CONSTRUCTION OF LARGE-PERIOD SYMPLECTIC MAPS BY INTERPOLATIVE METHODS

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October 2, 2009

Thanks to J. Scott Berg and Ronald Ruth

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7. Technical problems at large amplitude and how to deal with them
8. Proposal for further progress; meshless interpolation; quasi-random sequences.

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- To summarize evolution in systems where fields are complicated and integration in small time steps is difficult and costly. Examples:
  a) Very long wigglers for damping in advanced electron rings
  b) Insertion devices, helical or planar undulators, end fields
  c) FFAG accelerator, wavelength shifter for SR (G. Wüstefeld)
  d) Cases with important fringe fields, solenoidal field imposed on quadrupoles, etc.

COMPUTATIONAL FRAMEWORK

LEGO - A Class Library for Accelerator Design and Simulation (Y. Cai, SLAC-PUB-8011)

Single particle dynamics defined by Hamiltonian in the local frame of each lattice component.

LEGO is called for 1-turn tracking by a Fortran subroutine.

Interpolative map constructed from 1-turn tracking data alone.

CPU times quoted for a single 2.66 GHz processor.

EXAMPLE FOR THIS TALK

Electron ring, part of a study for ILC damping ring (Y. Cai, SLAC-PUB-11084 (2005)).

Racetrack form, 64 cells, primarily 90-degree FODO cells, $E = 5(GeV)$, $L = 960m$, $\epsilon_x = 47nm$. A scaling to more cells gives lower emittance, and a candidate for ILC damping ring.

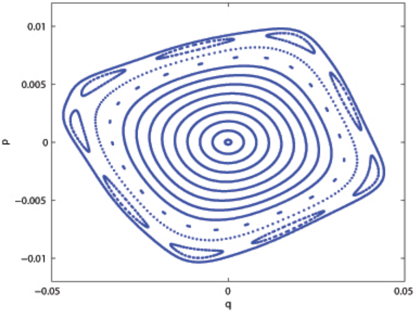


Figure: Phase plot from element-by-element tracking, 1000 turns, $\nu_x = 16.23$. Short term dynamic aperture just beyond outer curve.

FAILURE OF TAYLOR MAP IN THIS EXAMPLE

Spurious stochasticity, islands seen only vaguely.

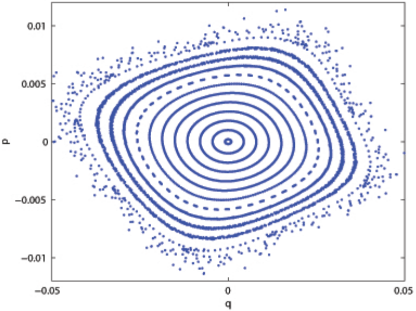


Figure: Phase plot from 10th order Taylor Map, 1000 turns, $\nu_x = 16.23$.

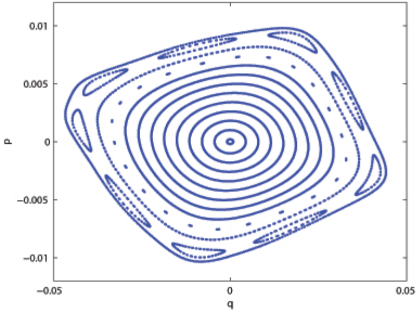
Does not help to go to 13th order, nor to make it symplectic by a generating function. Symplectified map not stochastic, and has islands, but at the wrong place.

SUCCESS OF SPLINE MAP IN THIS EXAMPLE

Tensor product cubic B-spline interpolation of 1-turn tracking data on 50×50 mesh.

$$(q, p) = M(q_0, p_0) = \sum_{i,j} (m_1, m_2)_{ij} B_i^{(1)}(q_0) B_j^{(2)}(p_0) \; . \qquad (1)$$

Agrees with tracking to graphical accuracy. Uses more large-amplitude information than is encoded in Taylor coefficients (i.e., in information at the origin).



PLOT OF B-SPLINE BASIS

B-splines with given knots form a basis for all splines with those knots. At any point only $k =$ (degree $+1$) B-splines are non-zero.

