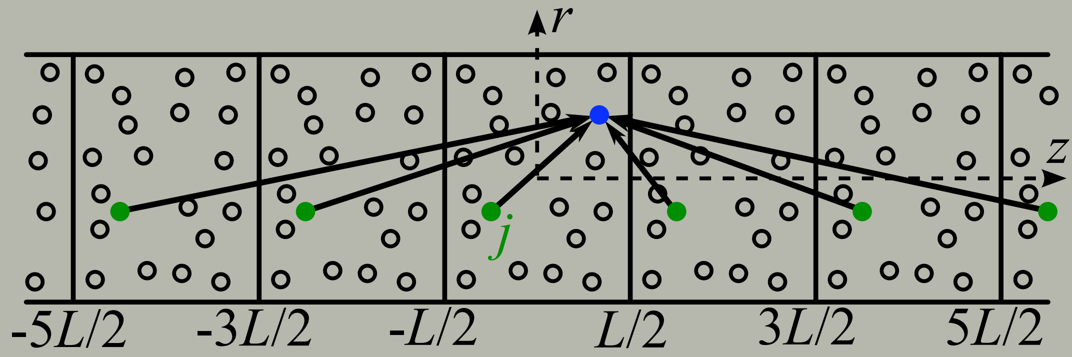
- The general relativistic formalism leads to the linearized Hamiltonian

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2} - \frac{\gamma}{\rho} x p_z + \frac{x^2}{2\rho^2} - \frac{K(s)}{2}\left(x^2 - y^2\right) + \frac{r_p}{\beta^2\gamma^2}\phi.$$

Dipole　　Quadrupole　　Coulomb potential

- In our molecular dynamics code (named "CRYSTAL"), it is possible to include the following lattice elements as well:
  - Solenoid
  - Nonlinear magnets
  - Dispersion-free bend
  - RF cavities (regular cavities & coupling cavities)
  - Wien filter
- The integration is performed in a symplectic manner.

Coulomb Potential Evaluation



- Periodic Boundary Condition: slice the beam in the longitudinal direction and assume that all *supercells* have an identical particle distribution in every integration step.
- Then, the scalar potential is evaluated from

$$\phi = \sum_j (\phi_{short}^{(j)} + \phi_{long}^{(j)}).$$

Ewald integral

$$\phi_{short}^{(j)} = \frac{1}{\sqrt{(x-x_j)^2 + (y-y_j)^2 + (z-z_j)^2}}$$

$$\phi_{long}^{(j)} = \frac{2}{L}\int_0^{\infty} \frac{\cosh(kz^{(j)}/L) J_0(kr^{(j)}/L) - 1}{e^k - 1} dk$$

where $z^{(j)} = |z - z_j|$ and $r^{(j)} = \sqrt{(x-x_j)^2 + (y-y_j)^2}$.

Comparison with Theory

Uniform Focusing Model : $H = \frac{p_x^2 + p_y^2 + p_z^2}{2} + \frac{1}{2}\kappa^2(x^2 + y^2) + \frac{r_p}{\beta^2\gamma^2}\phi$

Compare MD results with the Hasse-Schiffer theory, "Ann. Phys. (N.Y.) **203**, 419 (1990)".

Crystalline Structure	H-S Theory	MD code "CRYSTAL"
string	$0 < \lambda < 0.709$	$0 < \lambda < 0.7$
zigzag	$0.709 < \lambda < 0.964$	$0.7 < \lambda < 1.0$
1-shell	$0.964 < \lambda < 3.10$	$1.0 < \lambda < 3.1$
1-shell + string	$3.10 < \lambda < 5.7$	$3.1 < \lambda < 5.7$
2-shell	$5.7 < \lambda < 9.5$	$5.7 < \lambda < 9.5$
2-shell + string	$9.5 < \lambda < 13$	$9.5 < \lambda < 13$
3-shell		$13 < \lambda < 19$
3-shell + string	$\lambda = 19.9$	$19 < \lambda < 26$
4-shell	$\lambda = 26.6$	$26 < \lambda < 31$

Cooling Models

- Linear friction

$$\Delta p_q \equiv p_q^{out} - p_q^{in} = -f \cdot p_q^{in} \quad (q = x, y, z)$$

→ In the ideal equilibrium ($\Delta p_q = 0$), $p_q = 0$.

- Tapered cooling

$$\Delta p_z = -f \cdot (p_z^{in} - C_{xz} x^{in})$$

C_{xz} : tapering factor
(dependent on the lattice design)

→ In the ideal equilibrium ($\Delta p_z = 0$), $p_z = C_{xz} x$.

- Laser cooling

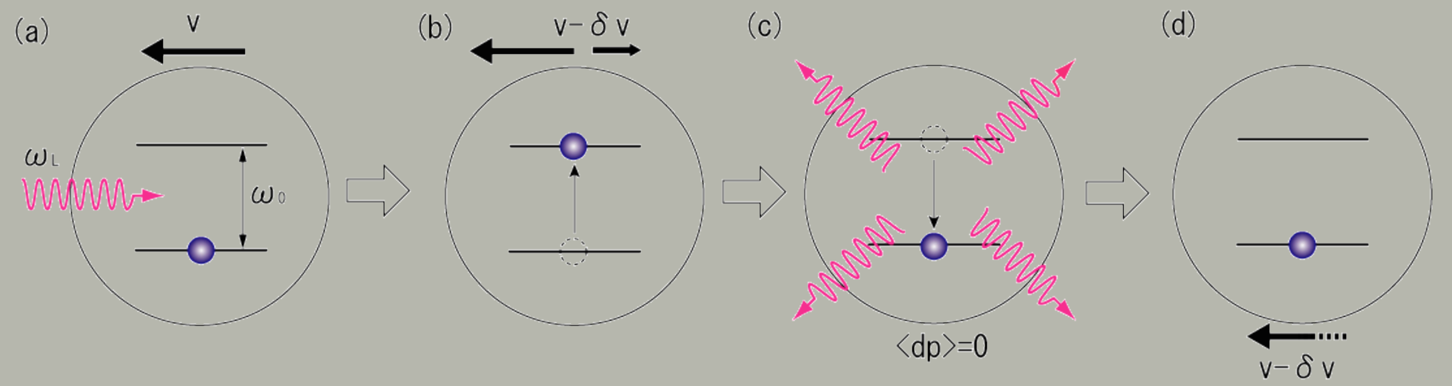
$$F_{\pm} = \pm \frac{1}{2} \hbar k_L \Gamma \frac{S_L}{1 + S_L + (2\delta_{\pm} / \Gamma)^2}$$

Saturation parameter : $S_L = S_0 \exp[-2(x^2 + y^2)/w^2]$

Laser detuning : $\delta_{\pm} \approx \omega_{\pm} \gamma [1 \mp \beta(1 + p_z/\gamma)] - \omega_0$

This frictional force operates along the direction of laser propagation.

