ROBUST STABILITY ANALYSIS OF ORBIT FEEDBACK CONTROLLERS

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Abstract

Closed loop stability of electron orbit feedback controllers is affected by mismatches between the accelerator model and the real machine. In this paper, the small gain theorem is used to express analytical criteria for closed loop stability in the presence of spatial uncertainty. It is also demonstrated how the structure of the uncertainty models affects the conservativeness of the robust stability results. The robust stability criteria are applied to the Diamond electron orbit controller and bounds on the allowable size of spatial uncertainties which guarantee closed loop stability is determined.

INTRODUCTION

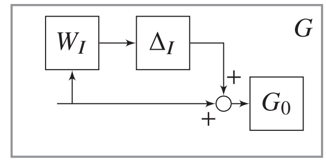
For electron orbit control, the nominal or "golden" response matrix is used to design the controller gains. The control system is referred to as "robust" if it is insensitive to differences between the actual system and the model of the system which is used to design the controller [1]. These differences are referred to as model-plant mismatch or simply uncertainty. The general approach for robust control design is to find a representation of the model uncertainty and then check for robust stability i.e. determine whether the system remains stable for all "real" processes within the uncertainty set. The following notation is adopted: Π is a set of possible perturbed plant models, $G_0(z)$ is the nominal plant model which belongs to the set Π and $G(z)$ is the perturbed model (representing the real plant) which also belongs to the set Π. A norm-bounded uncertainty description is used to describe the uncertainty i.e. the set Π is generated by allowing H_∞ norm bounded perturbations in the nominal plant i.e. Δ is a normalised perturbation with H_∞ norm ≤ 1 and the H_∞-norm of a discrete operator $M(z^{-1})$ is defined as

$$\|M\|_\infty := \max_{\omega \in [-\pi,\pi]} \sigma_{max}(G(e^{j\omega})) \quad (1)$$

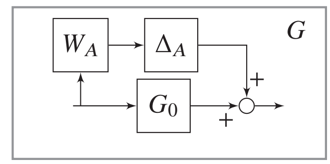
where M is linear, time-invariant and $\sigma_{max}(\cdot)$ indicates the largest singular value of the matrix.

Uncertainty can be classed as either parametric or unmodelled dynamics. Parametric uncertainty is where the model structure is known but some parameters are unknown or uncertain, whereas unmodelled dynamics refers to the case where dynamics have been neglected or missing. In this paper, "lumped" uncertainty is considered which represents one of several sources of parametric uncertainty and/or unmodelled dynamics. This type of uncertainty is usually represented as multiplicative uncertainty [1] as shown in Fig. 1a. In the case of multiplicative uncertainty,

$$\Pi_I : G(z) = G_0(z)(1 + W_I(z)\Delta_I(z)); \quad \|\Delta\|_\infty \leq 1 \forall \omega \quad (2)$$



(a) Model G_0 with multiplicative uncertainty.



(b) Model G_0 with additive uncertainty.

Figure 1: Uncertainty representations of real process G.

where the subscript I indicates "input" uncertainty. In the case of additive uncertainty,

$$\Pi_A : G(z) = G_0(z) + W_A(z)\Delta_A(z); \quad \|\Delta\|_\infty \leq 1 \forall \omega \quad (3)$$

which is shown in Fig. 1b. In each case, a weight $W(z)$ is introduced in order to normalise the perturbation to be less than 1 in magnitude at each frequency and to obtain the weight:

- for multiplicative uncertainty:

$$|W_I(e^{j\omega})| \geq \max\left|\frac{G(e^{j\omega}) - G_0(e^{j\omega})}{G_0(e^{j\omega})}\right| \quad \forall \omega \quad (4)$$

- for additive uncertainty:

$$|W_A(e^{j\omega})| \geq \max|G(e^{j\omega}) - G_0(e^{j\omega})| \quad \forall \omega. \quad (5)$$

The procedure for robust stability test is as follows:

1. The closed loop system is represented in the structure shown in Fig. 2 where Δ represents the uncertainty in the system and M is the transfer function matrix describing the closed loop "as seen by" Δ.

2. The **small gain theorem** for systems represented as Fig. 2, is applied. The theorem takes the form [1]:

Theorem 1 (Robust stability) *Assuming that the nominal system $M(e^{j\omega})$ is stable and that the perturbations Δ are stable, then the interconnected $M - \Delta$ system in Fig. 2 is stable for all perturbations Δ, satisfying $\|\Delta\|_\infty \leq 1$ (i.e. the system is robustly stable) if and only if*

$$\|M\|_\infty < 1. \quad (6)$$

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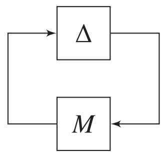

Figure 2: M-Δ structure for robust stability analysis.