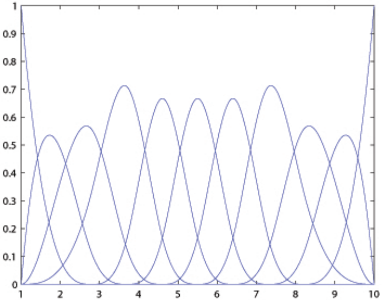


Figure: Cubic B-spline basis, 11 point interpolation, 8 distinct knots.

NATURAL DOMAIN OF THE MAP

The tensor product B-spline requires data on a rectangular grid, but the stable domain of the map is hardly rectangular. To make a spline we padded the data array by rough continuation of edge values. A more local interpolation instead of a spline tensor product is needed, and is under study.

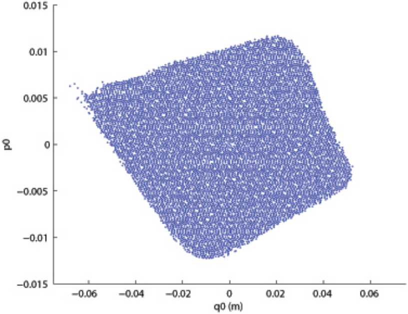


Figure: Initial conditions for which $|q| < 0.075$m after 40 turns.

SPLINE MAP AT 10^5 TURNS

Spline map fairly good to 10^5 turns, in spite of substantial violation of symplecticity.

$|\det \mathcal{M} - 1| \sim 10^{-3}$ at largest amplitudes, $\mathcal{M}$ =Jacobian matrix.

CPU time for 10^5 turns: 0.14 seconds. Scaled up to 6D: 38 seconds.

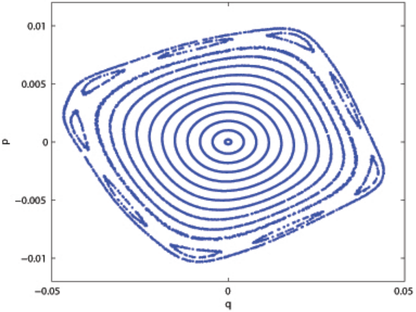


Figure: Phase plot from Spline Map, 10^5 turns, $\nu_x = 16.23$.

LONG-TERM BEHAVIOR OF NON-SYMPLECTIC MAP: SPLINE MAP AT 10^6 TURNS

Typical behavior of non-symplectic map after many iterations:

- Fuzziness instead of sharply defined invariant curves.
- Spurious damping (seen less at islands than for invariant curve circling origin).

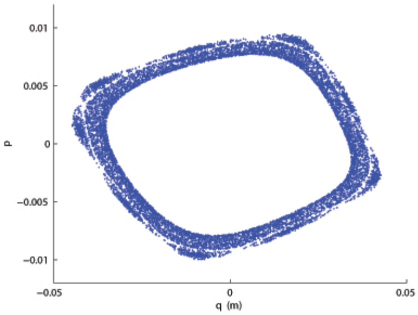


Figure: Spline Map at 10^6 turns, two initial conditions, $\nu_x = 16.23$.

CANONICAL TRANSFORM AND SYMPLECTIC CONDITION

Change of phase space variable $z_0 = (q_0, p_0) \longmapsto z = (q, p)$.
Define map M, Jacobian dM, matrix J:

$$z = M(z_0, t)\ , \quad dM = \begin{pmatrix} \partial q/\partial q_0 & \partial q/\partial p_0 \\ \partial p/\partial q_0 & \partial p/\partial p_0 \end{pmatrix}\ , \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ .$$

Two ways to characterize a Canonical Transform M:

1. Preserves the form of Hamiltonian equations of motion
2. Satisfies Symplectic Condition: $(dM)J(dM)^T = J$, all z_0. For 2D phase space, $\det(dM) = 1$

In particular, time evolution map is canonical. For 2D, note that vector field of Hamilton's equations $\dot{z} = J\partial H/\partial z$ is divergenceless, hence evolution is volume preserving, hence $\det(dM) = 1$.

ENFORCING SYMPLECTIC CONDITION BY GENERATING FUNCTION

Explicit time evolution map, fixed period T:

$$q = Q(q_0, p_0)\ , \quad p = P(q_0, p_0)\ . \tag{2}$$

Implicit map, defined by Generating Function $F(q, q_0)$:

$$p = F_1(q, q_0)\ , \quad p_0 = -F_2(q, q_0)\ , \tag{3}$$

where $\det F_{12}(q, q_0) \neq 0$ in region of interest. Solve (3b) for $q = \mathcal{Q}(q_0, p_0)$, substitute in (3a) to get $p = \mathcal{P}(q_0, p_0)$ as well.

