SIMULATION STUDIES & CODE VALIDATION FOR THE HEAD-TAIL INSTABILITY WITH SPACE CHARGE

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Abstract

The head-tail instability represents a potential intensity limitation for bunched beams in the synchrotrons of the FAIR project. Parametrical studies with numerical simulations over very long time scales are necessary in order to understand the effect of direct space charge, nonlinear synchrotron oscillations and image charges, which are all important in the FAIR synchrotrons. Existing analytic approaches either neglect space charge or describe simplified models, which require a numerical or experimental validation. For our simulation studies we use two different computer codes, HEADTAIL and PATRIC. In this work we verify models for wake-field kicks and space-charge effect using the analytic solution for head-tail mode frequencies and growth rates from the barrier airbag model.

INTRODUCTION

Modern synchrotrons, as SIS-100 and SIS-18 of the FAIR [1] complex, will operate with ion bunches under conditions of strong space charge, $\Delta Q_{\rm sc} \gg Q_s$, or moderate space charge, $\Delta Q_{\rm sc} \gtrsim Q_s$, where $\Delta Q_{\rm sc}$ is the shift of the betatron tune due to space charge and Q_s is the synchrotron tune. Transverse head-tail instability, which is one of the main concerns for the high-intensity operation, can be strongly modified by the effect of space charge. Classical theories, such as the model of Sacherer [2], do not include interactions of a head-tail mode with any incoherent tune spreads. Recent works [3, 4] propose approaches to treat head-tail modes with space charge. However, numerical simulations appear to be indispensable for a comprehensive stability analysis in different beam parameter regimes and with various collective effects taken into account.

A study for head-tail modes with space charge requires extensive parametrical scans of long time scale (tens of thousands of turns) runs. In the case of the weak head-tail instability it is not possible to scale up e.g. the impedance and hence reduce the run time, since mode coupling excites strong head-tail modes above the associated threshold. For reliable stability predictions it is thus necessary to use code modules, in this case primarily space-charge and wake-field implementations, which have been validated accurately, and are applicable for long time scale runs. In the present work we use two different particle tracking codes, `PATRIC` [5] and `HEADTAIL` [6], in order to take the advantage of different kinds of implementations and thus treat the task. The `PATRIC` code, a part of numerical development effort at GSI, was optimised for relatively short-term effects with well-resolved betatron oscillations and exact self-consistent space charge solvers, while the `HEADTAIL` code, created at CERN, was designed historically for longer-term phenomena, including electron-cloud effects, with an option of very fast once-per-turn modus. For the code validation in the range of moderate and strong space charge we suggest to use the model of an airbag bunch in barrier potential. This bunch model, being rather artificial, has a simple analytical solution [7], can be easily implemented in a simulation and, as we demonstrate here, gives very useful insight into physics of head-tail modes with incoherent interactions.

PHYSICAL MODEL

An analytical solution for head-tail modes in bunches with space charge has been derived in Ref. [7]. The model assumes the airbag distribution in phase space and the square-well (barrier) potential and thus a constant line density. The longitudinal momentum distribution has then two opposing flows of particle $\left[\delta(v_0 - v_{\rm b}) + \delta(v_0 + v_{\rm b})\right]$, the synchrotron tune in this bunch is $Q_s = v_{\rm b}/2\tau_{\rm b}Rf_0$, where $\tau_{\rm b}$ is the full bunch length in radian, R is the ring radius and f_0 is the revolution frequency. The model considers "rigid slices", i.e. only dipole oscillations without transverse emittance variation are included. It is also assumed that all betatron tune shifts are small compared to the bare tune $|\Delta Q| \ll Q_0$. The resulting tune shift due to space

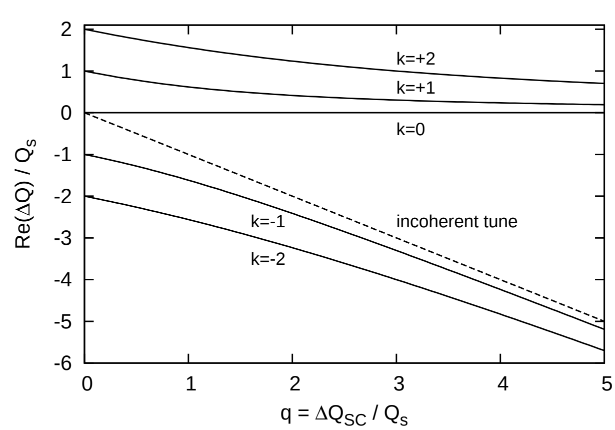

Figure 1: Tune shifts of five head-tail modes versus space charge parameter q as given by the airbag theory Eq. (1), the dashed line is the incoherent betatron tune $Q_0 - \Delta Q_{\rm sc}$.

