Applying an *hp*-Adaptive Discontinuous Galerkin Scheme to Beam Dynamics Simulations

TECHNISCHE UNIVERSITÄT DARMSTADT

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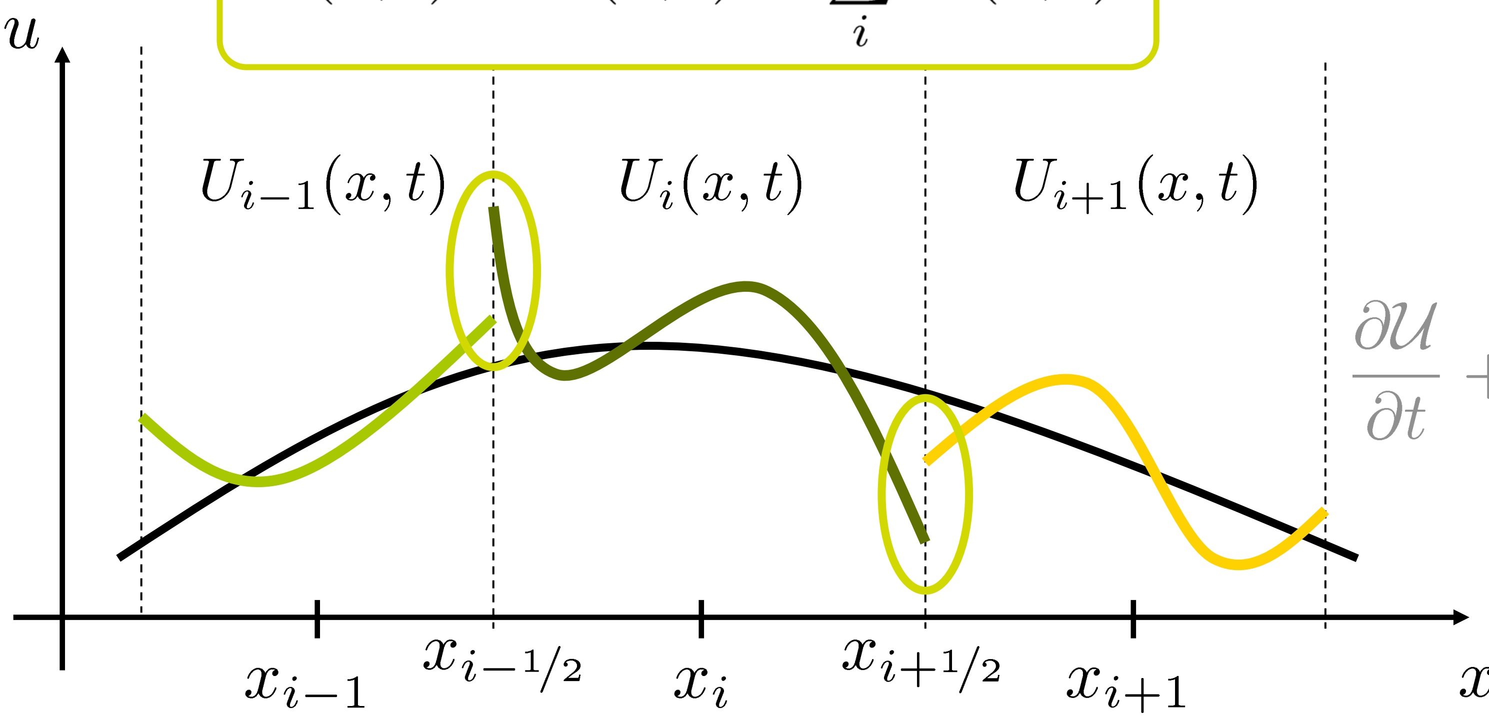
GRADUATE SCHOOL computational engineering

THE DISCONTINUOUS GALERKIN (DG) METHOD

- Is a Finite Element Method (FEM) with no continuity conditions across element boundaries