Doppler Laser Cooling

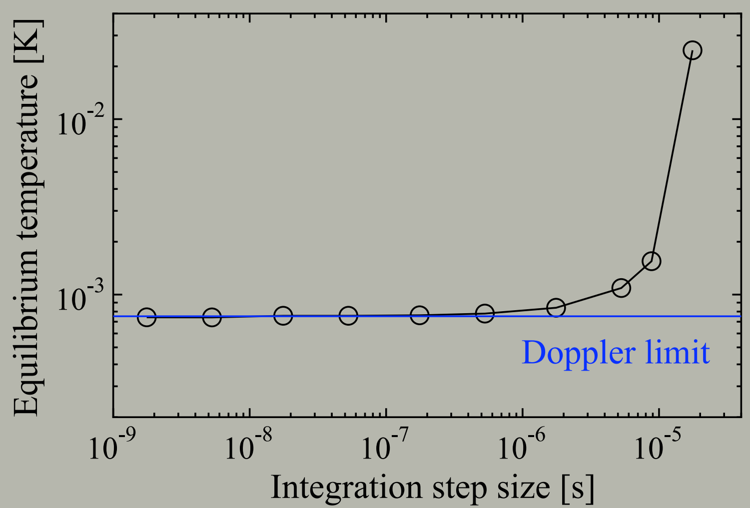


- Doppler limit

The equilibrium temperature reachable with the Doppler laser cooling is limited by the random nature of photon emission and absorption.

$$\frac{k_B T_D}{2} = \frac{\hbar\Gamma}{4} \quad \text{(1D case)}$$

Test Result (Monte-Carlo simulation)



Conditions for Beam Crystallization

Necessary Conditions

- Any cooler storage ring aiming at beam crystallization has to satisfy the following conditions:

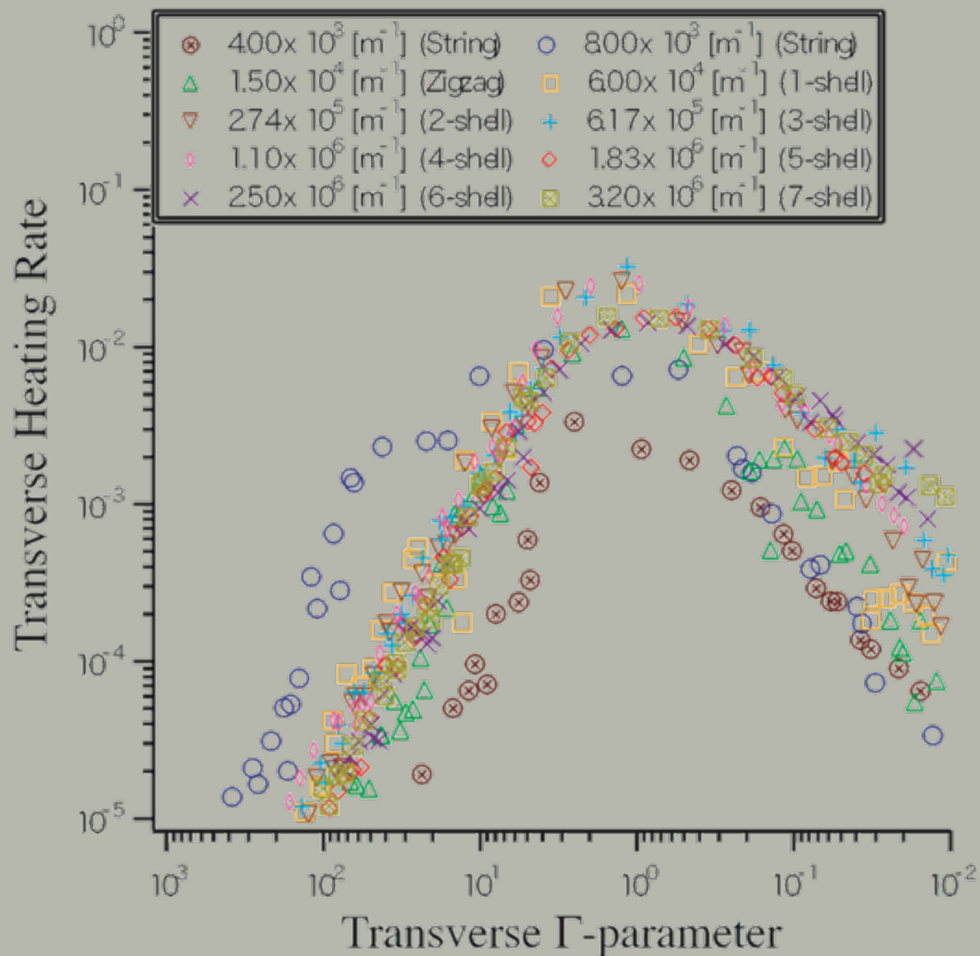
$$\gamma < \gamma_T \quad (\gamma_T : \text{transition energy})$$

$$\sigma_0 < 90\,[\text{deg.}] \quad (\sigma_0 : \text{bare betatron phase advance per lattice period})$$

- We need a sufficiently strong 3D cooling force to overcome heating from IBS.

- The cooling force has to be "tapered" to compensate the dispersive heating mechanism.