charge (without wake) is given by

$$\Delta Q = -\frac{\Delta Q_{\rm sc}}{2} \pm \sqrt{\frac{\Delta Q_{\rm sc}^2}{4} + k^2 Q_s^2} \,, \tag{1}$$

where "+" is for modes $k \geq 0$, $\Delta Q_{\rm sc}$ is defined as a positive value. These space-charge tune shifts are illustrated in Fig. 1, where we introduce a space charge parameter $q = \Delta Q_{\rm sc}/Q_s$.

The solution for weak head-tail instability is obtained in Ref. [7] assuming the wake potential $W(\tau) = W_0 \exp(-\alpha\tau)$ with a sufficiently short range compared to the bunch length, $\alpha\tau_{\rm b} \gg 1$. For the mode $k = 0$, which is not affected by space charge, the tune shift is given by

$$\Delta Q = \Delta Q_0(\alpha/\zeta + i) \,, \tag{2}$$

where we define ΔQ_0 as the growth rate for the mode $k = 0$,

$$\Delta Q_0 = -\frac{\zeta}{\alpha^2} \frac{\kappa W_0}{2Q_0} \,, \tag{3}$$

here $\zeta = \xi Q_0/\eta$ is the normalized chromaticity, $\Delta Q_\xi/Q = \xi \Delta p/p$, and

$$\kappa = \frac{I_0 q_{\rm ion}}{2\pi\gamma m\beta c\omega_0^2} \,, \tag{4}$$

where β, γ are relativistic parameters, I_0 is the beam current, $q_{\rm ion}$ is the ion charge and ω_0 is the angular revolution frequency.

For modes $k \neq 0$ this theory predicts

$$\Delta Q = -\Delta Q_{\rm sc} + \frac{\Lambda_0 \pm \sqrt{\Lambda_0^2 + 4k^2 Q_s^2 \Lambda_{\rm s}}}{2\Lambda_{\rm s}} \,, \tag{5}$$

where

$$\Lambda_0 = \Delta Q_0(\alpha/\zeta + i) + \Delta Q_{\rm sc} \,,$$
$$\Lambda_{\rm s} = 1 - \left[\frac{\Delta Q_0 \pi}{2\zeta Q_{\rm s}\tau_{\rm b}}\right]^2 .$$

SPACE CHARGE SIMULATIONS

In order to verify the space-charge implementation for long-time simulations with a particle tracking code, we have introduced the barrier-airbag bunch distribution in both PATRIC and HEADTAIL codes. For space charge, the frozen electric field model was used, i.e. a fixed potential configuration which follows the mass center for each single slice. This approach is justified for the "rigid-slice" regime, which is also a basis assumption for the airbag theory [Eq. (1)] we compare with. A homogeneous and round transverse bunch profile (K-V distribution) was used in simulations in this work, a linear rf force was assumed.

A good agreement between theory and simulations with space charge has been achieved with both codes. Figures 2 and 3 show the coherent bunch spectrum, which is the Fourier transform of the total transverse bunch offset, for moderate $q = 3$ and stronger $q = 8$ space charge. Prediction of the airbag theory is laid over in red.

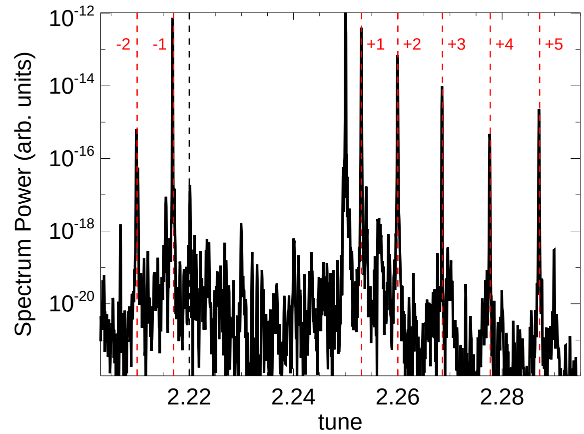

Figure 2: Dipole bunch spectrum from simulations (the HEADTAIL code, airbag bunch) with moderate space charge $q = 3$. The bare tune is Q_0=2.25, $Q_{\rm s} = 0.01$, the black dashed line is the incoherent tune ($Q_0 - \Delta Q_{\rm sc}$), red dashed lines are head-tail modes from Eq. (1)