$$\mathcal{U}(x,t) \approx U(x,t) = \sum_i U_i(x,t)$$



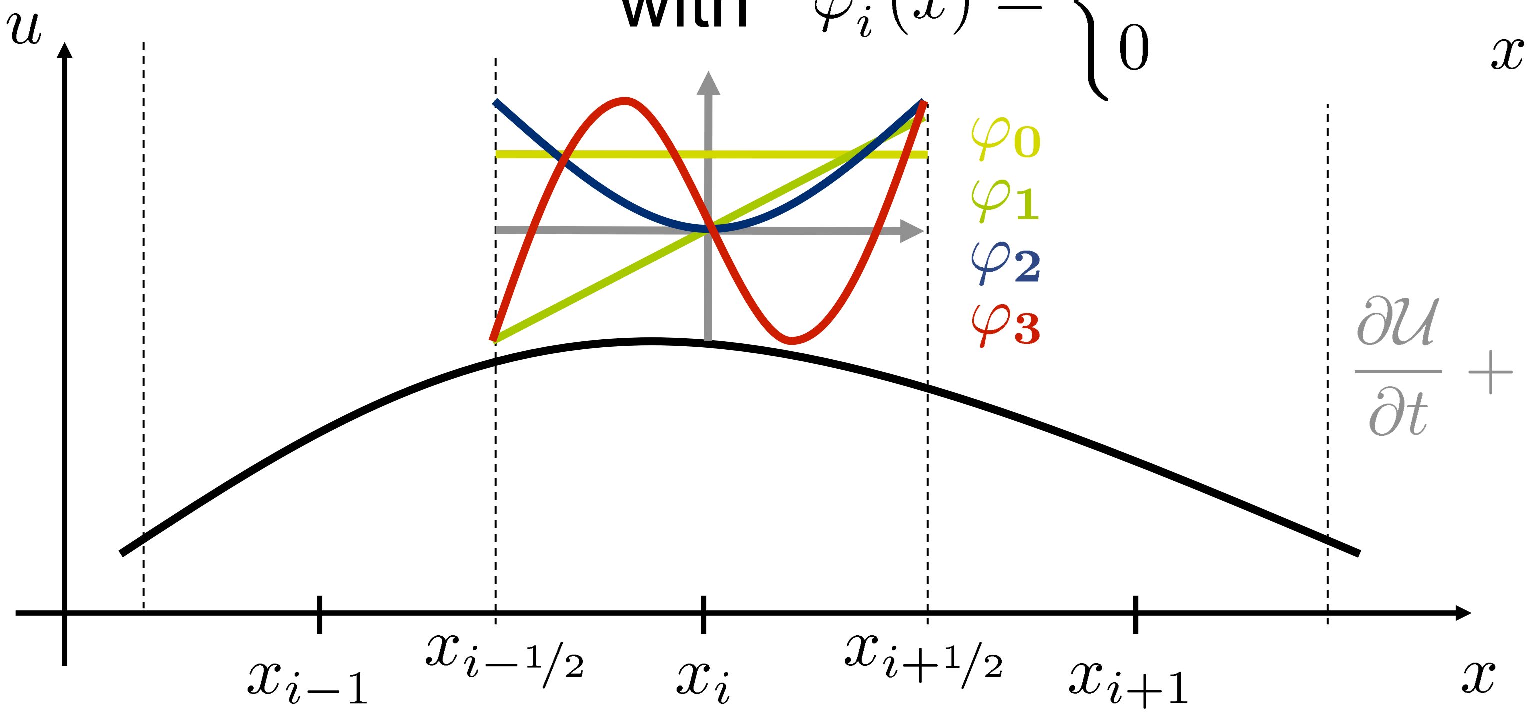
$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial f(\mathcal{U})}{\partial x} = 0$$

THE DISCONTINUOUS GALERKIN (DG) METHOD

- Basis functions with compact support in each I_j

$$\Phi_i(x) = \{\varphi_i^p(x)\} \quad \text{for} \quad p \in [0; P]$$

with
$$\varphi_i^p(x) = \begin{cases} \varphi^p(x - x_i) & x \in x_i, \\ 0 & x \notin x_i \end{cases}$$



$$\frac{\partial \mathcal{U}}{\partial t} + \frac{\partial f(\mathcal{U})}{\partial x} = 0$$

THE DISCONTINUOUS GALERKIN (DG) METHOD

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- DG-TD procedure

1. Approximate:

$$\mathcal{U}(x,t) \approx U(x,t) = \sum_{i,p} u_i^p(t) \varphi_i^p(x)$$

Time varying weighting coefficients

Space varying basis functions

2. Test and orthogonalize residual (Galerkin):

$$\sum_{i,p} \frac{\mathrm{d}u_i^p}{\mathrm{d}t} \int_{I_j} \mathrm{d}x \varphi_i^p \varphi_j^q + \int_{I_j} \mathrm{d}x \varphi_j^q \frac{\partial f(U)}{\partial x} = 0, \quad \forall j \in [1, N], \forall q \in [0, P]$$

$$\mathrm{M}_{ij}^{pq} \qquad \mathrm{S}_{ij}^{pq}$$

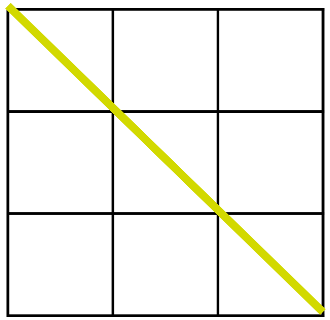
Mass Matrix

Stiffness Matrix

3. Evolve in time

THE DISCONTINUOUS GALERKIN (DG) METHOD

- Properties of the DG Mass and Stiffness Terms
  - Mass matrix $\mathbf{M}$:
    - compact support on I_j ➔ block-diagonal
    - orthogonal basis functions ➔ diagonal



$$\int_{I_i} \mathrm{d}x \varphi_i^p \varphi_j^q$$

- Stiffness matrix S:

$$\int_{I_j} \mathrm{d}x \varphi_j^q \frac{\partial f(U)}{\partial x} =$$

$$= h_i \left[U, x_{i+1/2}, t\right] \varphi_j^q(x_{i+1/2}) - h_i \left[U, x_{i-1/2}, t\right] \varphi_j^q(x_{i-1/2})$$

Apply Stokes' theorem

$f(U)$ is multivalued on element boundary

Introduce numerical fluxes (unique boundary value)

THE DISCONTINUOUS GALERKIN (DG) METHOD

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- Matrix form (semi-discrete):

$$\mathbf{M}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \mathbf{S}\mathbf{u} = 0$$

+ Mass matrix purely diagonal (trivially invertible)
  → explicit time stepping
+ Energy conservation (central fluxes[1])
+ Charge conservation (Cartesian grids, tensor product basis[2])
+ Locality, coupling via fluxes to direct neighbors only
  → highly suited for adaptive accuracy control

– Discontinuous solutions

1. M. Bernacki, S. Piperno: *A dissipation free DGTD method for the three-dimensional linearized Euler equations*, Journal of Comp. Acoustics 2006, Vol 14, No 4
2. E. Gjonaj et al.: *Conservation Properties of the Discontinuous Galerkin Method for Maxwell Equations*, International Conference on Electromagnetics in Advanced Applications (ICEAA) 2007

THE DISCONTINUOUS GALERKIN (DG) METHOD

- DG matrix formulation for Maxwell's equations (semi-discrete):