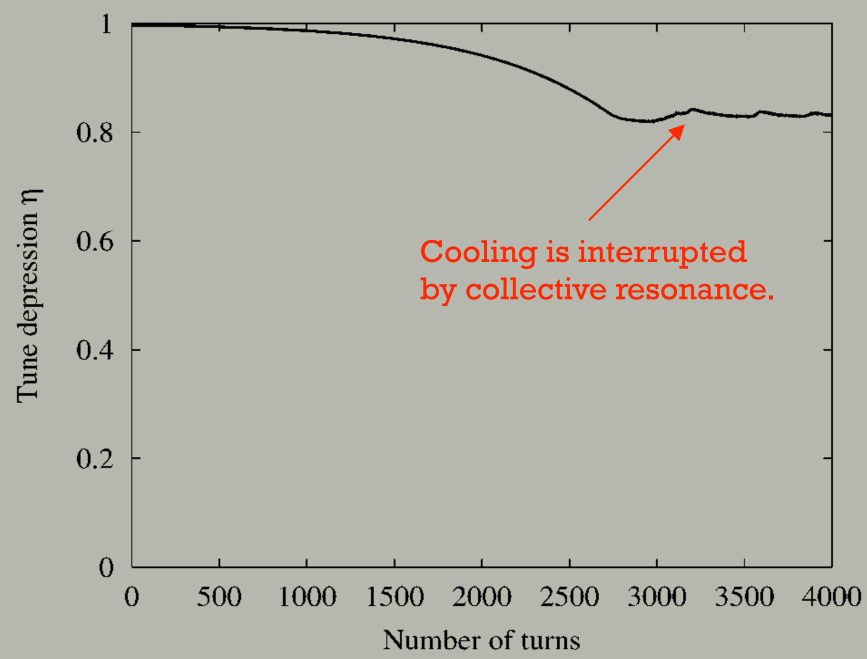
Collisional Heating



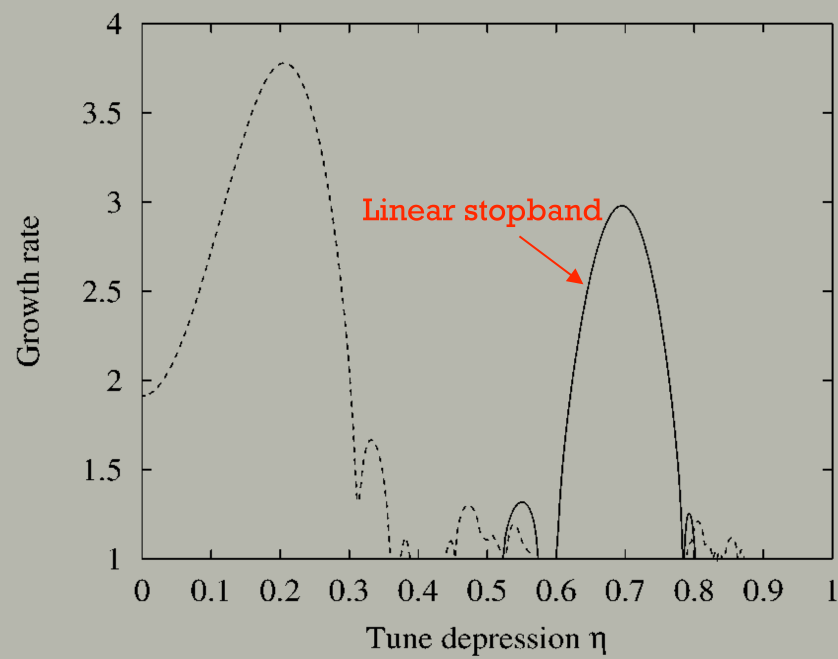
Coherent Resonance

TARN II (σ_0 = 108 deg.)

PIC simulation



Vlasov prediction

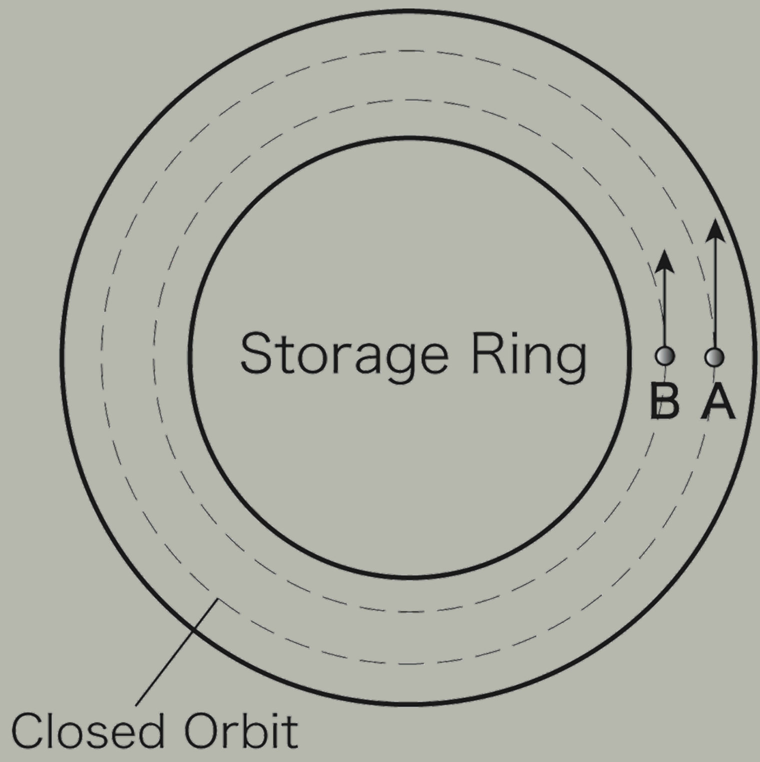


Dispersive Heating

- In a crystalline ground state, all particles have an identical revolution frequency.
- Because of a closed circular orbit, particle "A" must travel slightly faster than particle "B".
- Such a effect never occurs in plasma traps where Coulomb crystallization has been already achieved.

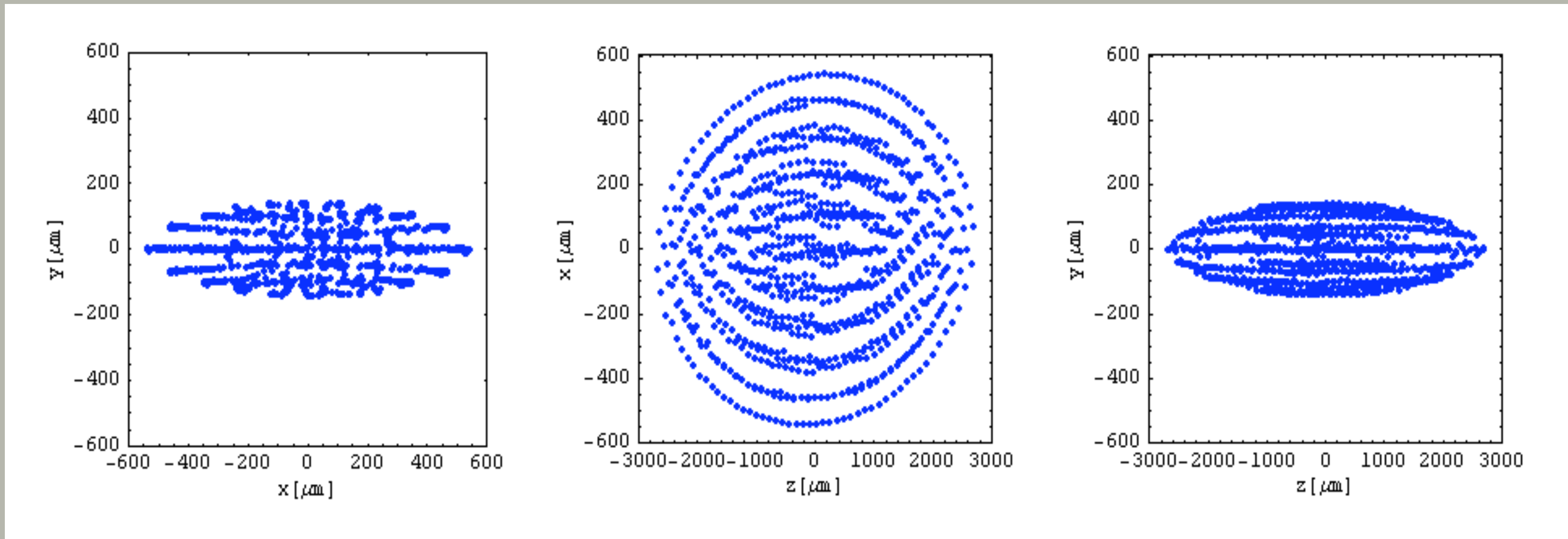
This term causes a serious trouble !