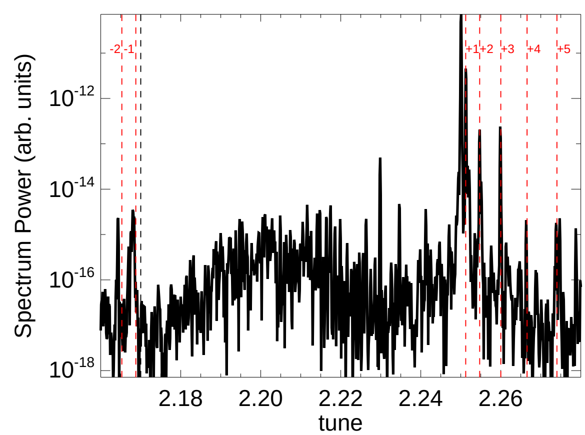

Figure 3: Dipole bunch spectrum from simulations (the PATRIC code, airbag bunch) with strong space charge $q = 8$. The legend and the tunes correspond to Fig. 2.

In our parametric studies we have found that even for moderate and weak space charge it is not enough to execute only one space-charge kick per turn. At least one kick per approximately 1 rad phase advance is necessary for an accurate description of space-charge effect. In the example presented here, $Q_0 = 2.25$, this means at least 16 kicks per turn, and for the SIS-100 tune $Q_0 = 18.7$ this means 120 kicks per turn.

In order to test the relevance of the airbag theory for realistic bunches we have performed simulations with a Gaussian bunch, i.e. Gaussian line density profile and Gaussian momentum distribution. For comparisons between bunch profiles, the line density of the airbag-bunch is taken to be equal to the density in the middle of the Gaussian bunch. Especially for strong space charge a surprisingly good agreement between the airbag theory and simulation results for the Gaussian bunch is found, see Fig. 4. The frequency shift of head-tail eigenmodes due to space charge seems to be a very robust effect, which does not depend on the details of the bunch form and distribution, especially for strong space charge.

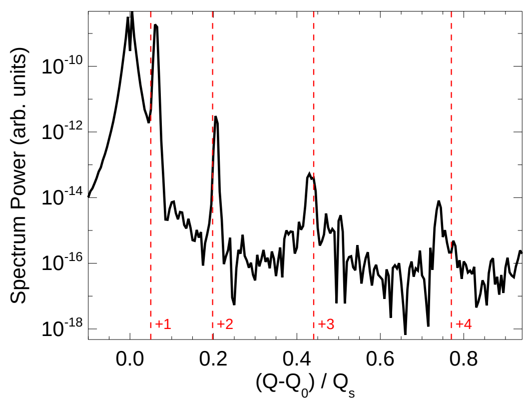

Figure 4: Dipole bunch spectrum from simulations for a Gaussian bunch with strong space charge $q = 20$. Red dashed lines are head-tail modes for the airbag bunch, Eq. (1). To clarify the notation we note that the tune shifts without space charge are $\Delta Q/Q_{\rm s} = 1$ for $k = 1$, $\Delta Q/Q_{\rm s} = 2$ for $k = 2$, etc.

INSTABILITY SIMULATIONS

For wake-field verifications the exponential wake potential $W(\tau) = W_0 \exp(-\alpha\tau)$ has been implemented in the codes. The wake momentum kick for a bunch slice τ_0 (or z_j) is calculated [8] as

$$\begin{aligned} \Delta x' &= \frac{2\pi\kappa}{R}\int_{-\tau_{\rm b}}^{\tau_0}\frac{\lambda(t,\tau)}{\lambda_0}\overline{x}(t,\tau)W(\tau_0-\tau)d\tau = \\ &= \frac{2\pi q_{\rm ion}\kappa}{\lambda_0 R^2}\sum_{i>j}N_i\overline{x}_i W(z_j - z_i)\,, \qquad (6) \end{aligned}$$

where $\lambda(t,\tau)$ is the bunch line density for the time t in the bunch position τ, λ_0 is the peak line density, $\overline{x}$ is the transverse slice offset, $x' = {\rm d}x/{\rm d}s$, N_i is the particle number in the ith slice, and κ is given in Eq. (4).