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\mathbf{M}_\epsilon \mathbf{e}\\ \mathbf{M}_\mu \mathbf{h}\end{pmatrix} + \underbrace{\begin{pmatrix}\mathbf{0} & -\mathbf{C}\\ \mathbf{C}^T & \mathbf{0}\end{pmatrix}}_{\mathbf{A}}\begin{pmatrix}\mathbf{e}\\ \mathbf{h}\end{pmatrix} = -\begin{pmatrix}\mathbf{j}\\ \mathbf{0}\end{pmatrix}$$

- $\mathbf{C}$ weak curl operator with $\mathbf{C} = \mathbf{C}^T$
- $\mathbf{A}$ is skew symmetric (for central fluxes)
  → pair-wise conjugate purely imaginary eigenvalues
  → symplectic time integration using e.g. three step leapfrog

LOCAL REFINEMENT TECHNIQUES

- *p*-Adaptation
  - Mathematically trivial for sets of hierarchical basis functions, e.g. the set of monomials

$$\mathbb{B}^{\mathrm{Mon}} = \{x^p\}, \forall p \in [0, P]$$

  but pair wise non-orthogonal → block-diagonal $\mathbf{M}$

  - Every set of pair wise orthogonal basis functions is inherently hierarchical, e.g. Legendre Polynomials

$$\mathbb{B}^{\mathrm{Leg}} = \{\mathcal{L}^p(x)\}, \forall p \in [0, P]$$

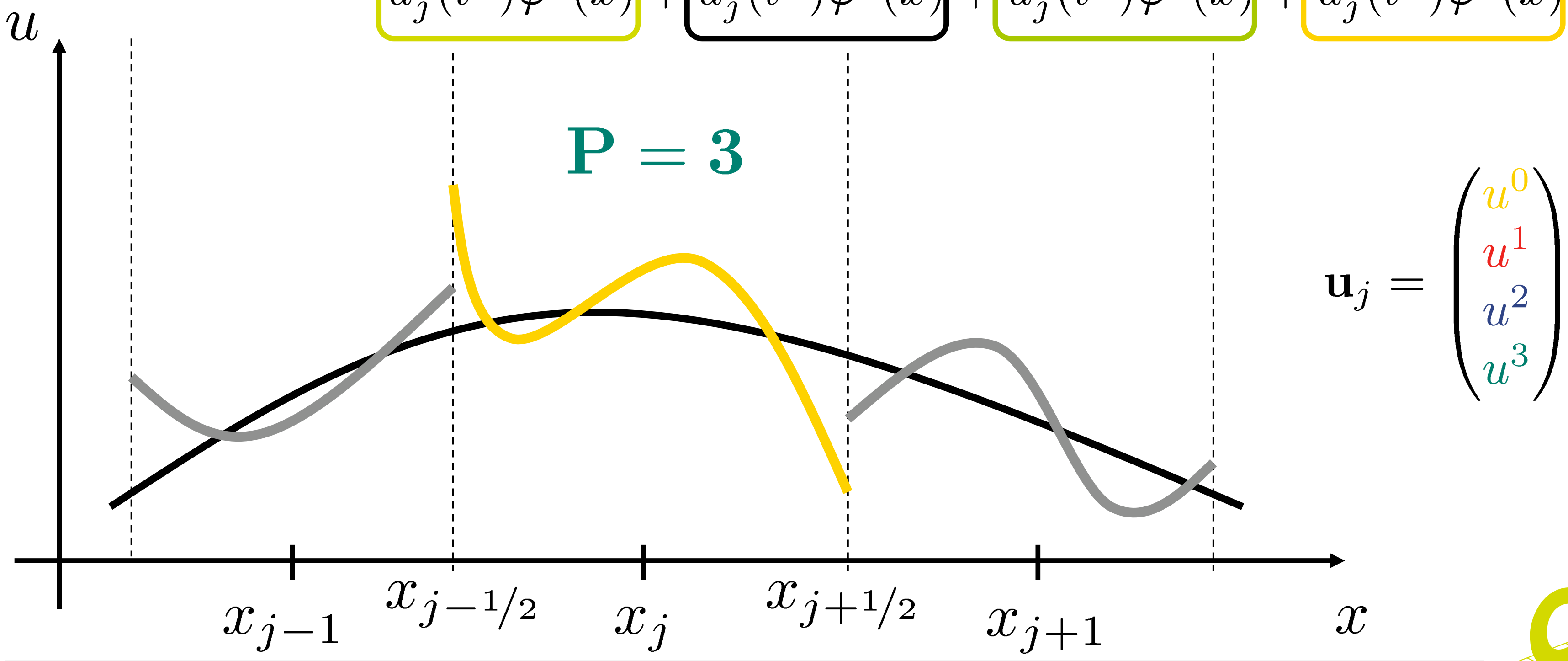
  - Memory and time efficient implementation "tricky"

LOCAL REFINEMENT TECHNIQUES

- *p*-Adaptation: Adapt local approximation order P

$$U_j(x, t^n) = \sum_p u_j^p(t^n)\varphi^p(x)$$

$$= u_j^0(t^n)\varphi^0(x) + u_j^1(t^n)\varphi^1(x) + u_j^2(t^n)\varphi^2(x) + u_j^3(t^n)\varphi^3(x)$$