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2} - \frac{\gamma}{\rho} x p_z + \frac{x^2}{2\rho^2} - \frac{K(s)}{2}\left(x^2 - y^2\right) + \frac{r_p}{\beta^2\gamma^2}\phi.$$

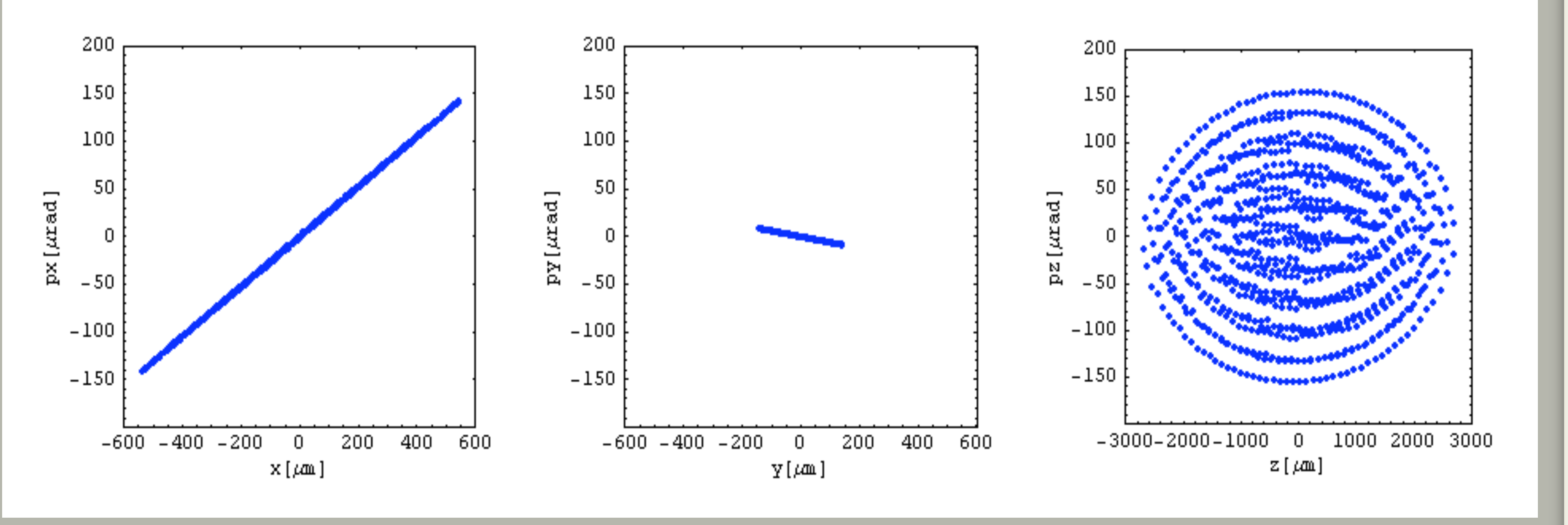


Tapered cooling is necessary !

Dispersive Motion



Phase Space

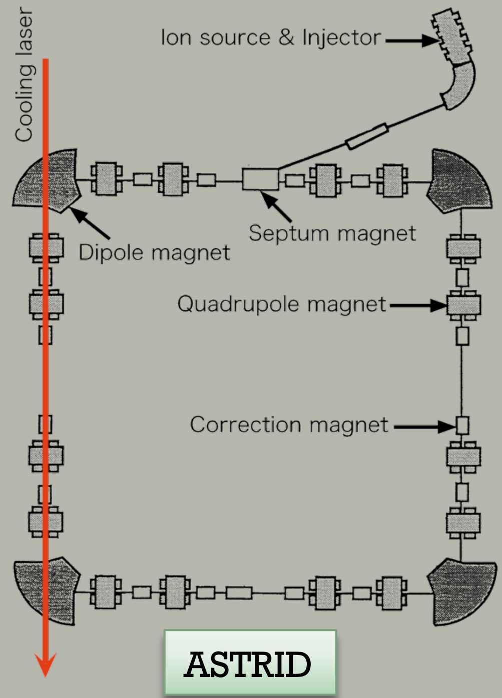


Toward Ultra-cold States of Beams

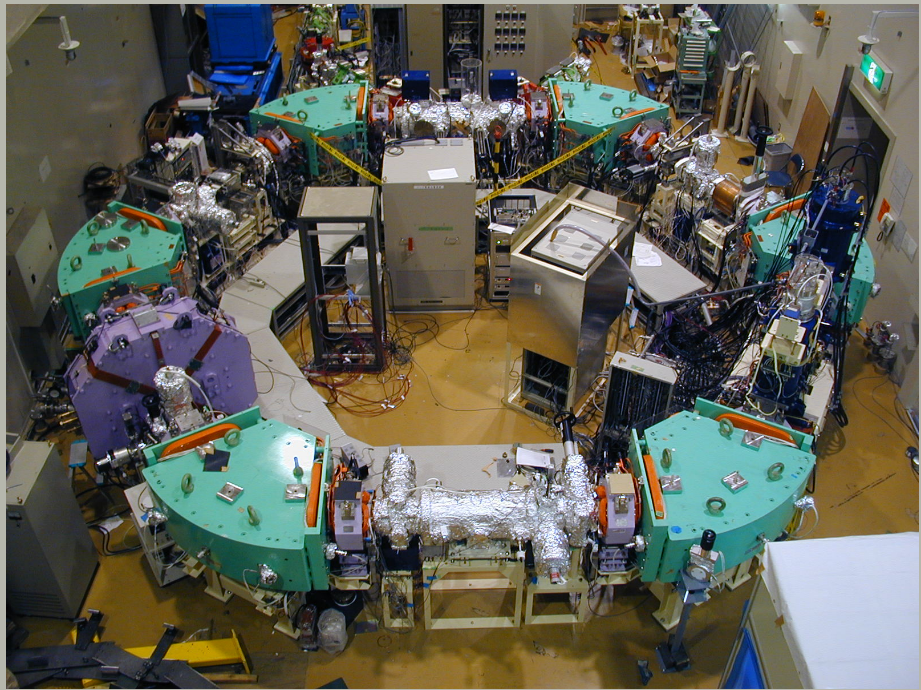
Past attempts

- In the 1990's, continuous efforts had been made at the Max Planck Institute, Heidelberg (TSR group) and Aarhus University (ASTRID group) to achieve beam crystallization.
- Although both groups succeeded in cooling heavy-ion beams with axial lasers, no crystalline states were reached.