If $F \in C^2$ then $(\mathcal{Q}, \mathcal{P}) : z_0 \mapsto z$ is symplectic!

Goal of this work: Determine F so that $(\mathcal{Q}, \mathcal{P}) = (Q, P) =$ time evolution map.

EARLIER WORK ON INTERPOLATED MAPS

Make symplectic map by relating a Mixed Variable Generating Function for period T to tracking information on period T.

- R.W., J.S. Berg, É. Forest, R.D. Ruth (1989 - 1997). Used polar coordinates; excluded small region of phase space due to a coordinate singularity. Fourier-spline representation of generator. Application to early LHC lattice. 10^7 turns (4D with energy modulation) in 3.6 hours (1995). Could study broad resonances at large amplitudes.

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- R.W., J.A. Ellison (1997-1999) Proved convergence of Fourier-spline series. Proposed convergent method in Cartesian coordinates using splines, first implemented June-August, 2009. Similar to idea of Berz (1991) using Taylor series.

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COMPETING NON-INTERPOLATIVE APPROACHES TO SYMPLECTIC MAPS:

- Make a mixed variable generator as a Taylor series (Dragt, Douglas, Yan, Berz, et al.)
- Irwin-Abell-Dragt "Jolt Factorization" (1991-2003) Lumped momentum kicks interleaved with linear symplectic maps.
- Both of these methods start with the Taylor map, and may not succeed if Taylor coefficients contain insufficient information.

APPLICATION OF MAPS TO COMPLEX FIELDS - EXAMPLE

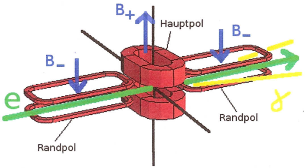


Abbildung 2.1: Skizze eines supraleitenden Wellenlängenschiebers

Figure: Wavelength shifter for SR, studied by symplectic map (Wüstefeld, Scheer)

CONSTRUCTION OF GENERATOR FROM TRACKING DATA - I

Map from element-by-element tracking $z = M(z_0)$ written as

$$(q, p) = (Q(q_0, p_0), P(q_0, p_0)) \ . \tag{4}$$

The gradient of the generator $F(q, q_0)$, when it exists, gives the same map in "mixed variables" (Hamilton-Jacobi theory):

$$(p, p_0) = (F_q(q, q_0), -F_{q_0}(q, q_0)) \ . \tag{5}$$

To relate the two descriptions, do a "partial inversion",

$$q = Q(q_0, p_0) \ \rightarrow \ p_0 = \Pi(q, q_0) \ , \tag{6}$$

supposing $\det Q_{p_0} \neq 0$. In practice, use Newton's method with guess from linear map.

CONSTRUCTION OF GENERATOR FROM TRACKING DATA - II

In this way we relate ∇F to the tracking map:

$$\nabla F = (F_q(q, q_0), F_{q_0}(q, q_0)) = (P(q_0, \Pi(q, q_0)), -\Pi(q, q_0)) \ . \quad (7)$$

We evaluate this on a mesh, interpolate by splines, and integrate the splines analytically along some convenient path to obtain

$$F(\zeta) = \int_{\zeta_0}^{\zeta} \nabla F(\zeta') \cdot d\zeta' \ , \quad \zeta = (q, q_0) \ . \qquad (8)$$

The integral is evaluated on a mesh, then interpolated by a spline of at least cubic degree, to give F as a C^2 function. Any $F \in C^2$ such that $\det F_{qq_0} \neq 0$ defines a symplectic transformation.

APPLICATION OF LINE INTEGRAL METHOD - I

Choose a tune, $\nu_x = 15.81$, for which the dynamic aperture is large, but phase curves highly distorted at large amplitudes.

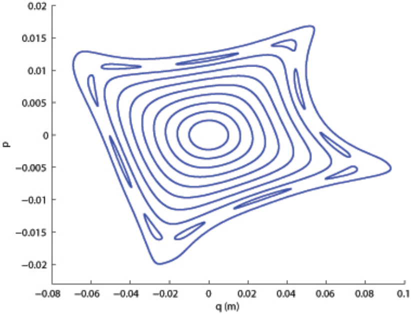


Figure: Phase plot from tracking, $\nu_x = 15.81$. Dynamic aperture (2000 turns) just beyond outer curve.