LOCAL REFINEMENT TECHNIQUES

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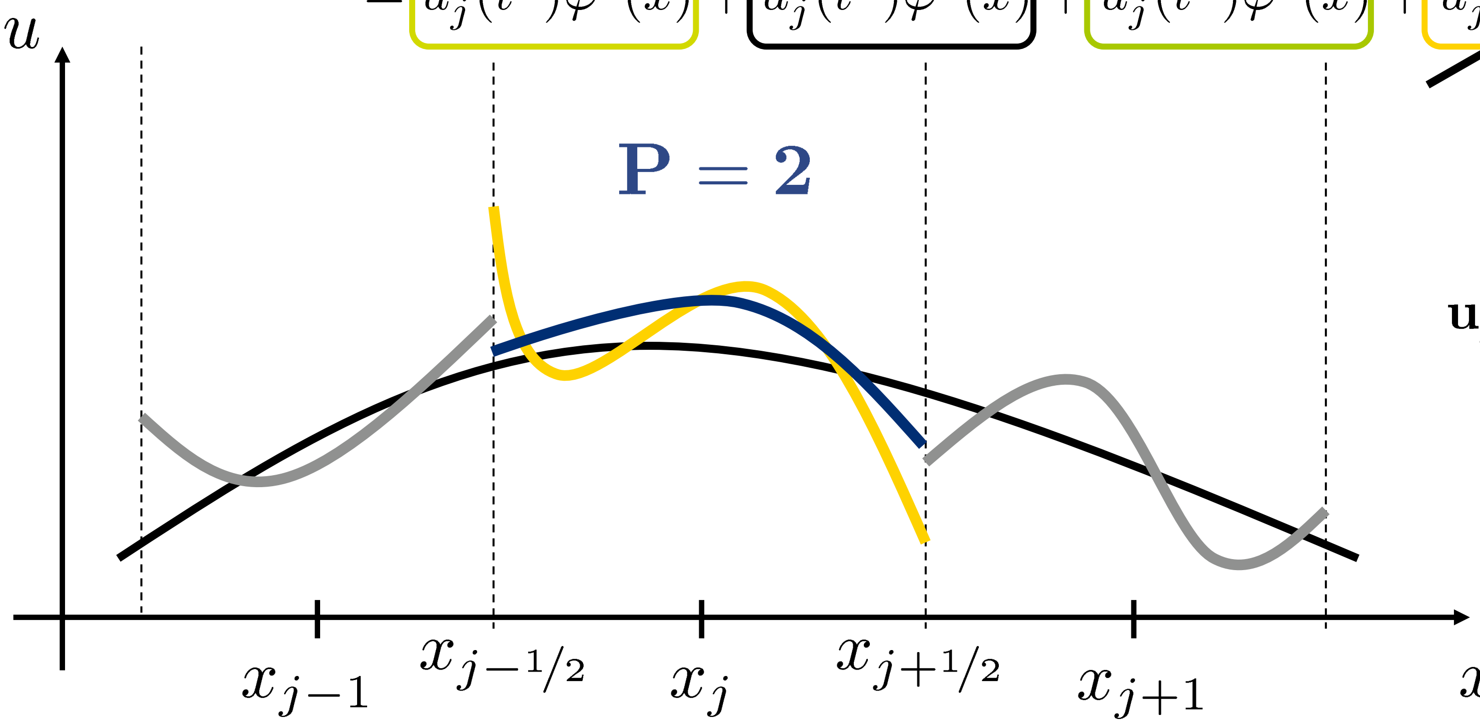
- *p*-Reduction

$$U_j(x,t^n) = \sum_p u_j^p(t^n)\varphi^p(x)$$

$$= u_j^0(t^n)\varphi^0(x) + u_j^1(t^n)\varphi^1(x) + u_j^2(t^n)\varphi^2(x) + u_j^3(t^n)\varphi^3(x)$$



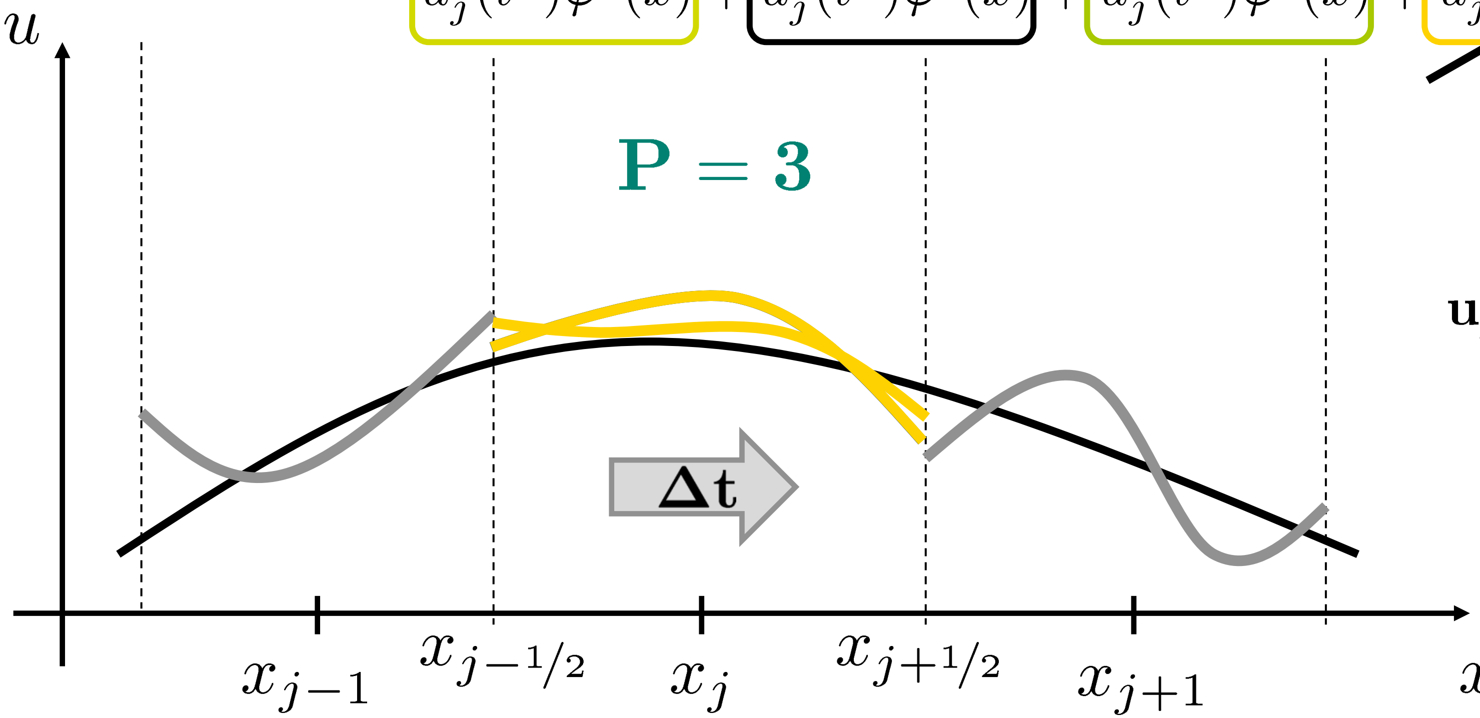
$$\mathbf{u}_j = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix}$$