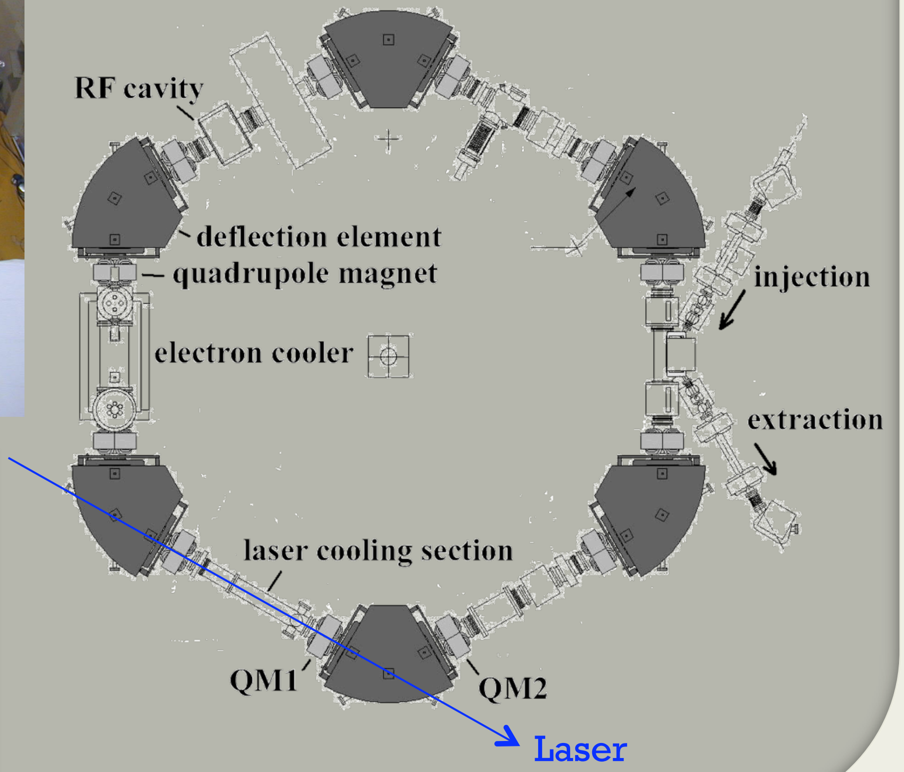
Lattice requirements			
Betatron phase advance	←	Too large	✖
Beam energy	←	OK	O
Cooling requirements			
3D cooling efficiency	←	Too low	✖
Tapering	←	NO	✖



S-LSR



Small Laser-equipped Storage Ring



Circumference	22.557 m
Superperiodicity	6
Ion Species	$^{24}Mg^{+}$, p
Kinetic Energy	~ 40 keV (Mg), 7 MeV (p)
Transition Gamma	1.67
Bending Curvature	1.05 m

Resonant Coupling Method

$$H = \frac{1}{2}(p_x^2 + \kappa_x^2 x^2) + \frac{1}{2}(p_x^2 + \kappa_y^2 y^2) + \frac{1}{2}(p_x^2 + \kappa_z^2 z^2) + \psi_c$$

Betatron oscillation

Synchrotron oscillation (**laser cooled**)

Controllable coupling potential

- When three degrees of freedom are independent of each other ($\psi_c = 0$), nothing takes place in x and y directions even if we strongly cool the z direction.
- Switch on the coupling potential to correlate the harmonic motions in the three directions. Linear coupling potentials should be employed for this purpose:

$$\psi_c = g_1 xy \cdot \delta_p(s - s_1) + g_2 xz \cdot \delta_p(s - s_2)$$

- Move the operating point onto coupling resonance:

$$\nu_x - \nu_y = \text{integer}, \quad \nu_x - \nu_z = \text{integer}$$

Coupling Sources

- Betatron-betatron coupling

  Skew quadrupole magnets; Solenoid magnets, etc.

- Synchro-betatron coupling

  Regular RF cavities placed at dispersive positions;
  Coupling RF cavities; Wien filters, etc.

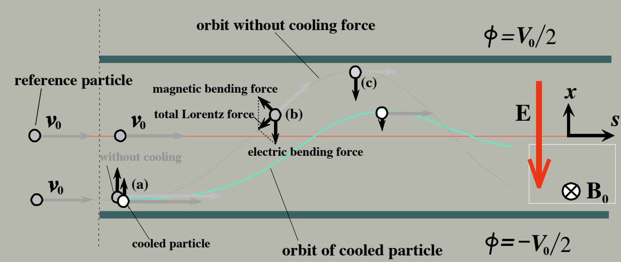
Rectangular cavity operating in a deflective mode (TM_{210}).

$$\mathbf{A} = \left(0, \quad 0, \quad g_c \cdot x \cdot \sin \omega t\right)$$

Direct vertical-longitudinal coupling can readily be generated by rotating this cavity around the axis by 90 deg.

$$H_{\text{Wien}} = \frac{p_x^2 + p_y^2 + p_z^2}{2} + \frac{1}{2}\mu_x^2 x^2 - \mu_x x p_z$$

The longitudinal linear friction can naturally be tapered by a Wien filter if momentum dispersion is finite in the cooling section.