In this paper, the robust stability test is applied to the closed loop for the Diamond electron orbit feedback system in the presence of uncertainty in the response matrix. Firstly, it is demonstrated how to apply the robust stability test to the system using the Singular Value Decomposition (SVD) of the response matrix and secondly the Fourier decomposition of the response matrix is used for the robust stability test.

ROBUST STABILITY ANALYSIS: SVD METHOD

Additive Uncertainty in the Response Matrix

Consider an additive uncertainty on the response matrix where the uncertainty is calculated from the difference between the golden and measured response matrices, such that

$$R = R_0 + W_R \Delta_R \tag{7}$$

and the weight W_R is introduced so that $\|\Delta_R\|_\infty \leq 1$ and found using Eq. 5. The perturbed closed loop system using an Internal Model Control (IMC) structure is represented in Fig. 3 where

$$\begin{aligned} U(z) &= -q(z) W_R \Psi D \Sigma^{-1} \left[g(z) \Phi^{\mathrm{T}} V(z) \right. \\ &\quad \left. + g(z) \Phi^{\mathrm{T}} R W_R^{-1} U(z) - g(z) \Sigma \Psi^{\mathrm{T}} W^{-1} U(z) \right] \\ U(z) &= -g(z) q(z) W_R \Psi D \Sigma^{-1} \Phi^{\mathrm{T}} V(z). \end{aligned} \tag{8}$$

Therefore the system "seen by" the uncertainty is the transfer matrix

$$M_R = -q(z) g(z) W_R \Psi D \Sigma^{-1} \Phi^{\mathrm{T}} \tag{9}$$

and for robust stability

$$\gamma_R = \|M_R\|_\infty \leq 1, \quad \forall \omega. \tag{10}$$

Multiplicative Uncertainty in Singular Values Σ_0

A multiplicative uncertainty, Δ_Σ in the singular values can be represented as

$$\begin{aligned} \Sigma &= \Sigma_0 \left(\Delta_\Sigma + I \right) \\ \Delta_\Sigma &= \Sigma_0^{-1} \Sigma - I \end{aligned} \tag{11}$$

where Σ_0 is derived from the SVD of the golden response matrix and Σ is derived from the SVD of the measured response matrix. In this case, the uncertainty is diagonal i.e.

$$\Delta_\Sigma = \mathrm{diag}\{\delta_\Sigma\}, \quad |\delta_{\sigma_i}| \leq 1 \quad \forall i. \tag{12}$$

The closed loop system with Δ_Σ is shown in Fig. 4 where

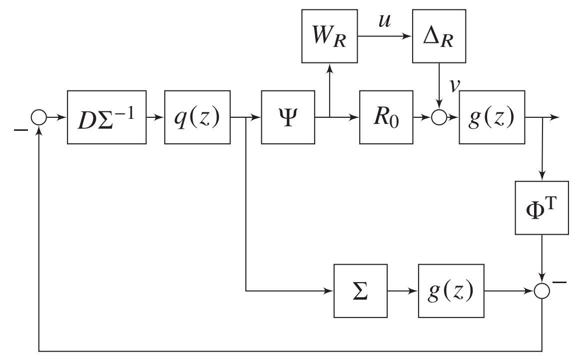

Figure 3: Additive uncertainty in R_0.

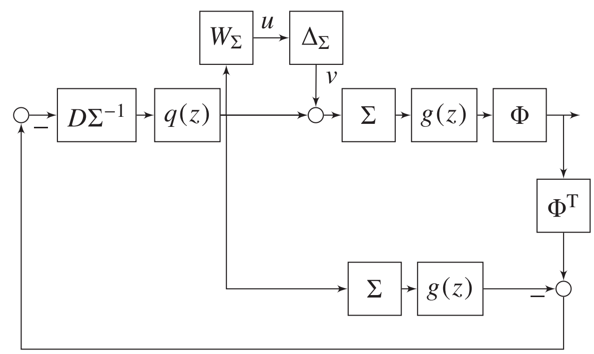

Figure 4: Multiplicative uncertainty in Σ_0.