The analytic solution for the exponential wake potential with the airbag bunch [Eqs. (2)-(5)] predicts the mode $k = 0$ to be the most unstable head-tail mode. As we illustrate in Fig. 5, $k \neq 0$ modes always have smaller growth rates than ΔQ_0. Here we normalize Im(ΔQ) by the growth rate of the $k = 0$ mode. In this example the $k = -1$ mode becomes more unstable than the $k = 0$ mode at $\Delta Q_0 \approx 0.003$, which is due to the fact that the model breaks down. As we can see in Fig. 6, after $\Delta Q_0 \approx 0.002$ (where $\Delta Q_0/Q_{\rm s} \approx 0.2$) the mode $k = -1$ does not approach the $k = 0$ mode anymore, although in reality they should cross and produce mode coupling, but the latter is not included in this theory. The observation that space charge suppresses the largest growth rate among $k \neq 0$ is not general and depends on parameters $\alpha\tau_{\rm b}$, $Q_{\rm s}$, etc. Here we take $\alpha\tau_{\rm b} = 40$ (in the simulations presented below as well), for other parameters space charge can cause an increase in the largest growth rate among $k \neq 0$. Finally, it should be noted that in a real bunch with space charge the $k < 0$ modes are strongly damped by any betatron- and synchrotron tune spread, additionally these modes are damped by the non-resonant dipole behaviour [9].

In the simulations we can observe only the dominant $k = 0$ mode, but it is possible to resolve real tune shifts also for $k \neq 0$, at least for small wake amplitudes. Our first comparison between simulations and the theory is for the case without space charge, see Fig. 6. Particle tracking simulations reproduces fairly well weak head-tail instability from the theory, both growth rates and real tune shifts. Only close to $\Delta Q_0 \approx 0.002$ discrepancies appear, because the airbag approach breaks down, as we discuss above.

The next step is to combine the wake-field effect with space charge. Results of this parametric scan are presented in Fig. 7, where a constant space charge tune shift $\Delta Q_{\rm sc} = 3Q_s$ was assumed. This is related to an increasing impedance, not increasing beam intensity. Also here we successfully reproduce the theory results for the head-tail instability with space charge. In agreement with the theory the simulations demonstrate that space charge does not affect the $k = 0$ mode and strongly changes other head-tail modes.

Another test for the relevance of the airbag theory for realistic bunches is to perform simulations with the exponential wake for a Gaussian bunch. Space charge intensity $q = 5$ in the middle of the bunch is assumed, $\alpha\tau_{\rm b} = 10$ is

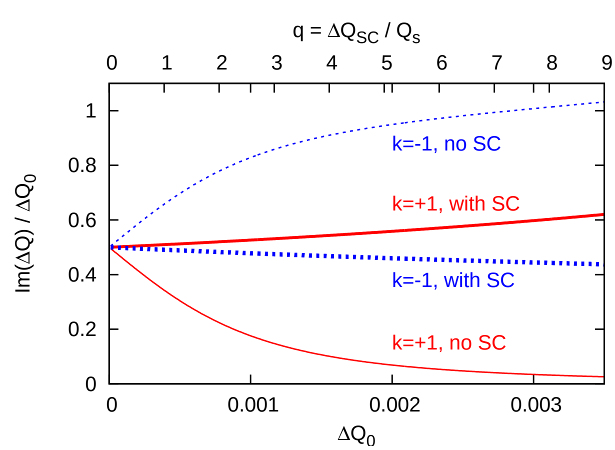

Figure 5: Growth rate given by the airbag theory Eqs. (2)-(5). The beam intensity increases along the horizontal axis, thus the wake amplitude (lower axis range) and space charge (upper axis range) grow proportionally.