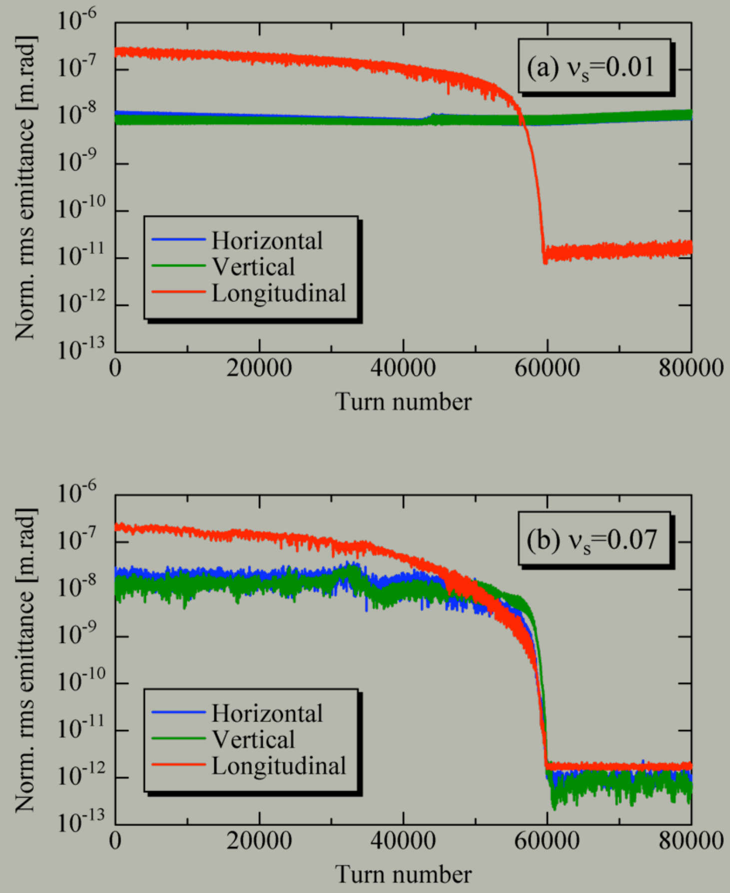
Laser Cooling @ S-LSR

3D laser cooling mode

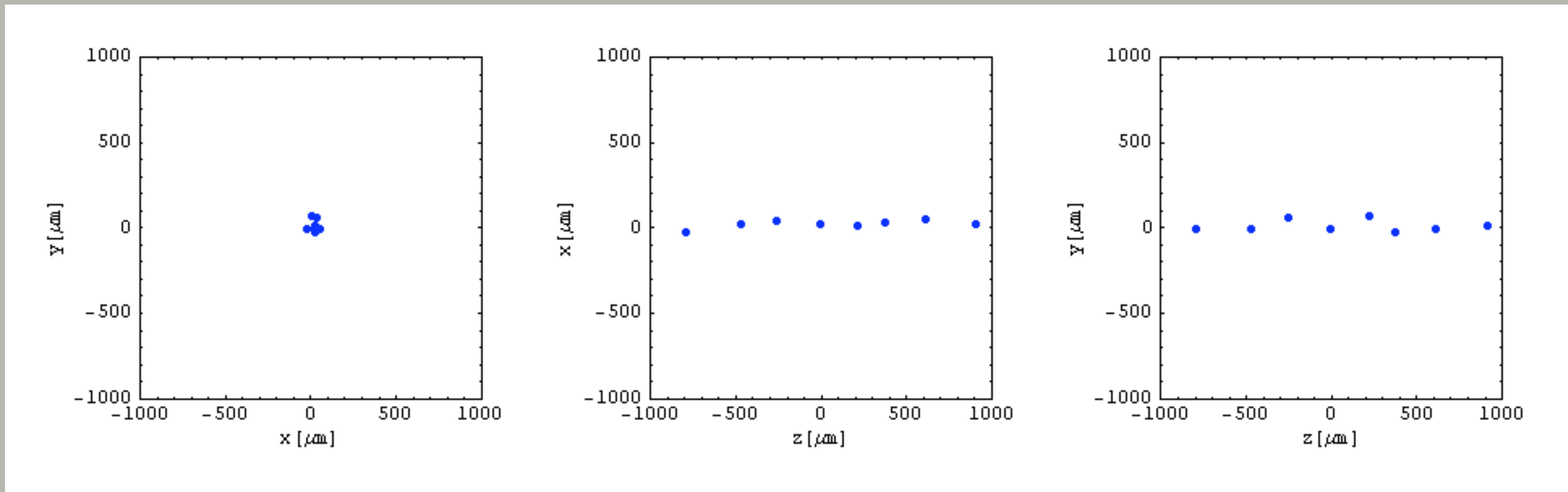
Ion Species	$^{24}Mg^{+}$
Kinetic Energy	35 keV
Bare Betatron Tunes	(2.067, 1.073)
RF Harmonic Number	100
Axial Length of Solenoid	0.8 m
Laser Saturation Parameter	1.0
Minimum Laser Spot Size	5 mm
Laser wavelength	280 nm

- A regular RF cavity sitting in a dispersive position is employed to activate longitudinal-horizontal coupling.
- A weak solenoid field is turned on to achieve horizontal-vertical coupling.
- The fractional parts of the two betatron tunes are set close to each other (in this example, 0.07).
- Switch on the RF cavity and a cooling laser.

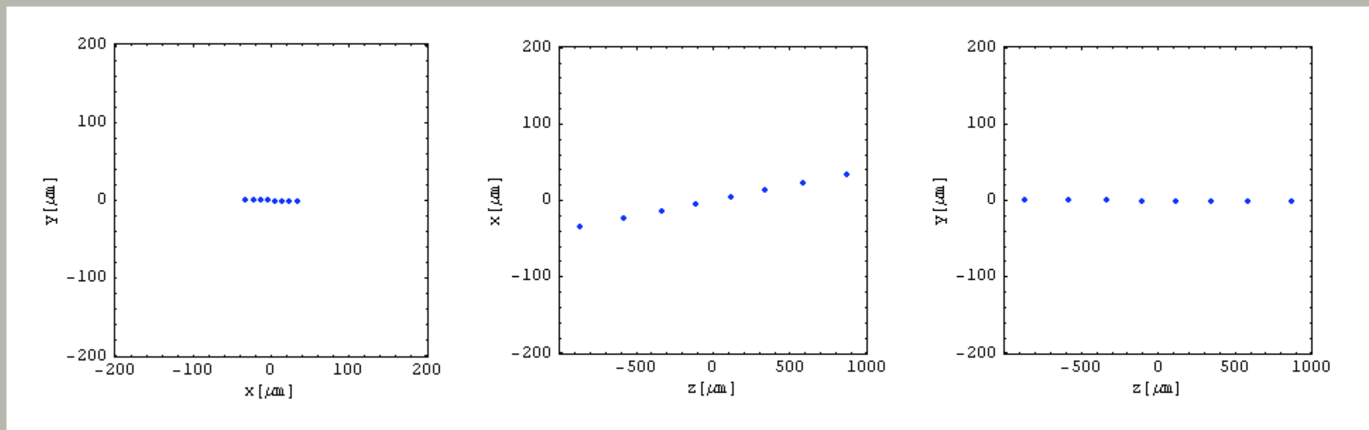
The ideal synchrotron tune for 3D laser cooling in the above operating mode is 0.07.