LOCAL REFINEMENT TECHNIQUES

- *p*-Enrichment

$$U_j(x,t^n) = \sum_p u_j^p(t^n)\varphi^p(x)$$

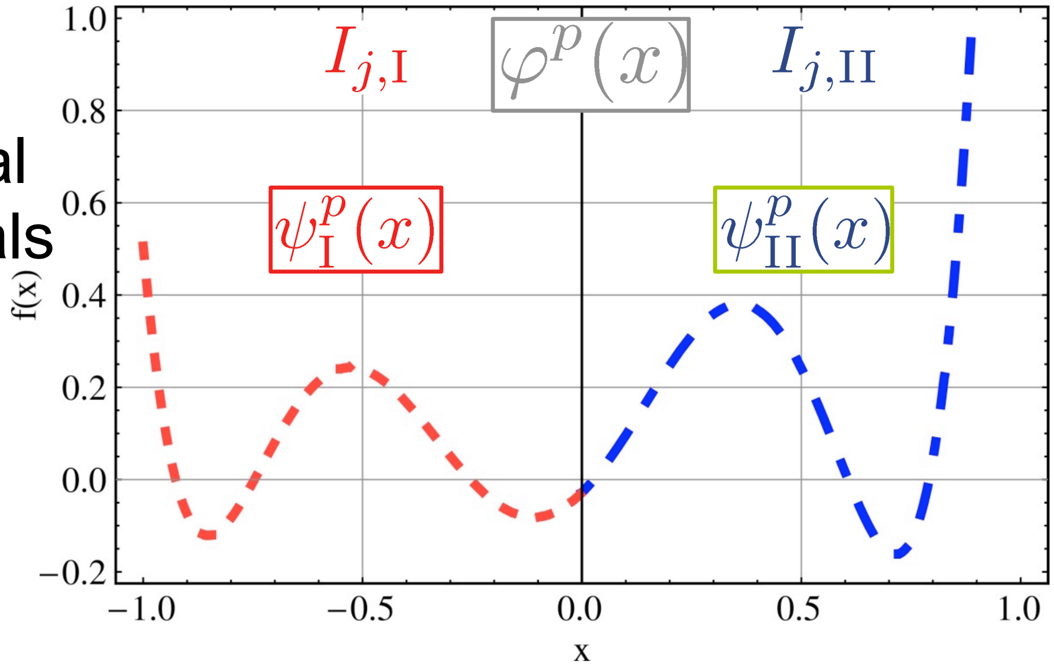
$$= u_j^0(t^n)\varphi^0(x) + u_j^1(t^n)\varphi^1(x) + u_j^2(t^n)\varphi^2(x) + u_j^3(t^n)\varphi^3(x)$$



$$\mathbf{u}_j = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \neq \mathbf{0} \end{pmatrix}$$

Methods for grid adaptations
Discontinuous Galerkin Method (DGM)

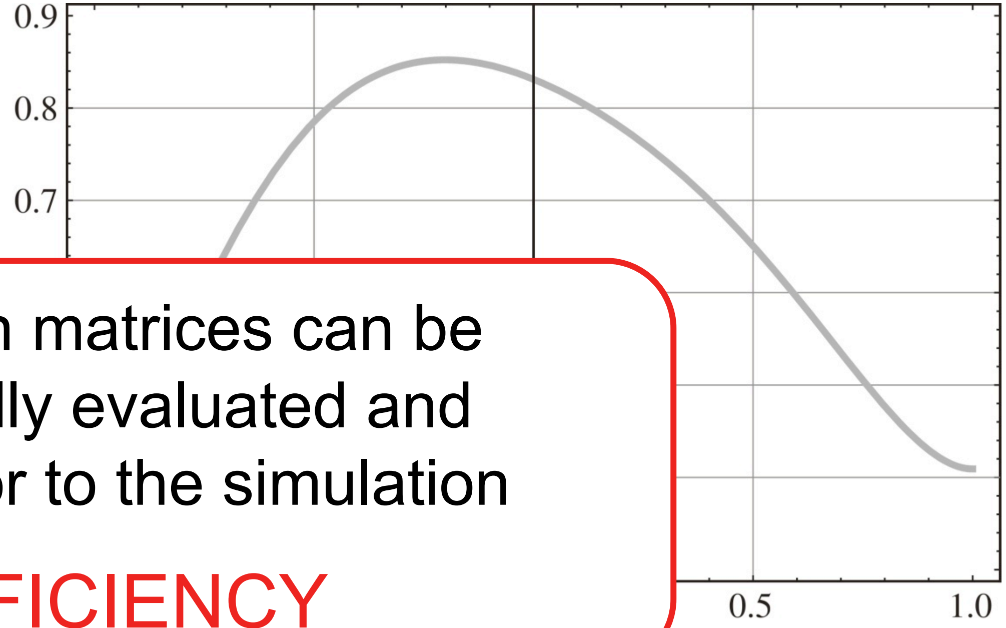
- *h*-refinement
  - Approximation in interval I_j is projected to intervals $I_{j,I}$ and $I_{j,II}$
  - Local projection matrices $\mathbf{P}_I$ and $\mathbf{P}_{II}$:
    - $\mathbf{P}_{\mathrm{I}}^{qp} = \langle \varphi^p, \psi_{\mathrm{I}}^q \rangle / \langle \psi_{\mathrm{I}}^q, \psi_{\mathrm{I}}^q \rangle$
    - $\mathbf{P}_{\mathrm{II}}^{qp} = \langle \varphi^p, \psi_{\mathrm{II}}^q \rangle / \langle \psi_{\mathrm{II}}^q, \psi_{\mathrm{II}}^q \rangle$