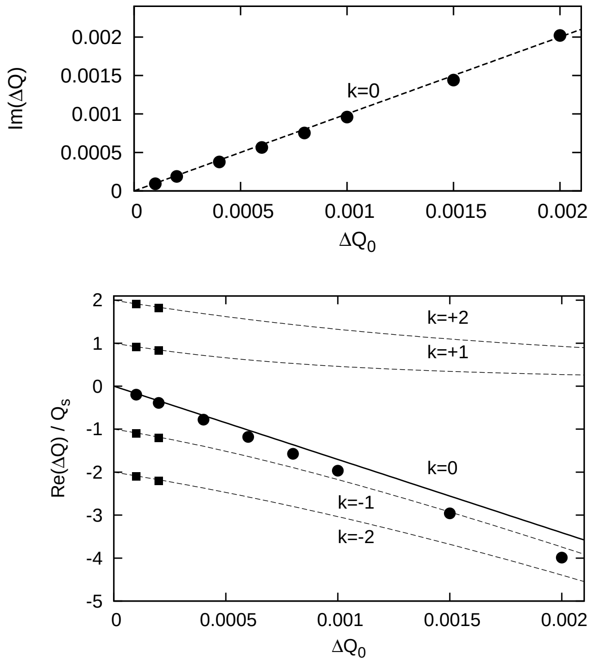

Figure 6: Growth rates (upper plot) and real tune shifts (bottom plot) of head-tail modes without space charge. Symbols: simulation output (the `PATRIC` code, airbag bunch); lines: airbag theory Eqs. (2)-(5). $\mathrm{Re}(\Delta Q)$ for $k \neq 0$ is shown only for small ΔQ_0.

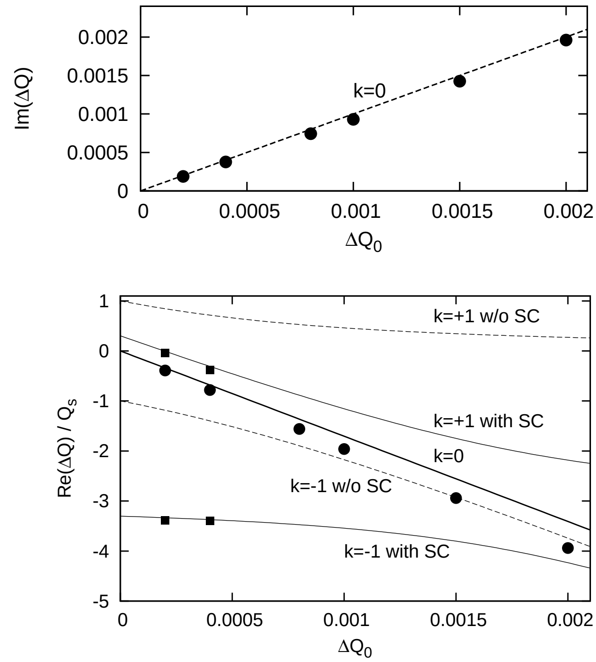

Figure 7: Growth rates (upper plot) and real tune shifts (bottom plot) of head-tail modes in the airbag bunch with space charge $q = 3$. The legend corresponds to Fig. 6. $\mathrm{Re}(\Delta Q)$ for $k = \pm 1$ is shown only for small ΔQ_0. The theory lines for cases with/without space charge are shown for comparison.

taken. Simulations with and without space charge showed a dominating $k = 0$ mode which is not affected by space charge. Figure 8 demonstrates that if growth rates and real tune shifts from simulations are multiplied by a factor 1.89, a good agreement with the airbag theory Eqs. (2) is achieved. This means that head-tail modes in a Gaussian bunch correspond, accurate to a constant factor, to the airbag theory, although the Gaussian bunch has totally different linear density and momentum distribution. Obviously, this scaling factor depends on bunch parameters and wake-field properties. Frequencies of $k \neq 0$ modes were difficult to resolve in this case, which may be related to the distribution type. However, one would suspect that this scaling factor is not effective for $k \neq 0$ modes due to effect of space charge. The appropriate way to clarify this issue may be simulations for parameters where $k \neq 0$ modes are the most unstable. This should be investigated in further studies.

Finally it is worth to mention that in our verification study we have proved that it is enough to apply the wake-field kick once per turn. Still, in the case with space charge numerous kicks are necessary for the space-charge module.