APPLICATION OF LINE INTEGRAL METHOD - II

Time for map construction including necessary tracking: 2.4 sec. The method works up to fairly large amplitudes, giving a phase plot that agrees with tracking to graphical accuracy.

Very stable for at least 10^9 turns, even for outer curve.

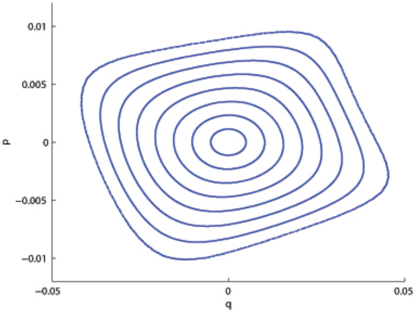


Figure: Phase plot from symplectic map, 10^7 turns. Iteration time $20 - 25$ sec. for 10^7 turns, per orbit.

APPLICATION OF LINE INTEGRAL METHOD - III

Why does it suddenly fail at still larger amplitudes? Inspect the locus of points (q, q_0) along orbits of the map, and compare to points at which solution $p_0 = \Pi(q, q_0)$ was achieved. Orbits approach forbidden region (Bermuda Triangle $\mathcal{B}$) in lower right corner. Also, we are forced to pad the spline data to fill in region $\mathcal{B}$.

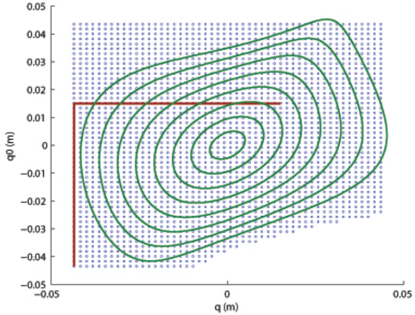


Figure: Green: Locus of (q, q_0) on orbits of previous figure. Blue: Points at which $\Pi(q, q_0)$ exists. Red: Path of integration.

APPLICATION OF LINE INTEGRAL METHOD - IV

Plot (q, q_0) curves of larger and larger amplitude until two curves cross ! Here we meet the limit of the region in which $F(q, q_0)$ exists. We might build F in the whole interior, but there are two technical annoyances: (1) Integration path hits the boundary (reason for sudden failure) (2) Spline requires even more padding and becomes less accurate.

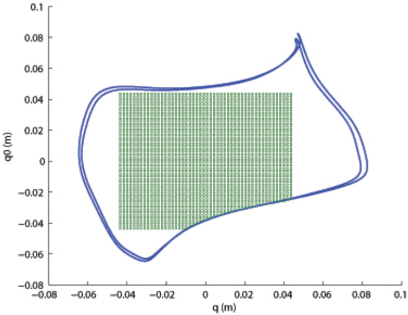


Figure: Two (q, q_0) curves that cross. Thus $\Pi(q, q_0)$ is not a single valued function.

APPLICATION OF LINE INTEGRAL METHOD - V

Enlargement showing crossing of (q, q_0) curves, near limit of existence of the generator.

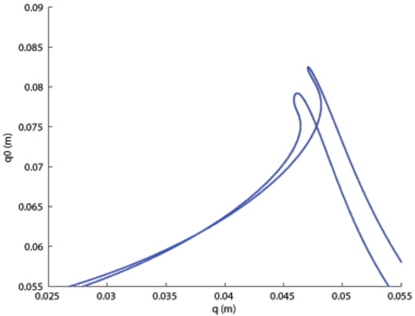


Figure: Two (q, q_0) curves that cross. Enlargement of previous figure.