$$\mathbf{u}_{j,\mathrm{I}} = \mathbf{P}_{\mathrm{I}} \mathbf{u}_j$$
$$\mathbf{u}_{j,\mathrm{II}} = \mathbf{P}_{\mathrm{II}} \mathbf{u}_j$$

Methods for grid adaptations
Discontinuous Galerkin Method (DGM)

- *h*-coarsening
  - The approximation in I_j is considered as piece-wise defi[...] $I_{j,\mathrm{I}}$ and [...]
  - Definition[...] projection[...]



Projection matrices can be analytically evaluated and stored prior to the simulation

EFFICIENCY

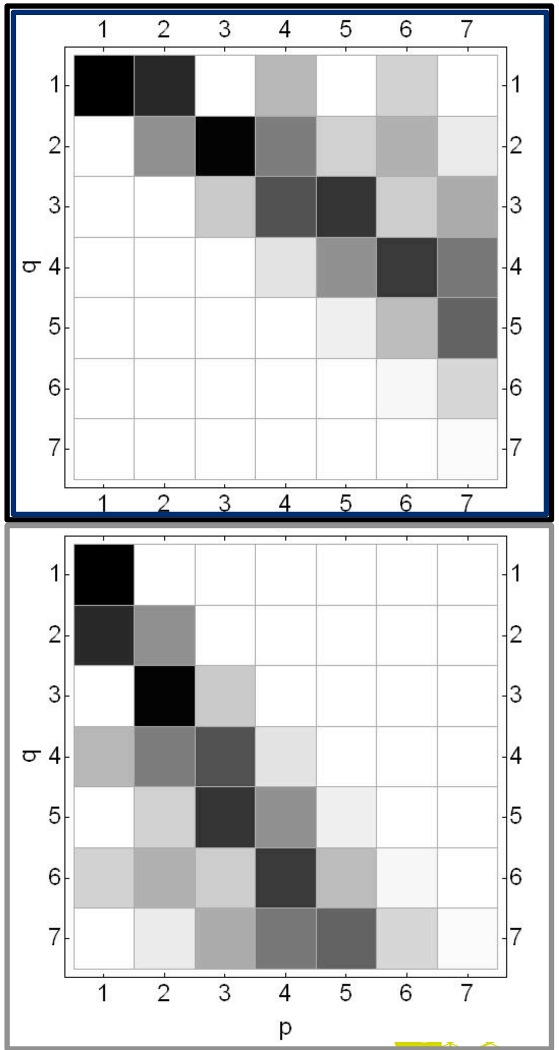
$$\mathbf{P}_c^{qp} = \langle \psi_{\mathrm{I}}^q + \psi_{\mathrm{II}}^q, \varphi^p \rangle / \langle \varphi^p, \varphi^p \rangle$$
$$= \left(\langle \psi_{\mathrm{I}}^q, \varphi^p \rangle + \langle \psi_{\mathrm{II}}^q, \varphi^p \rangle \right) / \langle \varphi^p, \varphi^p \rangle$$
$$= \mathbf{P}_{c,\mathrm{I}}^{qp} + \mathbf{P}_{c,\mathrm{II}}^{qp}$$