WAKE FIELD IN LONG BUNCHES

The validation of the wake-field module with the exponential wake potential $W(\tau) = W_0 \exp(-\alpha\tau)$ is not complete, since the most important case in e.g. SIS-synchrotrons — long bunches and long-range wake fields — is not included. In order to do this we again consider a problem which has an analytical solution. Take a coasting beam without effects of chromatic tune spread (zero chromaticity, or a space-charge induces large gap between coherent frequency and incoherent spectrum) and the resistive-wall impedance in the thick-wall regime. The latter implies that the skin depth $\delta_{\mathrm{sk}} = \sqrt{2c/Z_0\sigma_{\mathrm{rw}}\Omega_{\mathrm{cb}}}$ is small compared to the pipe wall thickness. This impedance is given by

$$\mathrm{Re}\Big(Z_{\mathrm{rw}}^{\perp}\Big) = \frac{L_{\mathrm{rw}}}{2\pi b^3} Z_0\, \delta_{\mathrm{sk}}\ , \tag{7}$$

where Z_0=376.7 Ohm, b is the pipe radius and $\Omega_{\mathrm{cb}} = Q_{\mathrm{cb}}\omega_0$ is the coherent frequency of the coasting beam. Since $Z_{\mathrm{rw}}^{\perp}$ is large for small frequencies as $1/\sqrt{\Omega}$, the eigenmode with the smallest frequency is the most unstable. As the coasting-beam unstable spectrum is represented by slow waves $\Omega_{\mathrm{cb}} = (n - Q_0)\omega_0$, the resistive-wall impedance mostly excites the eigenmode with

$$Q_{\mathrm{cb}} = 1 - Q_{\mathrm{f}} \tag{8}$$

and with the mode index n, which is the closest integer above the bare tune Q_0. Here Q_{f} is the fractional part of Q_0. The growth rate of a coasting-beam mode is

$$\mathrm{Im}(\Delta Q_{\mathrm{cb}}) = \frac{I_0 q_{\mathrm{ion}}}{4\pi c\gamma m Q_0 \omega_0} \mathrm{Re}\Big(Z^{\perp}\Big)\ . \tag{9}$$

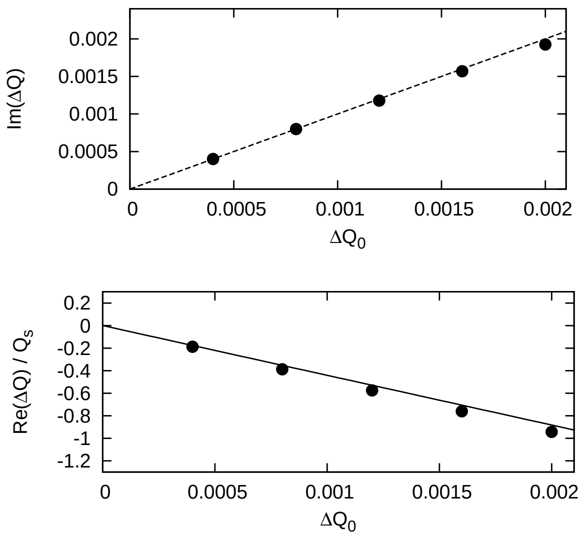

Figure 8: Growth rates and real tune shifts of the $k = 0$ head-tail mode with space charge $q = 5$. Circles are simulation results for a Gaussian bunch, both Im(ΔQ) and Re(ΔQ) are scaled by 1.89. Lines are given by the airbag theory.

In our verification simulations we consider a coasting beam with Gaussian momentum distribution, wake-field kicks are implemented according to Eq. (6). The resistive-wall wake function which corresponds to the impedance Eq. (7) is given by

$$W_{\rm rw}(z) \quad = \quad -\frac{cL_{\rm rw}}{b^3}\left(\frac{\beta}{\pi}\right)^{3/2}\sqrt{\frac{Z_0}{z\,\sigma_{\rm rw}}}\,. \qquad (10)$$

In our parametric study for coasting beams we compare the theory of the impedance Eq. (7) and the slow-wave Eqs. (8) and (9), on the one hand, with simulation results for the wake field Eq. (10), on the other hand. Figure 9 demonstrates this comparison, where the growth rate is normalized by the parameter $\Delta Q_{\rm rw0}$, which is the value of $\mathrm{Im}(\Delta Q_{\rm cb})$ calculated for $\Omega_{\rm cb} = \omega_0$, or formally $Q_{\rm f} = 0$. For this good agreement it was essential to include previous turns in the wake module Eq. (6), which seems to be obvious, since the "head" (talking in bunch terms) is strongly kicked by the "tail" from the previous turn in a coasting beam. Furthermore, the wake function Eq. (10) is a long-range potential, basically non-saturating. However, even for small $(1 - Q_{\rm f})$ it was enough to include three previous turns to achieve an adequate description.