$$\begin{aligned} U(z) = -W_\Sigma q(z) D \Sigma^{-1} & \left[g(z) V(z) + g(z) \Sigma W_\Sigma^{-1} U(z) - \right. \\ & \left. + g(z) \Sigma W_\Sigma^{-1} U(z) \right] \end{aligned} \tag{13}$$

so that

$$M_\Sigma = -q(z) g(z) W_\Sigma D \Sigma^{-1}. \tag{14}$$

Using the small gain theorem, for robust stability,

$$\gamma_\Sigma = \|M_\Sigma\|_\infty \leq 1 \quad \forall \omega. \tag{15}$$

Multiplicative Uncertainty in Singular Vectors Φ_0 *and* Ψ_0

If the uncertainty is considered to be in the left singular vectors, such that

$$\begin{aligned} \Phi &= \left(\Delta_\Phi + I \right) \Phi_0 \\ \Delta_\Phi &= \Phi \Phi_0^{\mathrm{T}} - I \end{aligned} \tag{16}$$

then given the structure in Fig. 5, the transfer matrix of the linear plant is

$$M_\Phi = -q(z) g(z) W_\Phi \Phi_0 D \Phi^{\mathrm{T}} \tag{17}$$

and using the small gain theorem

$$\gamma_\Phi = \|M_\Phi\|_\infty \leq 1 \quad \forall \omega \tag{18}$$

for robust stability.

Likewise, perturbed right singular vectors can be represented as

$$\begin{aligned} \Psi &= \Psi_0 \left(\Delta_\Psi + I \right) \\ \Delta_\Psi &= \Psi_0 \Psi^{\mathrm{T}} - I \end{aligned} \tag{19}$$

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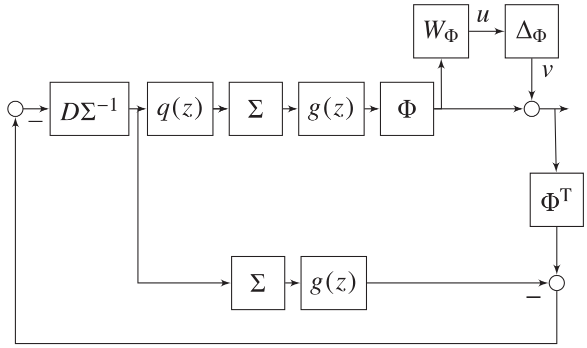

Figure 5: Multiplicative uncertainty in Φ_0.

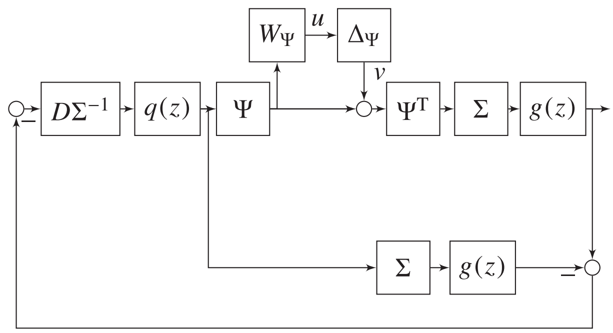

Figure 6: Multiplicative uncertainty in Ψ_0.

and given the structure in Fig. 6

$$M_\Psi = -q(z)g(z)W_\Psi \Psi_0 D \Psi^T \quad (20)$$

and robust stability

$$\gamma_\Psi = ||M_\Psi||_\infty \le 1 \quad \forall\omega. \quad (21)$$

Robust Stability Test Results

Fig. 7 shows the value of γ_R, γ_Σ, γ_Φ and γ_Ψ up to 5 kHz, where for each test, the bound is less than 1 and therefore the closed loop system is stable. The results can also be used to give a margin of stability. For additive uncertainty in the response matrix, the peak value over the range of frequencies is $\gamma_{R_{max}} = 0.2332$ occurring at DC. This means that the closed loop system is more sensitive to uncertainty at low frequencies. Given that

$$||\Delta_R||_\infty ||M_R||_\infty < 1 \quad (22)$$

and $\gamma_{R_{\max}} = 0.2332$, for robust stability

$$||\Delta_R||_\infty < 1/\gamma_{R_{\max}}. \quad (23)$$

This means that the uncertainty may increase by a factor of 4.3 before the worst-case uncertainty yields instability. Using the triangle inequality

$$\begin{aligned} ||R||_\infty &\le ||R_0||_\infty + ||\Delta_R||_\infty \\ ||R||_\infty - ||R_0||_\infty &\le \frac{1}{\gamma_{R_{\max}}} \end{aligned} \quad (24)$$

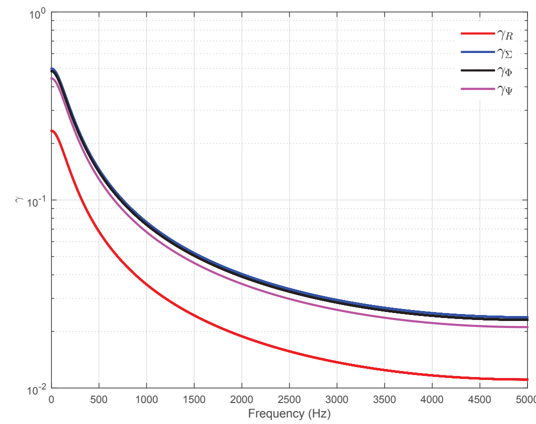

Figure 7: Size of γ_R (red), γ_Σ (blue), γ_Φ (black) and γ_Ψ (magenta) across frequencies up to 5 kHz using the singular value decomposition of the response matrix.

which gives an analytical bound on the variation of the response matrix from the nominal.