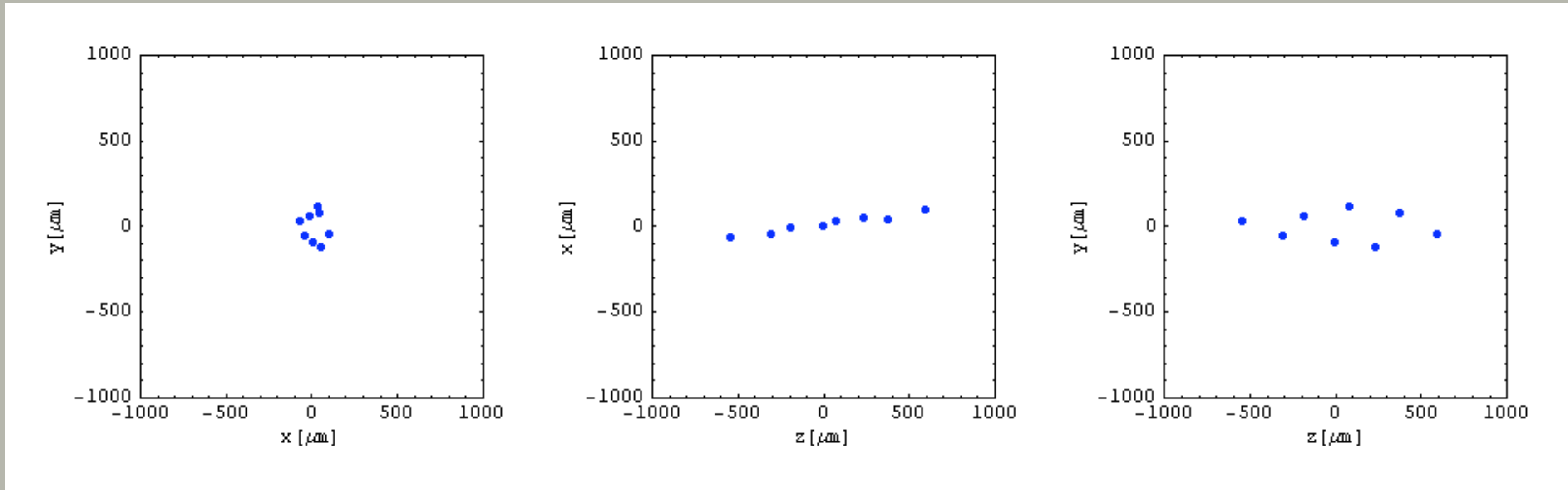
Final State



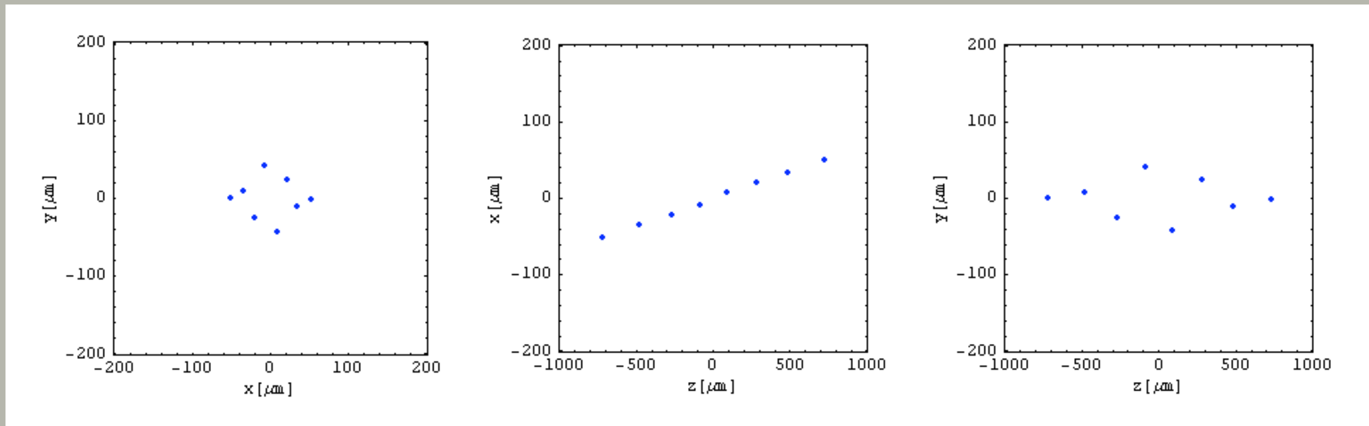
Ideal case



Final State (cont.)



Ideal case



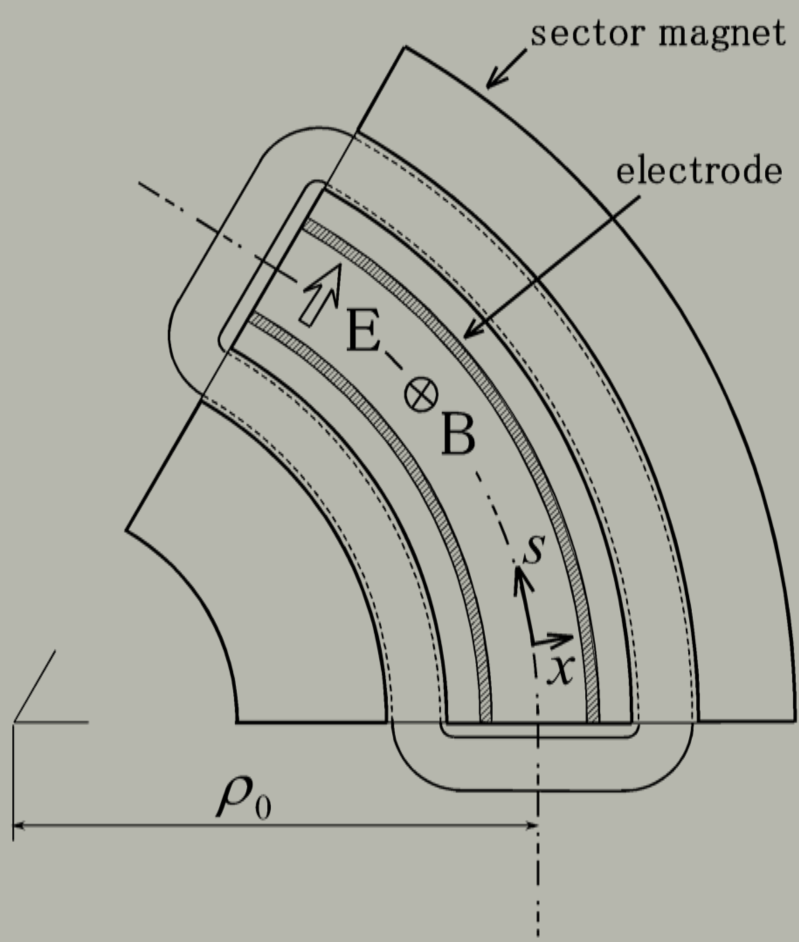
Dispersion-free Bending Element

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2} - \frac{\gamma}{\rho}\left(1 - \frac{eV_0}{\beta^2\gamma^2 E_0}\right)xp_z$$
$$+ \frac{1}{2}\left[1 - (1-n)\frac{eV_0}{\beta^2 E_0} + \frac{1}{\gamma^2}\left(\frac{eV_0}{\beta^2 E_0}\right)^2\right]\frac{x^2}{\rho^2}$$
$$+ \frac{1}{2}(1-n)\frac{eV_0}{\beta^2 E_0}\frac{y^2}{\rho^2} + \frac{r_p}{\beta^2\gamma^2}\phi$$