The coasting-beam mode type we could verify with the resulting beam-offset structure. Figure 10 shows an example of the dipole moment pattern (which is equal to the beam-offset for the dc beam) of an unstable beam from a `HEADTAIL` simulation. The dashed line is the fixed-location signal $\Delta_x(z)$, which is the natural output of our simulations, and the solid line is the snapshot signal $\Delta_x^{\rm snap}(z) = e^{-iQ_0 z/R}\Delta_x(z)$, which we would see at a certain time. The

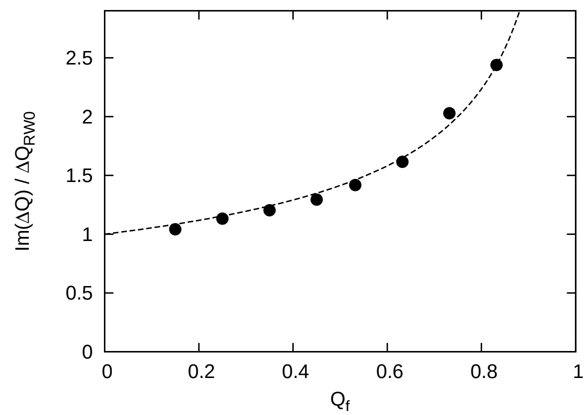

Figure 9: Growth rate of the resistive-wall instability in a coasting beam versus fractional part of the betatron tune. Dashed line is the analytic result Eq. (9) for the mode Eq. (8) and for the impedance Eq. (7). Circles: simulations with the wake field Eq. (10).

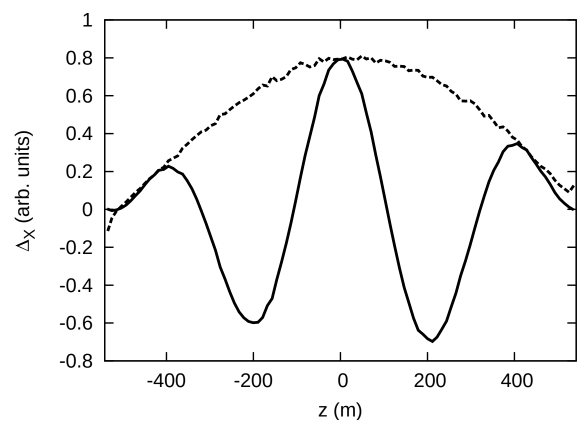

Figure 10: Transverse offset structure over the circumference of an unstable coasting beam from an exemplary simulation. Dashed line is the fixed-location signal, the solid line is the corresponding snapshot signal.

betatron tune is $Q_0 = 2.532$ here, accordingly, we observe the $n = 3$ mode.

CONCLUSIONS

Space-charge and wake-field modules, suitable for long time scale simulations of the weak head-tail instability, have been verified with the airbag-barrier model [7], which has an analytical solution for the exponential wake potential.

Simulations for moderate and strong space charge demonstrated a good agreement between `PATRIC` and `HEADTAIL` simulations and the theory. It was found that betatron oscillations should be well resolved in the particle tracking for a correct description of space-charge effects. Simulations with a Gaussian bunch showed that the airbag theory Eq. (1), although considering a simple bunch model, describes rather accurately effects of space charge, especially for strong space charge.

Instability simulations reproduced fairly well the theory predictions concerning growth rates and real tune shifts, both with and without space charge. Simulations with a Gaussian bunch demonstrated that the airbag theory Eqs. (2)-(5) can be very helpful even in understanding effects of space charge for the head-tail instability in realistic bunches. Our wake-field validation with a coasting beam has proved that the method can be surely used for long bunches and long-range wake potentials.

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