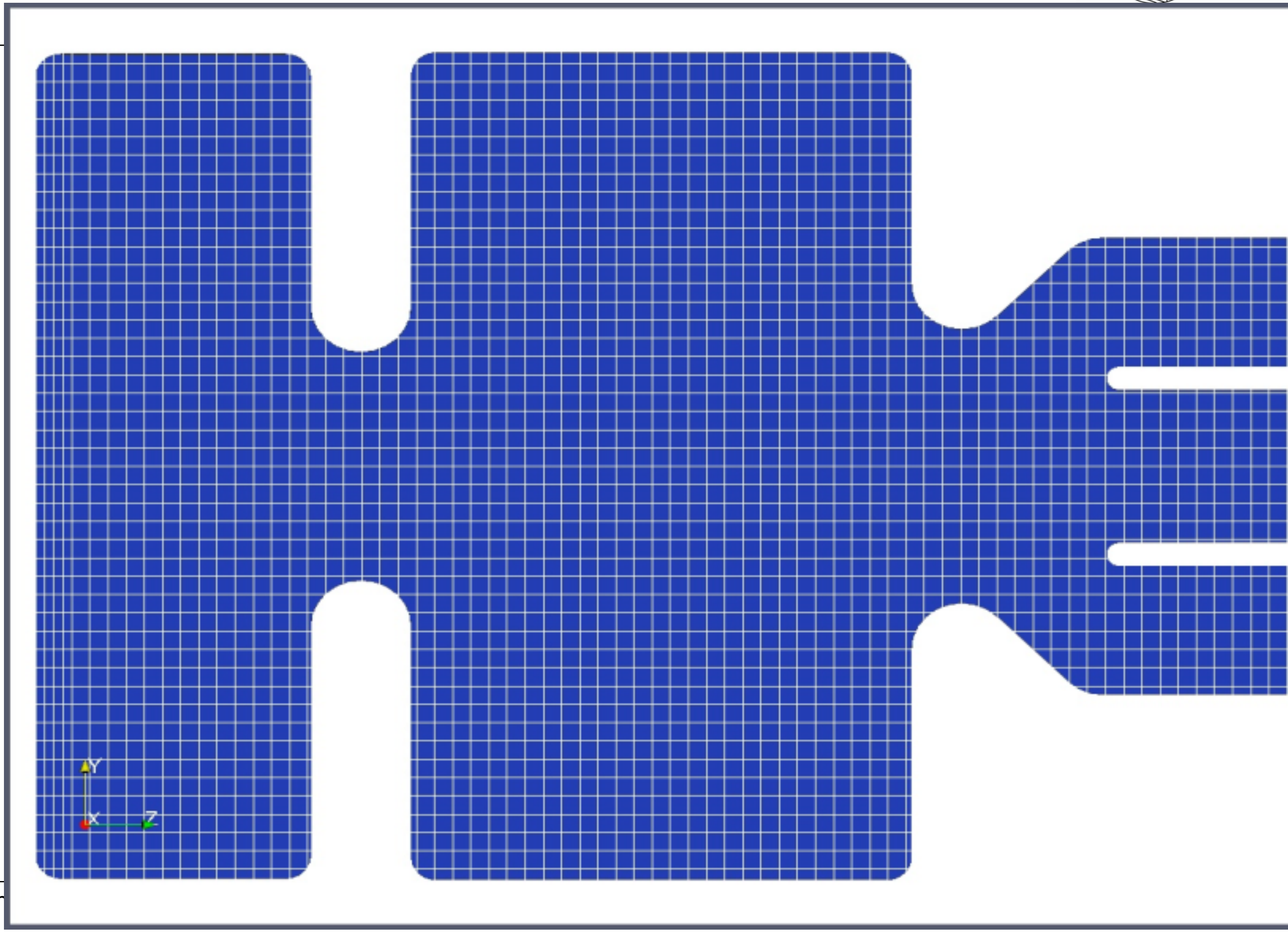
$$\mathbf{u}_j = \mathbf{P}_{c,\mathrm{I}} \mathbf{u}_{j,\mathrm{I}} + \mathbf{P}_{c,\mathrm{II}} \mathbf{u}_{j,\mathrm{II}}$$

LOCAL REFINEMENT TECHNIQUES

- *h*-Adaptation:
  - Projection based approach
  - Evaluation of analytical integral expressions once
  - Storage before time stepping starts
  - Upper / lower triangular matrix
    → efficient in-place calculation
  - Refinement: exact projection
  - Coarsening: no exact projection possible
    → moment matching approach

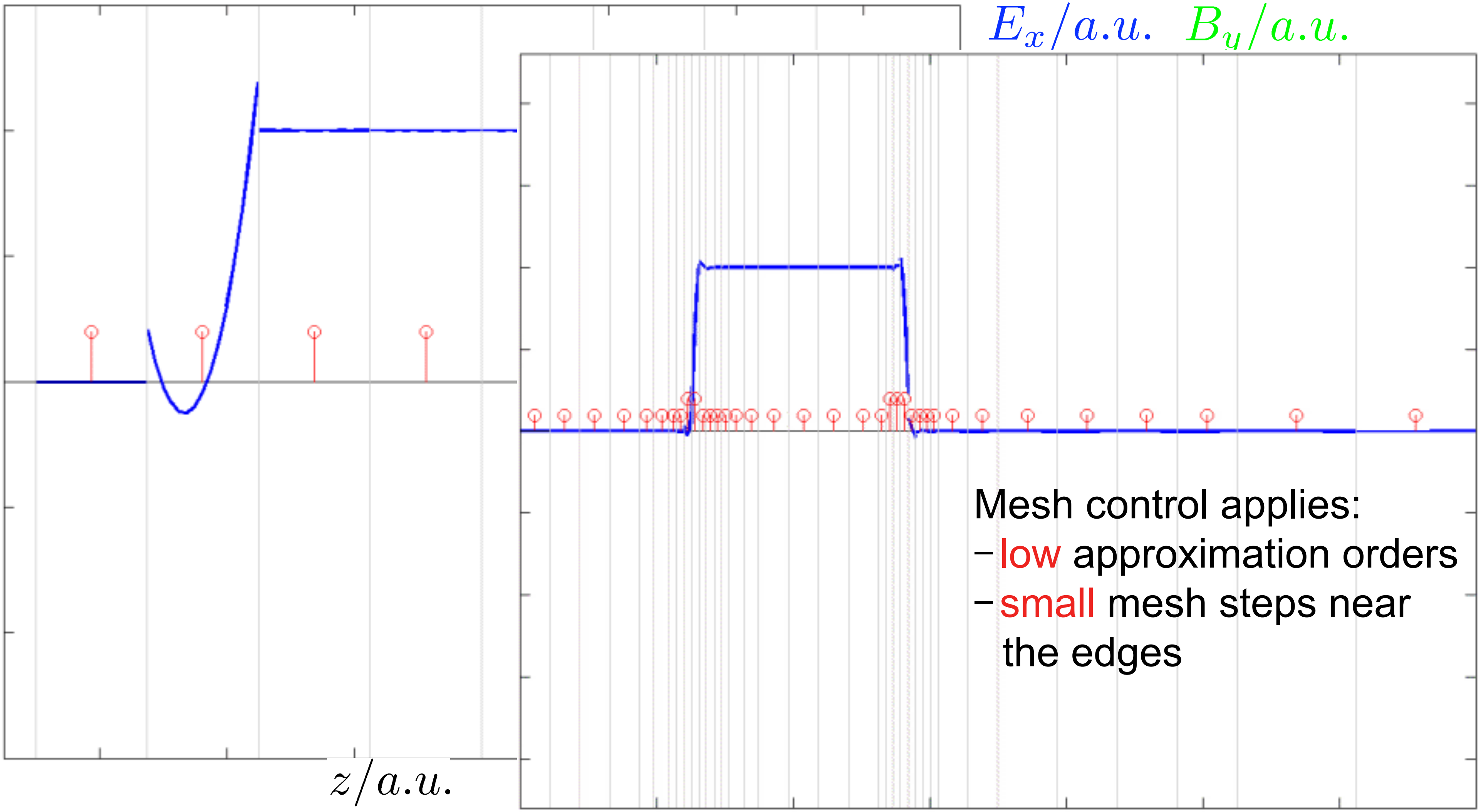
$$\mathbf{P}_I = 2\,\mathbf{P}^T_{c,I} \qquad \mathbf{P}_{II} = 2\,\mathbf{P}^T_{c,II}$$