From Fig. 7, γ_Σ is also at a maximum at DC where the peak value is at $\gamma_{\Sigma_{max}} = 0.4999$ which means that the uncertainty in the singular values can increase by a factor of 2 for the closed loop system to become unstable. As before, a bound can be placed on the size of the uncertainty, i.e.

$$||\Delta_\Sigma||_\infty \le 1/\gamma_{\Sigma_{max}}. \quad (25)$$

The uncertainty, Δ_Σ, is diagonal, so a bound on each singular value can be determined by observing that, for Eq. 25 to hold,

$$|\delta_{\Sigma_i}| \le 1/\gamma_{\Sigma_{max}}. \quad (26)$$

Since

$$\delta_{\Sigma_i} = \sigma_{0_i}^{-1}\sigma_i - 1 \quad (27)$$

then the bound on each singular value to ensure robust stability can be expressed as

$$\sigma_i \le \frac{\sigma_{0_i} + \sigma_{0_i}\gamma_{\Sigma_{max}}}{\gamma_{\Sigma_{max}}}. \quad (28)$$

From Fig. 7, the maximum bound on M_Φ and M_Ψ occurs at low frequencies and is $\gamma_{\Phi_{max}} = 0.4844$ and $\gamma_{\Psi_{max}} = 0.4431$ respectively. This means that the uncertainties Δ_Φ and Δ_Ψ can increase by a factor of 2.1 and 2.3 respectively before the worst case uncertainty yields instability. The uncertainties in Φ_0 and Ψ_0 are not diagonal, therefore a bound cannot be placed on the individual elements on the matrices Φ_0 and Ψ_0, but only placed on the matrix norm. From Eq. 16, a bound on Φ_0 is determined as

$$\begin{aligned} ||\Phi||_\infty &\le ||\Delta_\Phi \Phi_0||_\infty + ||\Phi_0||_\infty \\ ||\Phi||_\infty &\le ||\Delta_\Phi||_\infty ||\Phi_0||_\infty + ||\Phi_0||_\infty \\ \frac{||\Phi||_\infty - ||\Phi_0||_\infty}{||\Phi_0||_\infty} &\le ||\Delta_\Phi||_\infty \\ \frac{||\Phi||_\infty - ||\Phi_0||_\infty}{||\Phi_0||_\infty} &\le \frac{1}{\gamma_{\Phi_{max}}} \end{aligned} \quad (29)$$

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and likewise

$$\frac{\|\Psi\|_\infty - \|\Psi_0\|_\infty}{\|\Psi_0\|_\infty} \leq \frac{1}{\gamma_{\Psi_{max}}} \tag{30}$$

gives a bound on Ψ_0.

ROBUST STABILITY ANALYSIS: FOURIER METHOD

In [2] a harmonic decomposition of the response matrix is presented, where R is written as

$$R_0 = \hat{\Phi}_0 \hat{\Sigma}_0 \hat{\Psi}_0^{-\mathrm{T}} \tag{31}$$

where

$$\begin{aligned} \hat{\Sigma}_0 &= \mathrm{diag}_{f=-F,\ldots,F}\{\hat{\sigma}_f\} \\ \hat{\Phi}_0 &= \mathrm{diag}_{m=1,\ldots,M}\{\sqrt{\beta_m}\}\mathrm{Re}\left[\mathrm{e}^{if\tilde{\eta}_m}\right]_{mf} \\ \hat{\Psi}_0 &= \mathrm{diag}_{n=1,\ldots,N}\{\sqrt{\beta_n}\}\mathrm{Re}\left[\mathrm{e}^{-if\tilde{\eta}_n}\right]_{nf} \end{aligned} \tag{32}$$

and $\hat{\Phi}_0$ is determined by the beta function β_m and normalised phase advance η_m at BPM locations and $\hat{\Psi}_0$ is determined by the beta function β_n and normalised phase advance η_n at corrector locations and $\hat{\Sigma}_0$ is determined by the Fourier coefficients which depend on the tune ν such that

$$\hat{\sigma}_f = \frac{1}{2\pi}\frac{2\nu}{\nu^2 - f^2} \quad (f = 0, 1, 2, \ldots). \tag{33}$$