PROPOSED SOLUTION TO TECHNICAL PROBLEMS NEAR DYNAMIC APERTURE

- Use a better path or avoid the line integral by calculating F as Action Integral.
- Use a local interpolation method rather than tensor product B-spline, for instance a generalized Shepard method:

$$F(\zeta) = \sum_i P_i(\zeta) \frac{w_i(\zeta)}{\sum_j w_j(\zeta)} \ , \quad w_i(\zeta) = c(\zeta - \zeta_i) \|\zeta - \zeta_i\|^{-n} \ , \tag{9}$$

where n is a positive integer such as 4 or 6. Here $P_i(\zeta)$ is a polynomial that interpolates or approximates values of F at ζ_i and a few nearby sites. The factor $c(\zeta - \zeta_i)$ is a smooth cutoff that restricts the sum at any evaluation.
This formula interpolates P_i *and* its derivatives, and is globally smooth if $c(\zeta)$ is smooth. Works as well with scattered data !

GENERATOR AS ACTION INTEGRAL – HAMILTON'S PRINCIPAL FUNCTION

Integral of Lagrangian on orbit with initial value $z_0 = (q_0, p_0)$:

$$S(q_0, p_0, t) = \int_0^t \Big[p(\tau, z_0) \cdot \dot{q}(\tau, z_0) - H(z(\tau, z_0), \tau) \Big] d\tau \ . \tag{10}$$

Hamilton's brilliant idea (1830-1832): regard this as a function of initial and final q. Then

$$F(q, q_0, t) = S(q_0, \Pi(q, q_0, t), t) \ . \tag{11}$$

is the generator discussed heretofore, when $t = T$. See R.W., "Hamilton-Jacobi Equation" at Scholarpedia.com.

- Avoids line integral, for a big advantage in high-dimensions.
- Requires a modified tracking code to calculate action integral.

INTERPOLATION ON QUASI-RANDOM (LOW DISCREPANCY) SEQUENCE)

A possibly more efficient scheme in high dimensions is to interpolate data on scattered sites rather than on a mesh. In particular, interpolation on a quasi-random sequence (Sobol, Halton, $\cdots$) may enjoy the advantages of quasi - Monte Carlo quadrature over mesh based quadrature.

1. G. Fasshauer, "Meshless Approximation Methods with MatLab" (World Scientific,2007)
2. H. Niederreiter, "Random Number Generation and Quasi-Monte Carlo Methods (SIAM, 1992)
3. R.W., J. Ellison, K. Heinemann, G. Q. Zhang, EPAC08, TUPP109

We found in (3) that the number of interpolation sites could be reduced by a factor of 8 compared to a mesh, in a 2D example (re. Vlasov equation solution).

Expect a much bigger advantage in higher dimensions.

MANAGING LARGE SETS OF DATA AT SCATTERED SITES

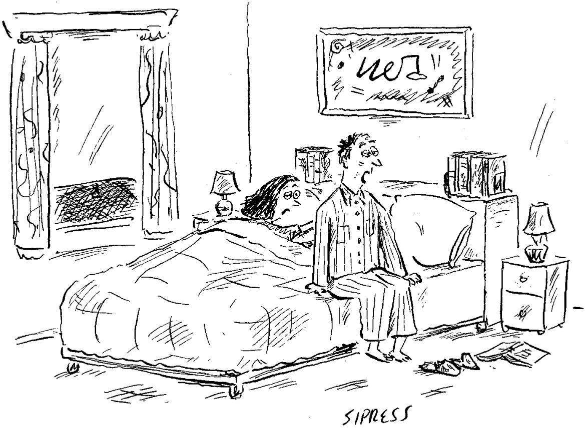
One needs to locate points and their neighbors; i.e., do a “range search”. To date, we define a grid and make lists of points in each cell. Can locate the cell to which any point belongs by taking integer parts of scaled coordinates of the point. Maybe awkward in high dimensions.

New technique: compressed bit map index FastBit (developed at LBNL). See crd.lbl.gov/ kewu/fastbit/. Set of vectors of bits, each vector corresponding to an attribute of data. Do bitwise logic on these vectors, at compressed level. Time is linear in the number of hits.

Used for searching for features in geologic data and various simulations.

OUTLOOK FOR FURTHER WORK

- Cases in which rectangular interpolation domains are sufficient (short of dynamic aperture, LHC, wigglers, etc.) can be handled by the line integral method and B-splines. Requires no modification of tracking code.
- Cases with strong nonlinearity, resulting in non-rectangular domains, may require action integral and local interpolation. Requires modified tracking code.
- Local interpolation on quasi-random sets may be advantageous in any case, and such interpolation could have wide applications outside this problem.



"I can't sleep. I think I'll get up and solve all my problems."