Y
X
Z

Examples and Applications

Autonomous *hp*-refinement for the DG method



Mesh control applies:

- **low** approximation orders
- **small** mesh steps near the edges

Examples and Applications

Autonomous *hp*-refinement for the DG method

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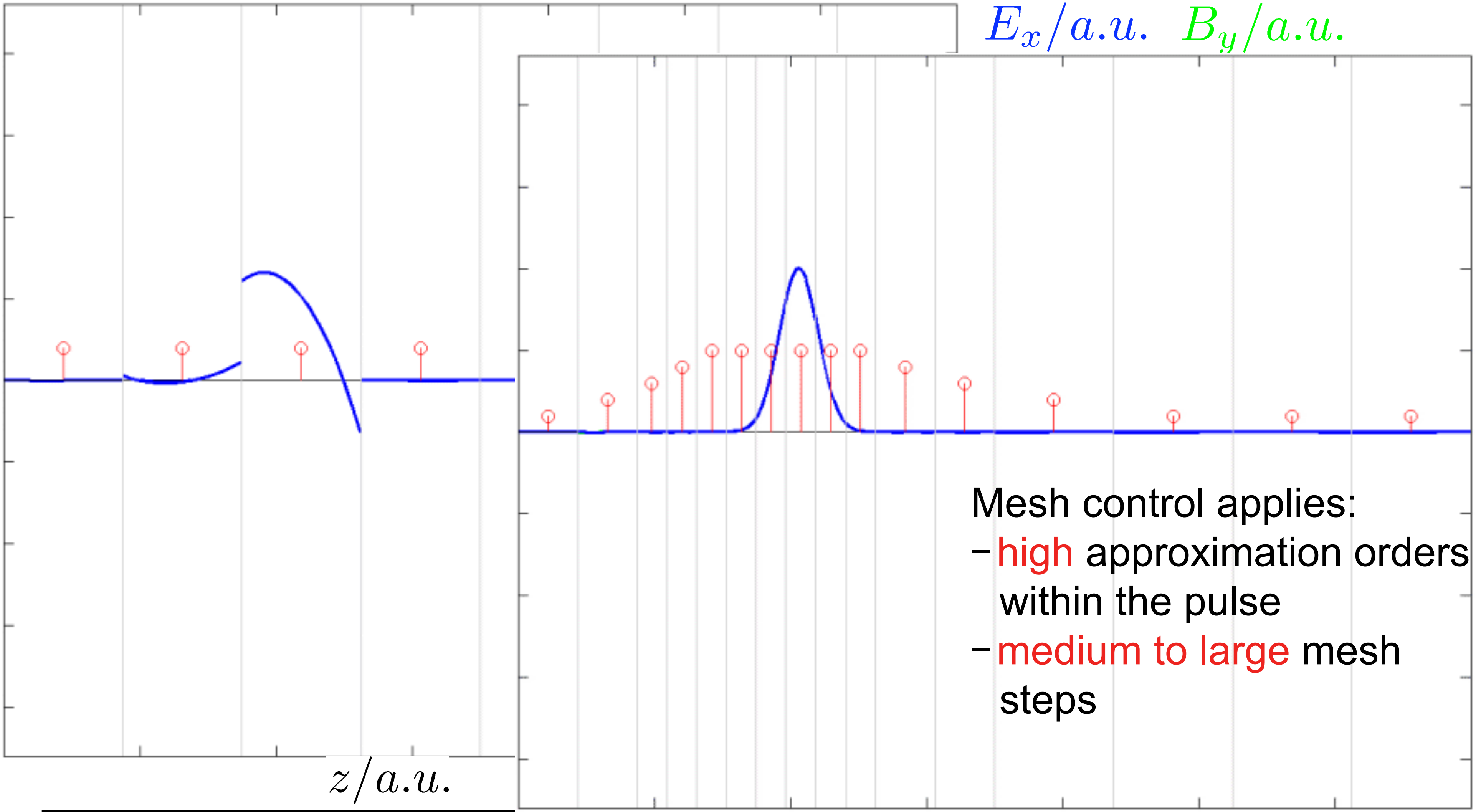

Mesh control applies:

- high approximation orders within the pulse
- medium to large mesh steps