n : electric-field index
V_0 / ρ : dipole electric-field strength

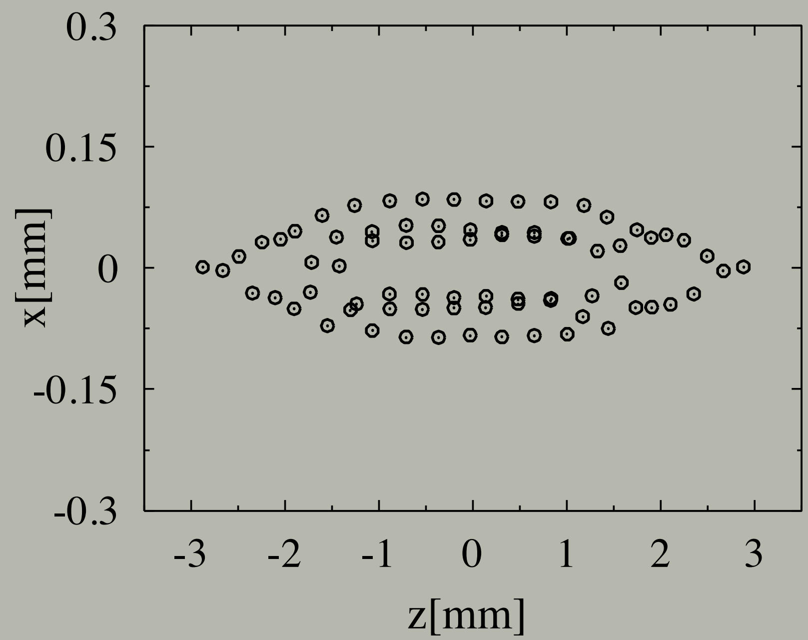
The cross term disappears when

$$\frac{eV_0}{\beta^2\gamma^2 E_0} = 1.$$



MD Example (Dispersion-free Mode)

Lattice	S-LSR
Ion Species	$^{24}Mg^{+}$
Kinetic Energy	35 keV
Bare Betatron Tunes	(2.07, 2.07)
Bare Synchrotron Tune	0.07
RF Harmonic Number	100
Coupling RF Field	200 V/m
Axial Length of Solenoid	0.8 m
Solenoid Field Strength	40 G
Magnetic Dipole Field	0.252 T
Electric Dipole Field	66.7 kV/m
Longitudinal Cooling	Linear Friction (**untapered**)
Transverse Cooling	None



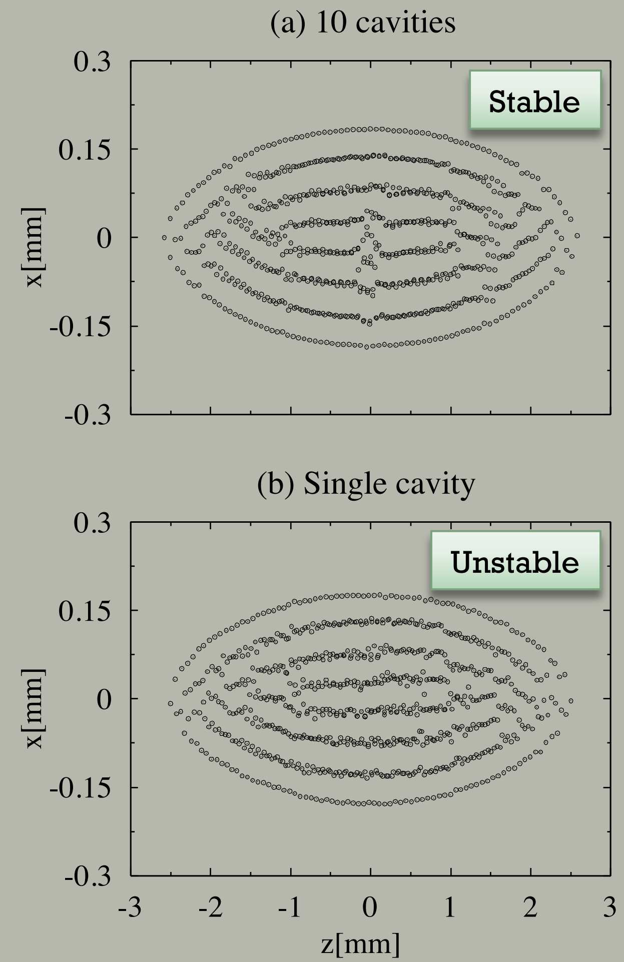
Normalized rms emittance : $\sim 4\times10^{-12}$ m · rad

- A coupling RF cavity operating in a deflective mode is introduced to extend the longitudinal cooling force to the horizontal direction. (Note that regular RF cavities cannot generate the synchro-betatron coupling potential in the dispersion-free mode.)
- As long as an "un-tapered" longitudinal cooling force is employed, it is generally impossible in a *dispersive* storage ring to reach such a 3D ordered state as shown here.

Another Example

Lattice	Test Ring (FDBFD)
Superperiodicity	10
Circumference	18.5 m
Transition Gamma	∞ (dispersion-free mode)
Ion Species	$^{24}Mg^{+}$
Kinetic Energy	35 keV
Bare Betatron Tunes	(1.46, 2.46)
Bare Synchrotron Tune	0.23
Number of Regular Cavities	1 or 10
Longitudinal Cooling	Linear Friction (**untapered**)
Transverse Cooling	Linear Friction

- A test storage ring with 10-fold symmetry has been assumed here.
- All lattice periods contain a single regular RF cavity.
- It is possible to form large multi-shell crystals without the tapered force when the ring is operated in the dispersion-free mode.
- To improve the stability of crystalline structures, all ten cavities must be excited which minimizes the lattice symmetry breakdown.



Normalized rms emittance : $< 1 \times 10^{-13}$ m · rad

Conclusions

- An MD simulation code (CRYSTAL) is developed to perform systematic studies of ultra-cold and crystalline ion beams.

- To reach a crystalline ground state, we need the following:
  1. *High-periodicity ring* (*low betatron tune per period*)
  2. *Compensation for dispersive heating* (*tapered cooling ; dispersion-free bend*)
  3. *Strong 3D cooling* (*longitudinal laser cooling + resonant coupling method*)

- 1D (string) and 2D (zigzag) crystalline beams can probably be generated even if the above three conditions are *weakly* broken. In contrast, the production of a large 3D (shell) crystal requires all conditions to be satisfied rather strictly.

- By cleverly combining state-of-the-art accelerator technologies, we can make an ultra-cold ion beam whose normalized rms emittance should be less than the order of 10^{-10} m or even lower.