Similar to the treatment using SVD, uncertainties can be included in the matrices of the harmonic decomposition where

$$\begin{aligned} \hat{\Sigma} &= \hat{\Sigma}_0\left(\Delta_{\hat{\Sigma}} + I\right) \\ \hat{\Phi} &= \left(\Delta_{\hat{\Phi}} + I\right)\hat{\Phi}_0 \\ \hat{\Psi} &= \hat{\Psi}_0\left(\Delta_{\hat{\Psi}} + I\right) \end{aligned} \tag{34}$$

and the associated transfer matrix M for each case is as described in Eq. 14, Eq. 17 and Eq. 20 but with the harmonic matrix counterpart to the SVD matrices i.e.

$$\begin{aligned} M_R &= -q(z)g(z)W_R\Psi\hat{D}\hat{\Sigma}^{-1}\hat{\Phi}^{\mathrm{T}} \\ M_{\hat{\Sigma}} &= -q(z)g(z)W_{\hat{\Sigma}}\hat{D}\hat{\Sigma}^{-1} \\ M_{\hat{\Phi}} &= -q(z)g(z)W_{\hat{\Phi}}\hat{\Phi}_0\hat{D}\hat{\Phi}^{\mathrm{T}} \\ M_{\hat{\Psi}} &= -q(z)g(z)W_{\hat{\Psi}}\hat{\Psi}_0\hat{D}\hat{\Psi}^{\mathrm{T}}. \end{aligned} \tag{35}$$

Fig. 8 shows the sizes of γ_R, $\gamma_{\hat{\Sigma}}$, $\gamma_{\hat{\Phi}}$ and $\gamma_{\hat{\Psi}}$ up to 5kHz, and for each case, the system is determined to be robustly stable for all frequencies. The peak value of $\gamma_{\hat{\Sigma}}$ is 0.3732 which means that the norm of the closed loop transfer matrix $M_{\hat{\Sigma}}$ can increase by a factor of 2.7 before the system becomes unstable. However this uncertainty in Fourier coefficients can also give a bound on the tune. In [2] the relationship between Fourier coefficients and the tune is given as

$$\Delta_{\hat{\Sigma}} = W_{\hat{\Sigma}}\Delta_\nu. \tag{36}$$

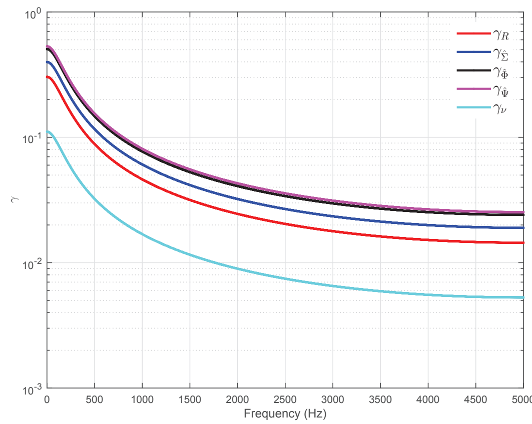

Figure 8: Size of γ_R (red), $\gamma_{\hat{\Sigma}}$ (blue), $\gamma_{\hat{\Phi}}$ (black), $\gamma_{\hat{\Psi}}$ (magenta) and γ_ν (cyan) across frequencies up to 5 kHz using the harmonic decomposition of the response matrix.

Since $\|\Delta_{\hat{\Sigma}}\|_\infty \leq 1/\gamma_{max_{\hat{\Sigma}}}$ then

$$\|\Delta_\nu\|_\infty = \frac{1}{\gamma_{max_{\hat{\Sigma}}}}\frac{1}{\|W_{\hat{\Sigma}}\|_\infty} \tag{37}$$

which is the maximum allowable size of uncertainty in the tune for the closed loop system to remain stable. The size of γ_ν is also shown in Fig. 8 and from the peak value, M_ν can increase by a factor of 9. This results corresponds to a tune change of 0.2 %, which means that for very small tune drifts, the closed loop system is guaranteed stable. The results are however conservative as it is assumed that the only source of uncertainty in the response matrix is from tube drift.

CONCLUSION

The small gain theorem was used to express analytical criteria for closed loop stability in the presence of several sources of spatial uncertainty. The robust stability test applied to the SVD matrices is useful for determining the closed loop stability when the controller is designed using SVD. However in order to determine bounds on beam parameters, the Fourier decomposition approach is better suited.

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